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On the Fractional Fractal Analysis of Multivariate Pointwise Lipschitz Oscillating Regularity

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Abstract: Classical Lipschitz regularity does not allow to capture possible different oscillating directional pointwise regularity behaviors in coordinate axes of functions f on $\mathbb{R}^d$, $d \geq 2$. To overcome this drawback, we use iterated fractional primitives to introduce a notion of multivariate pointwise Lipschitz oscillating regularity. We show a characterization in hyperbolic wavelet bases. As an application, we obtain the fractal print dimension of a given set of multivariate Lipschitz oscillating regularity, from the knowledge of fractional axes oscillating spaces to which f belongs.

Keywords: Multivariate Lipschitz oscillating regularity; Iterated fractional axes primitives; Print dimension.

1 Introduction

Multifractal analysis studies functions whose pointwise regularity varies from point to point. Classical multifractal analysis studies fractal set of points x where f has a given Lipschitz regularity at x. This regularity allows to denoise, classify and reconstruct signals [1,2,3].
However many multivariate functions have various directional and very oscillatory behaviors. These behaviors are important in image and signal analysis (see [4,5]).
Developing tools to deal with such problems seems to be one of the goals of multifractal analysis. The usefulness of wavelet techniques in multifractal analysis has been apparent for some years (see [6] and references therein). In this paper, we provide further insight into the interplay between wavelet analysis and multifractal analysis via fractional primitives.

Let us recall these notions. Let $0 < \alpha < 1$. Let $x \in \mathbb{R}^d$.A function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $Lip^\alpha(x)$ if there exists $C > 0$ such that

$$\exists\, C > 0 \ \forall\, y \qquad |f(y) - f(x)| \leq C|y-x|^\alpha \,. \tag{1}$$

The Lipschitz regularity of f at x is

$$\alpha_f(x) = \sup\{0 < \alpha < 1 : \quad f \in Lip^\alpha(x)\} \,. \tag{2}$$

This regularity doesn't capture oscillations of the function around x: it is the same for cusps such as $|y-x|^\gamma$, or very oscillatory behaviors such as

$$f_{\gamma,\beta}(y) = |y-x|^\gamma \sin(|y-x|^{-\beta}) \ \text{ with } 0 < \gamma < 1 \text{ and } \beta > 0 \,. \tag{3}$$

Let $0 < t < 1$. The fractional primitive of order t of a function f is $f_t := (I - \triangle)^{-t/2} f$, where the operator $(I - \triangle)^{-t/2}$ is the convolution operator, for which $\hat{f}_t(\xi) = (1 + |\xi|^2)^{-t/2} \hat{f}(\xi)$.
For t small enough, $\alpha_{f_{\gamma,\beta}}(x) = \gamma + (1+\beta)t$. The increase is $(1+\beta)t$ instead of t. This explains the following definition of [7].

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Definition 1 *The oscillating singularity exponents of f at x are*

$$\left(\alpha_f(x), \beta_f(x) = -1 + \frac{\partial}{\partial t}\alpha_{f_t}(x)|_{t=0}\right) .$$

Many signals on $\mathbb{R}^d, d \geq 2$, present various features and characteristics when measured in different directions. Unfortunately, the oscillating singularity exponents do not capture possible different directional regularity behaviors in coordinate axes. For example, they do not fit functions of the form $g(y_1, \cdots, y_d) = \prod_{i=1}^{d} |y_i - x_i|^{\alpha_i} \sin(|y_i - x_i|^{-\beta_i})$ where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d) \in (0,1)^d$ and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_d) \in (0,\infty)^d$.

In order to overcome this drawback, we propose first, to replace direct increments of f by iterated axes differences, and second we replace the fractional primitive of f by the iterated fractional axes primitives. We will therefore introduce a new notion of multivariate Lipschitz oscillating regularity for $\boldsymbol{\alpha} \in (0,1)^d$ and $\boldsymbol{\beta} \in (0,\infty)^d$. Let us explain this.

Let u_i denotes the i-th unit vector in $\mathbb{R}^d$. For $h \in \mathbb{R}$, write $\Delta_{h,i} f(x) = f(x + hu_i) - f(x)$. Let $\mathscr{D} = \{1, \cdots, d\}$. Let $\mathbf{s} = (s_1, \cdots, s_d)$ and $\mathbf{s}' = (s'_1, \cdots, s'_d)$. Write $\mathbf{s} \leq \mathbf{s}'$ if $s_i \leq s'_i$ for all $i \in \mathscr{D}$, and $\mathbf{s} < \mathbf{s}'$ if $s_i < s'_i$ for all $i \in \mathscr{D}$. Let also $\mathbf{0} = (0, \cdots, 0)$ and $\mathbf{1} = (1, \cdots, 1)$.

For $A = \{i_1, \cdots, i_k\} \subset \mathscr{D}$ and $\mathbf{h} = (h_1, \cdots, h_d) \in \mathbb{R}^d$, the iterated axes difference of f is defined as

$$\Delta_{\mathbf{h},A} f = \Delta_{h_{i_1}, i_1} \circ \cdots \circ \Delta_{h_{i_k}, i_k} f \, . \tag{4}$$

For example, for $d = 2$

$$\Delta_{h,\{1,2\}} f(x) = f(x+h) + f(x) - f(x_1, x_2 + h_2) - f(x_1 + h_1, x_2) \, . \tag{5}$$

For $d = 3$

$$\begin{aligned} \Delta_{h,\{1,2,3\}} f(x) &= f(x+h) - f(x_1+h_1, x_2+h_2, x_3) + f(x_1, x_2, x_3+h_3) - f(x) \\ &- f(x_1, x_2+h_2, x_3+h_3) + f(x_1, x_2+h_2, x_3) - \\ &- f(x_1+h_1, x_2, x_3+h_3) + f(x_1+h_1, x_2, x_3) \, . \end{aligned} \tag{6}$$

Definition 2 *Let $f \in L^\infty(\mathbb{R}^d)$. Let $\boldsymbol{\alpha} \in (0,1)^d$ and $x \in \mathbb{R}^d$. We say that $f \in R^{\boldsymbol{\alpha}}(x)$, if there exists $C > 0$ such that*

$$\forall (\emptyset \neq A = \{i_1, \cdots, i_k\} \subset \mathscr{D}) \ \forall \mathbf{h} = (h_1, \cdots, h_d) \in \mathbb{R}^d \quad |\Delta_{\mathbf{h},A} f(x)| \leq C \prod_{n=1}^{k} |h_{i_n}|^{\alpha_{i_n}} \, . \tag{7}$$

For $d = 1$, $R^{\boldsymbol{\alpha}}(x) = Lip^{\alpha}(x)$. For $d \geq 2$, $R^{\boldsymbol{\alpha}}(x)$ is a pointwise version of some anisotropic generalized Hölder classes of [9] for $p = \infty$. Moreover, contrary to [11,12,13], $R^{\boldsymbol{\alpha}}(x)$ yields extra information on the Lipschitz regularity of partial functions. For $d = 2$, $R^{\boldsymbol{\alpha}}(x)$ is more precise than that of [10].

