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Analytical Solution of Conformable Schrödinger Wave Equation with Coulomb Potential

Mohamed Ghaleb Al-Masaeed[1,*], *Eqab.M.Rabei*[2] *and Ahmed Al-Jamel*[2]

[1]Ministry of Education, Jordan
[2]Physics Department, Faculty of Science, Al al-Bayt University, P.O. Box 130040, Mafraq 25113, Jordan

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Abstract: In this paper, the conformable equation for hydrogen-like systems with conformable Coulomb's potential is constructed. Then, the conformable eigenfunctions and energy eigenvalues are obtained. The analytic solutions are expressed in terms of conformable spherical harmonics and conformable Laguerre functions that are appeared and defined in this work. Some aspects of the results are discussed. For instance, the probability density for the first three levels and different values of α are plotted, and it is observed that the probability density gradually converts to $\alpha = 1$ for all levels. The traditional version of this problem is recovered when the fractional parameter $\alpha = 1$. The set of conformable eigenfunctions could be useful as a basis for approximation methods developed for the conformable counterparts that appeared in conformable quantum mechanics.

Keywords: conformable Schrödinger equation, conformable spherical harmonics, conformable Legendre equation, hydrogen atom.

1 Introduction

The Schrödinger equation, which is the quantum equivalent of Newton's second law in classical mechanics, is an important conclusion in quantum mechanics for obtaining the wave function. Only a few idealized systems, like the hydrogen atom, may achieve the exact solutions to the Schrödinger equation. And the method of separating the variables was applied in order to solve it using three-dimensional spherical coordinates. It produces two equations, the first of which is a radial equation and the second of which is an angular equation. The radial equation's solution requires knowledge of the potential, while the angular equation's solution utilizes special functions, notably the associated Legendre equation [1].
Recently, the application of fractional calculus has emerged as one of the most exciting topics in a variety of physical science fields. The concept of a non-integer order of derivatives originally appeared in correspondence between L'Hospital and Leibniz in 1695, when L'Hospital inquired as to what was meant by $\frac{d^n f}{dx^n}$ if $n = \frac{1}{2}$ [2, 3]. A fractional derivative has since been described in a few research publications. Most of them gave definitions of the fractional derivative in integral form. There are several definitions of a fractional derivative such as Hadamard [4], Riemann-Liouville [2, 3], Caputo [5], Riesz [6, 7] , Weyl [8], Grünwald [9], Chen [10], and Riesz-Caputo [11]. Many works on fractional calculus with diverse definitions have been created in recent years; for example, [12–22].
A novel derivative idea, the conformable derivative, was proposed a few years ago by Khalil et al. [23]. The conformable derivative of f with order $0 < \alpha \leq 1$ is defined by [23]

$$D^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{1}$$

where $f : [0, \infty) \to R$. This definition generally satisfies the standard properties of the traditional derivative, which makes it attractive for researchers. Some of these properties are [23]

* Corresponding author e-mail: moh.almssaeed@gmail.com

–$D^\alpha(af+bg) = aD^\alpha(f) + bD^\alpha(g)$ for all real constant a, b
–$D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$
–$D^\alpha(t^p) = pt^{p-\alpha}$ for all p
–$D^\alpha(\frac{f}{g}) = \frac{gD^\alpha(f)-fD^\alpha(g)}{g^2}$
–$D^\alpha(c) = 0$ with c is constant.

Using the definitions of conformable derivative, one can show that for a wavefunction $\psi(s)$ the following relations hold true

$$D^\alpha[\psi(s)] = s^{1-\alpha}\psi(s) \tag{2}$$

and

$$D^\alpha[D^\alpha\psi(s)] = (1-\alpha)s^{1-2\alpha}\psi'(s) + s^{2-2\alpha}\psi''(s). \tag{3}$$

For further knowledge about the properties and applications of this type of derivative, we refer you to [24–29] and references therein. The conformable derivative does not satisfy zero-order, semigroup, or the Generalized Leibniz rule, but the conformable fractional derivative still contains components of the ordinary derivative. The conformable derivative is hence called a local operator [30].
The conformable calculus was used to solve the conformable Bohr Hamiltonian appropriate for triaxial nuclei which involved the Kratzer potential as an analytical solution in [31]. In addition, The conformable calculus was used a new category of critical point symmetries associated with zeros of conformable Bessel functions to describe spectra of nuclei around the critical point to obtain the exact eigenvalue and eigenfunction solutions of local fractional Bohr-Mottelson Hamiltonian (with infinite square well potential) in [32]. In special relativity, the conformable derivative was used to study the effect of deformation of special relativity studied by conformable derivative [33], and in quantum mechanics to study its effect on the formation of quantum-mechanical operators [34]. In addition, the annihilation and creation operators are used to quantize the conformable harmonic oscillator [35], and the Bateman damping system is quantized with conformable derivative in ref [36]. Recently the conformable operator is used to extend the approximation methods in quantum mechanics (variational method [37], perturbation theory [38] and WKB approximation [39].
The purpose of this paper is to solve the conformable Schrödinger equation for the hydrogen atom with conformable Coulomb's potential.

2 Conformable Schrödinger Equation For Hydrogen Atom

The conformable Schrödinger equation in 3D-spherical coordinates is given by

$$\left(\nabla^{2\alpha} - \frac{2m^\alpha}{\hbar_\alpha^{2\alpha}}(V_\alpha(r^\alpha) - E^\alpha)\right)\psi_\alpha(r^\alpha, \theta^\alpha, \varphi^\alpha) = 0. \tag{4}$$

Applying the separation of variables by assuming that $\psi_\alpha(r^\alpha, \theta^\alpha, \varphi^\alpha) = R_{n\ell\alpha}(r^\alpha)Y_{\ell\alpha}^{m\alpha}$, we obtain two equations. The first one reads as

$$\frac{1}{Y_{\ell\alpha}^{m\alpha}\sin(\theta^\alpha)}D_\theta^\alpha[\sin(\theta^\alpha)D_\theta^\alpha Y_{\ell\alpha}^{m\alpha}] + \frac{1}{Y_{\ell\alpha}^{m\alpha}\sin^2(\theta^\alpha)}D_\varphi^{2\alpha}Y_{\ell\alpha}^{m\alpha} = -\alpha^2\ell(\ell+1). \tag{5}$$

This equation is called a conformable angular equation of the Schrödinger equation. The second equation is the conformable radial equation:

$$D_r^\alpha[r^{2\alpha}D_r^\alpha R_\alpha] + \left[\frac{2m^\alpha r^{2\alpha}}{\hbar_\alpha^{2\alpha}}(E^\alpha - V_\alpha(r^\alpha)) - \alpha^2\ell(\ell+1)\right]R_\alpha = 0. \tag{6}$$

The solution for this equation depends on knowing the conformable potential $V_\alpha(r^\alpha)$.

2.1 Conformable spherical harmonics

Consider the Schrödinger equation of the form [40]

$$\frac{\hat{p}_\alpha^2}{2m^\alpha}\psi_\alpha(x,t) = (E^\alpha - V_\alpha(\hat{x}_\alpha))\psi_\alpha(x,t). \tag{7}$$

The coordinate and the momentum operators are defined as

$$\hat{x}_\alpha = x, \quad \hat{p}^\alpha = -i\hbar_\alpha^\alpha \nabla^\alpha. \tag{8}$$

where $\hbar_\alpha^\alpha = \frac{h}{(2\pi)^