Definition 3 *Let $f \in L^2(\mathbb{R}^d)$. The iterated fractional axes primitives of order $\mathbf{t} = (t_1, \cdots, t_d) \in (0,1)^d$ of f is the function $f_{\mathbf{t}}(y) = \prod_{i=1}^{d} (I - \frac{\partial^2}{\partial_i^2})^{-t_i/2} f$, where the operator $\prod_{i=1}^{d} (I - \frac{\partial^2}{\partial_i^2})^{-t/2}$ is the convolution operator which amounts multiplying the Fourier transform of f with $\prod_{i=1}^{d} (1 + |\xi_i|^2)^{-t_i/2}$.*

Definition 4 *Let $\boldsymbol{\alpha} \in (0,1)^d$ and $\boldsymbol{\beta} \in (0,\infty)^d$. A function f has multivariate rectangular Lipschitz oscillating regularity $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, and we write $f \in M^{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)$ if*

$$\forall \, \varepsilon > 0 \qquad f \in R^{(\alpha_1 - \varepsilon, \cdots, \alpha_d - \varepsilon)}(x) \, . \tag{8}$$

and

$$\exists \, \mathbf{a} > \mathbf{0} \ \forall \, \varepsilon > 0 \ \forall \, \mathbf{0} < \mathbf{t} \leq \mathbf{a} \qquad f_{\mathbf{t}} \in R^{(\alpha_1 + t_1(1+\beta_1) - \varepsilon, \cdots, \alpha_d + t_d(1+\beta_d) - \varepsilon)}(x) \, . \tag{9}$$

In 1988, Rogers [21] has replaced the Hausdorff dimension by a multivariate dimension. If D is a rectangle of $\mathbb{R}^d$, let $L_1(D), \cdots, L_d(D)$ be the edge-lengths of D, with $L_d(D) \leq \cdots \leq L_1(D)$. For $A \subset \mathbb{R}^d$ and $\varepsilon > 0$, a family (D_n) of rectangles is a ε-cover of A and we write $(D_n) \in \mathscr{C}_\varepsilon(A)$ if $\forall \, n \ L_1(D_n) < \varepsilon$ and $A \subset \bigcup_n D_n$.
For $\varepsilon > 0$ and $(\delta_1, \cdots, \delta_d) \geq \mathbf{0}$, let

$$\mathscr{H}_\varepsilon^{(\delta_1, \cdots, \delta_d)}(A) = \inf\{ \sum_{n \in \mathbb{N}} \prod_{i=1}^{d} (L_i(D_n))^{\delta_i} \ : \ (D_n)_n \in \mathscr{C}_\varepsilon(A)\} \, . \tag{10}$$

Set the measure

$$\mathscr{H}^{(\delta_1,\cdots,\delta_d)}(A) = \sup_{\varepsilon>0} \mathscr{H}_{\varepsilon}^{(\delta_1,\cdots,\delta_d)}(A) \tag{11}$$

The print dimension of A (see [16,21]) is the set defined as

$$printA = \{(\delta_1,\cdots,\delta_d) \geq \mathbf{0} \ : \ \mathscr{H}^{(\delta_1,\cdots,\delta_d)}(A) > 0\} \,. \tag{12}$$

In the next section, we prove a characterization of multivariate Lipschitz oscillating regularity in hyperbolic wavelet bases. As an application, in Section 3, given $\mathbf{0} < \boldsymbol{\alpha} < \mathbf{1}$ and $\boldsymbol{\beta} > \mathbf{0}$, we obtain the fractal print dimension of sets of the form

$$E_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \{x : f \in M^{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)\} \tag{13}$$

from the knowledge of fractional axes oscillating spaces to which f belongs.

2 Characterization of multivariate oscillating regularities in hyperbolic wavelet bases

Let ψ_{-1} (resp. ψ_1) be a father (resp. mother wavelet). Put $[j] = j$ if $j \in \mathbb{N}_0$ and $[-1] = 0$. Put $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$. If $j \in \mathbb{N}_{-1}$ and $k \in \mathbb{Z}$, write

$$\psi_{j,k}(t) = \begin{cases} \psi_1(2^j t - k) & \text{if } j \in \mathbb{N}_0 \,, \\ \psi_{-1}(t-k) & \text{if } j = -1 \,. \end{cases} \tag{14}$$

Then $\left(2^{[j]/2}\psi_{j,k}(t)\right)_{j\in\mathbb{N}_{-1},k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$ (see [14,19,20]).

For $\mathbf{j} \in \mathbb{N}_{-1}^d$, put $[\mathbf{j}] = ([j_1],\cdots,[j_d])$ and $|\mathbf{j}| = \sum_{i=1}^{d} j_i$.

For $\mathbf{k} \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$, put

$$\Psi_{\mathbf{j},\mathbf{k}}(x) = \prod_{i=1}^{d} \psi_{j_i,k_i}(x_i) \,. \tag{15}$$

Then $\{2^{|[\mathbf{j}]|/2}\,\Psi_{\mathbf{j},\mathbf{k}} \ : \ \mathbf{k} \in \mathbb{Z}^d, \mathbf{j} \in \mathbb{N}_{-1}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ called hyperbolic wavelet basis [8,15,22,23,24]. Thus any function $f \in L^2(\mathbb{R}^d)$ can be written as

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{j}\in\mathbb{N}_{-1}^d} C_{\mathbf{j},\mathbf{k}} \Psi_{\mathbf{j},\mathbf{k}} \tag{16}$$

with

$$C_{\mathbf{j},\mathbf{k}} = 2^{|[\mathbf{j}]|} \int_{\mathbb{R}^d} \Psi_{\mathbf{j},\mathbf{k}}(y) f(y)\, dy \,. \tag{17}$$

Let $-\mathbf{1}$ denotes the vector $(-1,\cdots,-1)$ in $\mathbb{R}^d$.

Remark 1 *Using wavelet arguments similar to those of [7,5], we can replace $f_{\mathbf{t}}$ in property (9) by the pseudo-iterated fractional axes primitives of order $\mathbf{t} = (t_1,\cdots,t_d)$ of f given by*

$$\tilde{f}_{\mathbf{t}} = \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{j}\in\mathbb{N}_{-1}^d} \tilde{C}_{\mathbf{t},\mathbf{j},\mathbf{k}} \Psi_{\mathbf{j},\mathbf{k}} \tag{18}$$

with

$$\tilde{C}_{\mathbf{t},\mathbf{j},\mathbf{k}} = 2^{-[j_1]t_1 - \cdots - [j_d]t_d} C_{\mathbf{j},\mathbf{k}} \,. \tag{19}$$

We will prove the following theorem that allows to characterize pointwise multivariate oscillating regularities.

Theorem 1 *Let $\boldsymbol{\alpha} \in (0,1)^d$ and $\boldsymbol{\beta} \in (0,\infty)^d$.*

1.If $f \in M^{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)$ then

$$\forall\, \mathbf{k} \in \mathbb{Z}^d \quad \left|C_{-\mathbf{1},\mathbf{k}}\right| \leq C \,, \tag{20}$$

$$\forall\, \mathbf{k} \in \mathbb{Z}^d \ \ \forall\, \mathbf{j} \in \mathbb{N}_{-1}^d \backslash \{-\mathbf{1}\} \quad \left|C_{\mathbf{j},\mathbf{k}}\right| \leq C \prod_{i=1}^{d} 2^{-[j_i]\alpha_i} (1 + \left|k_i - 2^{[j_i]} x_i\right|)^{\alpha_i} \,. \tag{21}$$

and

$$\exists \mathbf{a} > \mathbf{0}\, \forall \varepsilon > 0\, \forall \mathbf{0} < \mathbf{t} \leq \mathbf{a}\, \forall \mathbf{j} \in \mathbb{N}_{-1}^d \backslash \{-\mathbf{1}\}\, \forall \mathbf{k} \in \mathbb{Z}^d \ \left|C_{\mathbf{j},\mathbf{k}}\right| \leq C \prod_{i=1}^{d} 2^{-[j_i](\alpha_i + t_i\beta_i - \varepsilon)} (1 + \left|k_i - 2^{[j_i]} x_i\right|)^{\alpha_i + t_i\beta_i - \varepsilon} \,. \tag{22}$$

2.*Let f be uniformly Lipschitz, in the sense that $f \in L^\infty$ and*

$$\exists C > 0 \;\; 0 < \varepsilon < 1 \quad \forall\, y,z \in \mathbb{R}^d \qquad |f(y) - f(z)| \le C|y-z|^\varepsilon \,. \tag{23}$$

If both (20) *and* (21) *hold then*

$$\forall\, \boldsymbol{\alpha}' < \boldsymbol{\alpha} \qquad f \in R^{\boldsymbol{\alpha}'}(x)\,. \tag{24}$$

Remark 2 *The proof is done for $d = 1$ (see [7, 17, 18, 5]). Note also that for any d, we have no need to prove property* (22) *because it can de deduced from property* (21) *by replacing the $C_{\mathbf{j},\mathbf{k}}$ by the $\tilde{C}_{\mathbf{t},\mathbf{j},\mathbf{k}}$.*

Proof of Theorem 1

1.Let $\mathbf{j} \in \mathbb{N}^d_{-1}$ and $\mathbf{k} \in \mathbb{Z}^d$.

–If $\mathbf{j} = -\mathbf{1}$ then

$$|C_{-\mathbf{1},\mathbf{k}}| = |\int f(y) \prod_{i=1}^{d} \psi_{-1}(y_i - k_i)dy| \le ||f||_{L^\infty(\mathbb{R}^d)} ||\psi_{-1}||^d_{L^\infty(\mathbb{R})}\,.$$

–Assume that $\mathbf{j} \in \mathbb{N}^d_0$.
Relations (5), and (6), together with induction and the fact that the integral of ψ_1 vanishes yield

$$\forall\, \mathbf{j} \in \mathbb{N}^d_0 \qquad C_{\mathbf{j},\mathbf{k}} = 2^{|\mathbf{j}|} \int \Delta_{y-x,\mathscr{D}} f(x) \Psi_{\mathbf{j},\mathbf{k}}(y)\, dy\,. \tag{25}$$

Since $f \in R^\alpha(x)$ then

$$\begin{aligned} |C_{\mathbf{j},\mathbf{k}}| &\le C2^{|\mathbf{j}|} \int \left(\prod_{i=1}^{d} |y_i - x_i|^{\alpha_i} \right) |\Psi_{\mathbf{j},\mathbf{k}}(y)|\, dy \\ &= C \prod_{i=1}^{d} \left(2^{j_i} \int |y_i - x_i|^{\alpha_i} |\psi_{j_i,k_i}(y_i)|\, dy_i \right) \\ &\le C \prod_{i=1}^{d} \left(2^{j_i} \int \left(|y_i - k_i 2^{-j_i}|^{\alpha_i} + |k_i 2^{-j_i} - x_i|^{\alpha_i} \right) |\psi_{j_i,k_i}(y_i)|\, dy_i \right)\,. \end{aligned}$$

The localization of the wavelet ψ_1 yields (21).

–Assume that $\mathbf{j} \ne -\mathbf{1}$ and $\mathbf{j} \notin \mathbb{N}^d_0$. For convenience of notation, we will consider cases $d = 2$ and $d = 3$. Our method extends to the general case following the same induction idea.

–Case $d = 2$.

•If $j_1 \ge 0$ and $j_2 = -1$ then

$$\begin{aligned} C_{\mathbf{j},\mathbf{k}} &= 2^{j_1} \int (f(y) + f(x) - f(x_1,y_2)) \Psi_{\mathbf{j},\mathbf{k}}(y)\, dy \\ &= 2^{j_1} \int (\Delta_{y-x,\{1,2\}} f(x) + f(y_1,x_2)) \Psi_{\mathbf{j},\mathbf{k}}(y) dy \\ &= 2^{j_1} \int \Delta_{y-x,\{1,2\}} f(x) \Psi_{\mathbf{j},\mathbf{k}}(y) dy \;+\; 2^{j_1} \int f(y_1,x_2) \Psi_{\mathbf{j},\mathbf{k}}(y) dy \end{aligned}$$

Since $\int \psi_{-1} = 1$ then

$$C_{\mathbf{j},\mathbf{k}} = 2^{j_1} \int \Delta_{y-x,\{1,2\}} f(x) \Psi_{\mathbf{j},\mathbf{k}}(y) dy + 2^{j_1} \int f(y_1,x_2) \psi_{j_1,k_1}(y_1) dy_1\,. \tag{26}$$

Since $f \in R^\alpha(x)$, then as above, thanks to the localization of both ψ_{-1} and ψ_1, the first term decays as in (21). On the other hand, since $x = (x_1,x_2)$ and $f \in R^{(\alpha_1,\alpha_2)}(x)$ then $f(.,x_2) \in Lip^{\alpha_1}(x_1)$. Therefore Remark 2 yields

$$\begin{aligned} 2^{j_1} |\int f(y_1,x_2) \psi_{j_1,k_1}(y_1) dy_1| &\le C2^{-j_1\alpha_1} (1 + |k_1 - 2^{j_1} x_1|)^{\alpha_1} \\ &\le C \prod_{i=1}^{2} 2^{-[j_i]\alpha_i} (1 + \left|k_i - 2^{[j_i]} x_i\right|)^{\alpha_i} \;\; \text{(because } [-1] = 0) \,. \end{aligned}$$

•If $j_1 = -1$ and $j_2 \ge 0$ then by similarity $C_{\mathbf{j},\mathbf{k}}$ satisfies (21).

–Case $d = 3$.

•If $(j_1, j_2) \in \mathbb{N}^2_0$ and $j_3 = -1$, then using (6) and the fact that $\int \psi_1 = 0$

$$C_{\mathbf{j},\mathbf{k}} = 2^{||\mathbf{j}||} \int_{\mathbb{R}^3} (\Delta_{y-x,\{1,2,3\}} f(x) + f(y_1,y_2,x_3)) \Psi_{\mathbf{j},\mathbf{k}}(y) dy$$

where $y=(y_1,y_2,y_3)$. As above the term $2^{|[\mathbf{j}]|}\int_{\mathbb{R}^3}\Delta_{y-x,\{1,2,3\}}f(x)\Psi_{\mathbf{j},\mathbf{k}}(y)dy$ satisfies (21).
On the other hand, since $\int\psi_{-1}=1$ then
$2^{|[\mathbf{j}]|}\int_{\mathbb{R}^3}f(y_1,y_2,x_3)\Psi_{\mathbf{j},\mathbf{k}}(y)dy=2^{j_1+j_2}\int_{\mathbb{R}^2}f(y_1,y_2,x_3)\Psi_{(j_1,j_2),(k_1,k_2)}(y_1,y_2)dy_1dy_2$.
Since $x=(x_1,x_2,x_3)$ and $f\in R^{(\alpha_1,\alpha_2,\alpha_3)}(x)$ then $f(.,.,x_3)\in R^{(\alpha_1,\alpha_2)}(x_1,x_2)$. From the above computation done in the case $d=2$

$$2^{j_1+j_2}|\int_{\mathbb{R}^2}f(y_1,y_2,x_3)\Psi_{(j_1,j_2),(k_1,k_2)}(y_1,y_2)dy_1dy_2|\le C\prod_{i=1}^{2}(2^{-[j_i]}+\left|k_i2^{-[j_i]}-x_i\right|)^{\alpha_i}\ .$$

Since $j_3=-1$ and $[-1]=0$ then

$$2^{j_1+j_2}|\int_{\mathbb{R}^2}f(y_1,y_2,x_3)\Psi_{(j_1,j_2),(k_1,k_2)}(y_1,y_2)dy_1dy_2|\le C\prod_{i=1}^{3}(2^{-[j_i]}+\left|k_i2^{-[j_i]}-x_i\right|)^{\alpha_i}\ .$$

- Of course, the cases ($(j_1,j_3)\in\mathbb{N}_0^2$ and $j_2=-1$) and ($(j_2,j_3)\in\mathbb{N}_0^2$ and $j_1=-1$) are similar.
- If $j_1\ge 0$ and $j_2=j_3=-1$, then using (6) and the fact that $\int\psi_1=0$

$$C_{\mathbf{j},\mathbf{k}}=2^{|[\mathbf{j}]|}\int_{\mathbb{R}^3}(\Delta_{y-x,\{1,2,3\}}f(x)+f(y_1,y_2,x_3)+f(y_1,x_2,y_3)-f(y_1,x_2,x_3))\Psi_{\mathbf{j},\mathbf{k}}(y)dy\ .$$

The term $2^{|[\mathbf{j}]|}\int_{\mathbb{R}^3}\Delta_{y-x,\{1,2,3\}}f(x)\Psi_{\mathbf{j},\mathbf{k}}(y)dy$ satisfies (21).
On the other hand, since $\int\psi_{-1}=1$ and $|[\mathbf{j}]|=j_1=|[(j_1,j_2)]|$ then

$$2^{|[\mathbf{j}]|}\int_{\mathbb{R}^3}f(y_1,y_2,x_3)\Psi_{\mathbf{j},\mathbf{k}}(y)dy=2^{|[(j_1,j_2)]|}\int_{\mathbb{R}^2}f(y_1,y_2,x_3)\Psi_{(j_1,-1),(k_1,k_2)}(y_1,y_2)dy_1dy_2\ .$$

Since $x=(x_1,x_2,x_3)$ and $f\in R^{(\alpha_1,\alpha_2,\alpha_3)}(x)$ then $f(.,.,x_3)\in R^{(\alpha_1,\alpha_2)}(x_1,x_2)$. From the above computation done in the case $d=2$

$$2^{|[(j_1,j_2)]|}|\int_{\mathbb{R}^2}f(y_1,y_2,x_3)\Psi_{(j_1,j_2),(k_1,k_2)}(y_1,y_2)dy_1dy_2|\le C\prod_{i=1}^{2}(2^{-[j_i]}+\left|k_i2^{-[j_i]}-x_i\right|)^{\alpha_i}\ .$$

Since $j_3=-1$ and $[-1]=0$ then

$$2^{|[\mathbf{j}]|}|\int_{\mathbb{R}^2}f(y_1,y_2,x_3)\Psi_{(j_1,j_2),(k_1,k_2)}(y_1,y_2)dy_1dy_2|\le C\prod_{i=1}^{3}(2^{-[j_i]}+\left|k_i2^{-[j_i]}-x_i\right|)^{\alpha_i}\ .$$

Of course, similarly the same decay holds for $2^{|[\mathbf{j}]|}\int_{\mathbb{R}^3}f(y_1,x_2,y_3)\Psi_{\mathbf{j},\mathbf{k}}(y)dy$.
On the other hand, since $x=(x_1,x_2,x_3)$ and $f\in R^{(\alpha_1,\alpha_2,\alpha_3)}(x)$ then $f(.,x_2,x_3)\in Lip^{\alpha_1}(x_1)$. Therefore Remark (2) yields

$$\begin{aligned}2^{j_1}|\int f(y_1,x_2,x_3)\psi_{j_1,k_1}(y_1)dy_1| &\le C2^{-j_1\alpha_1}(1+\left|k_1-2^{j_1}x_1\right|)^{\alpha_1}\\ &\le C\prod_{i=1}^{3}2^{-[j_i]\alpha_i}(1+\left|k_i-2^{[j_i]}x_i\right|)^{\alpha_i}\ .\end{aligned}$$

- Of course, cases ($j_1=j_3=-1$ and $j_2\ge 0$) and ($j_1=j_2=-1$ and $j_3\ge 0$) are similar.

The proof of (22) follows from that of (21) using (19).

2. For convenience of notation, we consider the case $d=2$. Our method extends to the general case following the same ideas.

Remark 3 *If $f\in L^\infty(\mathbb{R}^2)$ then*

$$\forall\, x\in\mathbb{R}^2\quad |\Delta_{\mathbf{h},\{1\}}f(x)|\le 2||f||_\infty\le C\le C|h_1|^{\alpha_1'}\quad\forall\,|h_1|\ge 1\ ,$$

$$\forall\, x\in\mathbb{R}^2\quad |\Delta_{\mathbf{h},\{2\}}f(x)|\le 2||f||_\infty\le C\le C|h_2|^{\alpha_2'}\quad\forall\,|h_2|\ge 1$$

and using (5)

$$\forall\, x \in \mathbb{R}^2 \quad |\Delta_{\mathbf{h},\{1,2\}} f(x)| \le 4||f||_\infty \le C \le C\prod_{i=1}^{2} |h_i|^{\alpha_i'} \quad \forall\, (|h_1|,|h_2|) \in [1,\infty)^2 \,.$$

So in order to check that $f \in R^{\alpha'}(x)$, *it suffices to prove that*

$$|\Delta_{\mathbf{h},\{1\}} f(x)| \le C\,|h_1|^{\alpha_1'} \quad \forall\, |h_1| \le 1 \,,$$

$$|\Delta_{\mathbf{h},\{2\}} f(x)| \le C\,|h_2|^{\alpha_2'} \quad \forall\, |h_2| \le 1$$

and

$$|\Delta_{\mathbf{h},\{1,2\}} f(x)| \le C\prod_{i=1}^{2} |h_i|^{\alpha_i'} \quad \forall\, (|h_1|,|h_2|) \notin [1,\infty)^2 \,.$$

Since f is uniformly Lipschitz, then (23) holds.
If $\mathbf{j} = (j_1, j_2) \in \mathbb{N}_0^2$ or $(j_1 \ge 0, j_2 = -1)$, then for all $\mathbf{k} = (k_1,k_2) \in \mathbb{Z}^2$

$$\begin{aligned}
|C_{\mathbf{j},\mathbf{k}}| &= 2^{|[\mathbf{j}]|}\left|\int (f(y) - f(k_1 2^{-j_1}, y_2))\Psi_{\mathbf{j},\mathbf{k}}(y)\, dy\right| \\
&\le C2^{|[\mathbf{j}]|}\int |y_1 - k_1 2^{-j_1}|^{\delta} |\Psi_{\mathbf{j},\mathbf{k}}(y)|\, dy \\
&\le C2^{j_1}\int |y_1 - k_1 2^{-j_1}|^{\delta} |\psi_{j_1,k_1}(y_1)|\, dy_1 \;\; 2^{[j_2]}\int |\psi_{j_2,k_2}(y_2)|\, dy_2 \,.
\end{aligned}$$

Using the localization of the wavelet ψ_1

$$|C_{\mathbf{j},\mathbf{k}}| \le C2^{-\delta j_1} \,. \tag{27}$$

Analogously, if $\mathbf{j} \in \mathbb{N}_0^2$ or $(j_1 = -1, j_2 \ge 0)$, then for all $\mathbf{k} \in \mathbb{Z}^2$

$$|C_{\mathbf{j},\mathbf{k}}| \le C2^{-\delta j_2} \,. \tag{28}$$

Relations (27) and (28) imply that

$$\forall \mathbf{j}\ \forall \mathbf{k}\ \forall\, \theta \in [0,1] \qquad \left|C_{\mathbf{j},\mathbf{k}}\right| \le C2^{-\delta\theta[j_1]}2^{-\delta(1-\theta)[j_2]}. \tag{29}$$

Put $\delta_1 = \delta\theta$ and $\delta_2 = (1-\theta)\delta$. By (21)

$$\exists\, C > 0\ \ \forall 0 < \sigma < 1\ \forall \mathbf{j}\ \forall \mathbf{k} \quad \left|C_{\mathbf{j},\mathbf{k}}\right| \le C\prod_{i=1}^{2} 2^{-(1-\sigma)\delta_i[j_i]}\left(2^{-[j_i]} + \left|k_i 2^{-[j_i]} - x_i\right|\right)^{\sigma\alpha_i} .$$

Then

$$\left|C_{\mathbf{j},\mathbf{k}}\right| \le C\prod_{i=1}^{2} \mu_{\alpha_i,\delta_i,j_i,k_i,x_i} \tag{30}$$

where

$$\mu_{\alpha,\delta,j,k,t} = 2^{-(1-\sigma)\delta[j]}2^{-\sigma\alpha[j]}\left(1 + \left|k - t2^{[j]}\right|\right)^{\sigma\alpha} . \tag{31}$$

For $h \in \mathbb{R}$, put

$$\Delta_h \psi_{j,k}(t) = \psi_{j,k}(t+h) - \psi_{j,k}(t) \,. \tag{32}$$

Clearly

$$\Delta_{\mathbf{h},\{1\}} \Psi_{\mathbf{j},\mathbf{k}}(x) = \psi_{j_2,k_2}(x_2)\Delta_{h_1}\psi_{j_1,k_1}(x_1) \,, \tag{33}$$

$$\Delta_{\mathbf{h},\{2\}} \Psi_{\mathbf{j},\mathbf{k}}(x) = \psi_{j_1,k_1}(x_1)\Delta_{h_2}\psi_{j_2,k_2}(x_2) \tag{34}$$

and

$$\Delta_{\mathbf{h},\{1,2\}} \Psi_{\mathbf{j},\mathbf{k}}(x) = \prod_{i=1}^{2} \Delta_{h_i}\psi_{j_i,k_i}(x_i) \,. \tag{35}$$

Set

$$Z(t) = \sum_{k\in\mathbb{Z}} \left|\psi_{-1,k}(t)\right| , \tag{36}$$

$$V(h,t) = \sum_{k\in\mathbb{Z}} \left|\Delta_h \psi_{-1,k}(t)\right| , \tag{37}$$

$$T_{\delta,\alpha}(h,t) = \sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,-1,k,t} \left|\Delta_h \psi_{-1,k}(t)\right| , \tag{38}$$

$$S_{\delta,\alpha}(t) = \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,j,k,t} \left|\psi_{j,k}(t)\right| , \tag{39}$$

$$W_{\delta,\alpha}(t) = \sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,-1,k,t} \left|\psi_{-1,k}(t)\right| \tag{40}$$

and

$$R_{\delta,\alpha}(h,t) = \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,j,k,t} \left|\Delta_h \psi_{j,k}(t)\right| . \tag{41}$$

Split f as

$$f = f_{-\mathbf{1}} + F + G + H \tag{42}$$

where

$$f_{-\mathbf{1}} = \sum_{\mathbf{k}\in\mathbb{Z}^2} C_{-\mathbf{1},\mathbf{k}}\, \Psi_{-\mathbf{1},\mathbf{k}} , \tag{43}$$

$$F = \sum_{\mathbf{j}\in\mathbb{N}_0^2}\sum_{\mathbf{k}\in\mathbb{Z}^2} C_{\mathbf{j},\mathbf{k}} \Psi_{\mathbf{j},\mathbf{k}} , \tag{44}$$

$$G = \sum_{j_1\in\mathbb{N}_0}\sum_{\mathbf{k}\in\mathbb{Z}^2} C_{(j_1,-1),\mathbf{k}} \Psi_{(j_1,-1),\mathbf{k}} \tag{45}$$

and

$$H = \sum_{j_2\in\mathbb{N}_0}\sum_{\mathbf{k}\in\mathbb{Z}^2} C_{(-1,j_2),\mathbf{k}} \Psi_{(-1,j_2),\mathbf{k}} . \tag{46}$$

Properties (30), (33), (34) and (35) imply that

$$\left|\Delta_{\mathbf{h},\{1\}} F(x)\right| \le C R_{\delta_1,\alpha_1}(h_1,x_1) S_{\delta_2,\alpha_2}(x_2) , \tag{47}$$

$$\left|\Delta_{\mathbf{h},\{2\}} F(x)\right| \le C S_{\delta_1,\alpha_1}(x_1) R_{\delta_2,\alpha_2}(h_2,x_2) , \tag{48}$$

$$\left|\Delta_{\mathbf{h},\{1,2\}} F(x)\right| \le C \prod_{i=1}^{2} R_{\delta_i,\alpha_i}(h_i,x_i) , \tag{49}$$

$$\left|\Delta_{\mathbf{h},\{1\}} G(x)\right| \le C R_{\delta_1,\alpha_1}(h_1,x_1) W_{\delta_2,\alpha_2}(x_2) , \tag{50}$$

$$\left|\Delta_{\mathbf{h},\{2\}} G(x)\right| \le C S_{\delta_1,\alpha_1}(x_1) T_{\delta_2,\alpha_2}(h_2,x_2) , \tag{51}$$

$$\left|\Delta_{\mathbf{h},\{1,2\}} G(x)\right| \le C R_{\delta_1,\alpha_1}(h_1,x_1) T_{\delta_2,\alpha_2}(h_2,x_2) , \tag{52}$$

$$\left|\Delta_{\mathbf{h},\{1\}} H(x)\right| \le C T_{\delta_1,\alpha_1}(h_1,x_1) S_{\delta_2,\alpha_2}(x_2) , \tag{53}$$

$$\left|\Delta_{\mathbf{h},\{2\}} H(x)\right| \le C W_{\delta_1,\alpha_1}(x_1) R_{\delta_2,\alpha_2}(h_2,x_2) , \tag{54}$$

$$\left|\Delta_{\mathbf{h},\{1,2\}} H(x)\right| \le C T_{\delta_1,\alpha_1}(h_1,x_1) R_{\delta_2,\alpha_2}(h_2,x_2) , \tag{55}$$

$$\left|\Delta_{\mathbf{h},\{1\}} f_{-1}(x)\right| \leq CV(h_1,x_1)Z(x_2)\,, \tag{56}$$

$$\left|\Delta_{\mathbf{h},\{2\}} f_{-1}(x)\right| \leq CZ(x_1)V(h_2,x_2)\,, \tag{57}$$

$$\left|\Delta_{\mathbf{h},\{1,2\}} f_{-1}(x)\right| \leq CV(h_1,x_1)V(h_2,x_2)\,, \tag{58}$$

The following lemma together with Remark 3 and properties (47) to (67) yield (24).

Lemma 1 *There exists $C > 0$ such that*

$$\forall\, t \qquad Z(t) \leq C\,, \tag{59}$$

$$\forall\, |h| \geq 1 \;\; \forall\, t \qquad V(h,t) \leq C\,, \tag{60}$$

$$\forall\, t \qquad S_{\delta,\alpha}(t) \leq C\,, \tag{61}$$

$$\forall\, t \qquad W_{\delta,\alpha}(t) \leq C\,, \tag{62}$$

$$\forall\, |h| \geq 1 \;\; \forall\, t \qquad T_{\delta,\alpha}(h,t) \leq C|h|^{\sigma\alpha}\,, \tag{63}$$

$$\forall\, |h| \geq 1 \;\; \forall\, t \qquad R_{\delta,\alpha}(h,t) \leq C|h|^{\sigma\alpha}\,, \tag{64}$$

$$\forall\, 0 < |h| \leq 1 \;\; \forall\, t \qquad V(h,t) \leq C|h|\,, \tag{65}$$

$$\forall\, 0 < |h| \leq 1 \;\; \forall\, t \qquad T_{\delta,\alpha}(h,t) \leq C|h| \tag{66}$$

and

$$\forall\, 0 < |h| \leq 1 \;\; \forall\, t \qquad R_{\delta,\alpha}(h,t) \leq C|h|^{\sigma\alpha+(1-\sigma)\delta}\,. \tag{67}$$

Proof of Lemma 1: For $N > 1$, the one variable function $\phi_N(t) = \sum_{k\in\mathbb{Z}} (1+|t-k|)^{-N}$ is bounded. Thanks to the localization of ψ_{-1}

$$Z(t) \leq C\phi_N(t)\,.$$

This yields (59).
Of course (60) is a consequence of

$$V(h,t) \leq Z(t) + Z(t+h) \leq C\,.$$

Clearly

$$\mu_{\alpha,\delta,j,k,t} \leq C2^{-\delta(1-\sigma)[j]}(2^{-\sigma\alpha[j]} + \left|k2^{-[j]} - t\right|^{\sigma\alpha}) \tag{68}$$

and

$$\mu_{\alpha,\delta,j,k,t} \leq C2^{-\delta(1-\sigma)[j]}2^{-\sigma\alpha[j]}(1+\left|2^{[j]}t-k\right|)^{\sigma\alpha}\,. \tag{69}$$

Thanks to (69) and the localization of wavelets ψ_1 and ψ_{-1}

$$\sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,j,k,t}\left|\psi_{j,k}(t)\right| \leq C2^{-\delta(1-\sigma)[j]}2^{-[j]\sigma\alpha}\phi_{N-\sigma\alpha}(2^{[j]}t)$$

So

$$\sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,j,k,t}\left|\psi_{j,k}(t)\right| \leq C2^{-\delta(1-\sigma)[j]}2^{-[j]\sigma\alpha}\,. \tag{70}$$

Therefore (61) and (62) hold.
On the other hand

$$\left|\Delta_h\psi_{j,k}(t)\right| \leq \left|\psi_{j,k}(t)\right| + \left|\psi_{j,k}(t+h)\right|\,. \tag{71}$$

Again

$$\sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,j,k,t} \left|\psi_{j,k}(t+h)\right| \le C2^{-\delta(1-\sigma)[j]}2^{-\sigma\alpha[j]} \sum_{k} \frac{\left(1+\left|2^{[j]}t-k\right|\right)^{\sigma\alpha}}{\left(1+\left|2^{[j]}(t+h)-k\right|\right)^{N}} .$$

By triangle and Hölder inequalities

$$\left(1+\left|2^{[j]}t-k\right|\right)^{\alpha\sigma} \le C\left((1+\left|2^{[j]}(t+h)-k\right|)^{\alpha\sigma} + (2^{[j]}|h|)^{\alpha\sigma}\right) .$$

Then

$$\sum_{k} \mu_{\alpha,\delta,j,k,t} \left|\psi_{j,k}(t+h)\right| \le C2^{-\delta(1-\sigma)[j]}2^{-\sigma\alpha[j]}\phi_{N-\sigma\alpha}(2^{[j]}(t+h)) + C2^{-\delta(1-\sigma)[j]}|h|^{\alpha\sigma}\phi_{N}(2^{[j]}(t+h)) .$$

This result together with (70) and (71) imply that

$$\sum_{k} \mu_{\alpha,\delta,j,k,t} \left|\Delta_h \psi_{j,k}(t)\right| \le C2^{-\delta(1-\sigma)[j]}(2^{-\sigma\alpha[j]} + |h|^{\sigma\alpha}) . \tag{72}$$

–If $|h| \ge 1$ then

$$\sum_{k} \mu_{\alpha,\delta,j,k,t} \left|\Delta_h \psi_{j,k}(t)\right| \le C2^{-(1-\sigma)\delta[j]}|h|^{\sigma\alpha} .$$

This yields both (63) and (64).

–Assume now that $0 < |h| \le 1$. Let $J \in \mathbb{N}_0$ such that $2^{-J} \le |h| < 2.\,2^{-J}$. Split $R_{\delta,\alpha}(h,t)$ as

$$R_{\delta,\alpha}(h,t) = \sum_{j=0}^{J}\sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,j,k,t} \left|\Delta_h \psi_{j,k}(t)\right| + \sum_{j=J+1}^{\infty}\sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,j,k,t} \left|\Delta_h \psi_{j,k}(t)\right| . \tag{73}$$

Since $\alpha > 0, 0 < \sigma < 1$ and $\delta > 0$ then relation (72) yields

$$\sum_{j=J+1}^{\infty}\sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,j,k,t} \left|\Delta_h \psi_{j,k}(t)\right| \le C|h|^{\sigma\alpha+(1-\sigma)\delta} . \tag{74}$$

Let us bound the first sum in (73). If wavelets ψ_1 and ψ_{-1} are of class C^1 then by the mean value theorem

$$\left|\Delta_h \psi_{j,k}(t)\right| \le |h| \sup_{u\in[t,t+h]} \left|\psi'_{j,k}(u)\right| ,$$

where $[t,t+h]$ is the segment between t and $t+h$.
Thanks to the localization of ψ'_1 and ψ'_{-1}

$$\left|\psi'_{j,k}(u)\right| \le \frac{C2^{[j]}}{\left(1+\left|2^{[j]}u-k\right|\right)^{N}} .$$

(a)if $k \notin [2^{[j]}t, 2^{[j]}(t+h)]$, then $\left|2^{[j]}u-k\right| \ge \min\left\{\left|2^{[j]}t-k\right|, \left|2^{[j]}(t+h)-k\right|\right\}$, so

$$\frac{1}{\left(1+\left|2^{[j]}u-k\right|\right)^{N}} \le C\left(\frac{1}{\left(1+\left|2^{[j]}t-k\right|\right)^{N}} + \frac{1}{\left(1+\left|2^{[j]}(t+h)-k\right|\right)^{N}}\right) .$$

It follows that

$$\sum_{k\notin[t,t+h]} \left|\Delta_h \psi_{-1,k}(t)\right| \le C|h| . \tag{75}$$

Also, an argument similar to that of (72) yields

$$\sum_{k\notin[2^{[j]}t,2^{[j]}(t+h)]} \mu_{\alpha,\delta,j,k,t} \left|\Delta_h \psi_{j,k}(t)\right| \le C2^{-(1-\sigma)\delta[j]}2^{[j]}|h|(2^{-\sigma\alpha[j]} + |h|^{\sigma\alpha}) . \tag{76}$$

(b)Since $j \leq J$ and $2^{-J} \leq |h| < 2.\ 2^{-J}$ then there are maximum two integers k in $[2^{[j]}t, 2^{[j]}(t+h)]$. For these k's we have $|t - k2^{-[j]}| \leq |h|$. So

$$\sum_{k\in[t,t+h]} \left|\Delta_h \psi_{-1,k}(t)\right| \leq C|h| \tag{77}$$

and (68) implies that

$$\sum_{k\in[2^{[j]}t, 2^{[j]}(t+h)]} \mu_{\alpha,\delta,j,k,t} \left|\Delta_h \psi_{j,k}(t)\right| \leq C2^{-(1-\sigma)\delta[j]} 2^{[j]} |h| (2^{-\sigma\alpha[j]} + |h|^{\sigma\alpha}) . \tag{78}$$

It follows from (75) and (77) that

$$\sum_{k\in\mathbb{Z}} \left|\Delta_h \psi_{-1,k}(t)\right| \leq C|h| \tag{79}$$

and so (65) holds.
Besides, it follows from (76) and (78) that

$$\sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,j,k,t} \left|\Delta_h \psi_{j,k}(t)\right| \leq C2^{-(1-\sigma)\delta[j]} 2^{[j]} |h| (2^{-\sigma\alpha[j]} + |h|^{\sigma\alpha}) . \tag{80}$$

In the particular case where $j = -1$, we obtain

$$\sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,-1,k,t} \left|\Delta_h \psi_{-1,k}(t)\right| \leq C|h|(1 + |h|^{\sigma\alpha}) \leq 2C|h| \text{ because } |h| \leq 1 \tag{81}$$

and so (66) holds.
Since $(\alpha, \sigma, \delta) \in (0,1)^3$, then (80) yields

$$\sum_{j=0}^{J} \sum_{k\in\mathbb{Z}} \mu_{\alpha,\delta,j,k,t} \left|\Delta_h \psi_{j,k}(t)\right| \leq C|h|^{\alpha\sigma + \delta(1-\sigma)} . \tag{82}$$

Both (74) and (82) yield (67).

□

3 Print dimension of multivariate Lipschitz oscillating regularities

Given $\mathbf{0} < \boldsymbol{\alpha} < \mathbf{1}$ and $\boldsymbol{\beta} > \mathbf{0}$, we will obtain the print dimension of the set $E_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ given in (13), from the knowledge of fractional axes oscillating spaces to which f belongs.

For $\mathbf{k}, \mathbf{j}$, let

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(\mathbf{j}, \mathbf{k}) = \prod_{i=1}^{d} [k_i 2^{-j_i}, (k_i + 1)2^{-j_i}).$$

Set

$$C_{\boldsymbol{\lambda}} = C_{\mathbf{j},\mathbf{k}} \text{ and } \boldsymbol{\Lambda}_{\mathbf{j}} = \{\boldsymbol{\lambda}(\mathbf{j}, \mathbf{k}) \ : \ \mathbf{k} \in \mathbb{Z}^d\} . \tag{83}$$

Definition 5 *(see [12])*
Let $p < 0$ and $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}^d$. A bounded function f belongs to $O_p^{\mathbf{s}}$ if

$$\forall\, \varepsilon > 0 \ \exists\, C > 0 \ \exists\, J \geq 0 \ \forall\, |[\mathbf{j}]| \geq J \qquad 2^{(\varepsilon-1)|[\mathbf{j}]| + p\sum_{i=1}^{d} s_i j_i} \sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}_{\mathbf{j}}} \left(\sup_{\boldsymbol{\lambda}'\subset\boldsymbol{\lambda}} |C_{\boldsymbol{\lambda}'}| \right)^p \leq C . \tag{84}$$

where the supremum is over $\boldsymbol{\lambda}' \in \boldsymbol{\Lambda}_{\mathbf{j}'}$ with $\mathbf{j}' \geq \mathbf{j}$.

For $\mathbf{0} < \boldsymbol{\alpha} < \mathbf{1}$, set

$$\mathscr{B}_{\boldsymbol{\alpha}} = \{x : f \in R^{\boldsymbol{\alpha}}(x)\} . \tag{85}$$

From property (21) and arguments similar to those of Theorem 4 in [12], we easily obtain the following result.

Proposition 1 *Let $p<0$ and $\mathbf{s}=(s_1,\cdots,s_d)\in\mathbb{R}^d$. Let $\mathbf{0}<\boldsymbol{\alpha}<\mathbf{1}$. If $f\in O_p^{\mathbf{s}}$ then print $\mathscr{B}_{\boldsymbol{\alpha}}$ is included in the set of $\boldsymbol{\delta}=(\delta_1,\cdots,\delta_d)\geq\mathbf{0}$ such that either $\delta_d\leq\max_{n\in\mathscr{D}}\xi_n$, or $\delta_{d-1}+\delta_d\leq\max_{n_1\neq n_2}(\xi_{n_1}+\xi_{n_2}),\cdots$, or $\delta_2+\cdots+\delta_d\leq\max_{n_1\neq\cdots\neq n_{d-1}}(\xi_{n_1}+\cdots+\xi_{n_{d-1}})$ or $\delta_1+\cdots+\delta_d\leq\xi_1+\cdots+\xi_d$, where $\xi_i=(\alpha_i-s_i)p+1$.*

We introduce the following definition.

Definition 6 *Let $p<0$, $\mathbf{s}=(s_1,\cdots,s_d)\in\mathbb{R}^d$ and $\mathbf{0}<\mathbf{t}=(t_1,\cdots,t_d)<\mathbf{1}$. We say that f belongs to the fractional oscillation space $O_p^{\mathbf{s},\mathbf{t}}$ if $\tilde{f}_{\mathbf{t}}\in O_p^{\mathbf{s}}$, where $\tilde{f}_{\mathbf{t}}$ is the pseudo-iterated fractional axes primitives $\tilde{f}_{\mathbf{t}}$ of order $\mathbf{t}$ of f given by (18) and (19).*

Define $\boldsymbol{\eta}(p,\mathbf{t})=(\eta_1(p,\mathbf{t}),\cdots,\eta_d(p,\mathbf{t}))$ by

$$f\in O_p^{\frac{\boldsymbol{\eta}(p,\mathbf{t})}{p}+\boldsymbol{\varepsilon},\mathbf{t}}\quad\forall\,\boldsymbol{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_d)>\mathbf{0}\tag{86}$$

and

$$f\notin O_p^{\frac{\boldsymbol{\eta}(p,\mathbf{t})}{p}-\boldsymbol{\varepsilon},\mathbf{t}}\quad\forall\,\boldsymbol{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_d)>\mathbf{0}\,.\tag{87}$$

From (13) and Definition 4, we deduce the following result.

Theorem 2 *Let $p<0$. Let $\mathbf{0}<\boldsymbol{\alpha}<\mathbf{1}$ and $\boldsymbol{\beta}>\mathbf{0}$. Then print $E_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is included in the set of $\boldsymbol{\delta}\geq\mathbf{0}$ such that either $\delta_d\leq\max_{n\in\mathscr{D}}\theta_n$, or $\delta_{d-1}+\delta_d\leq\max_{n_1\neq n_2}(\theta_{n_1}+\theta_{n_2}),\cdots$, or $\delta_2+\cdots+\delta_d\leq\max_{n_1\neq\cdots\neq n_{d-1}}(\theta_{n_1}+\cdots+\theta_{n_{d-1}})$ or $\delta_1+\cdots+\delta_d\leq\theta_1+\cdots+\theta_d$, where $\theta_i=\limsup_{\mathbf{t}\to 0^+}(\alpha_i+t_i(1+\beta_i))p-\boldsymbol{\eta}_i(p,\mathbf{t})+1$.*

Conflict of Interest

The authors declare that there is no conflict regarding the publication of this paper.

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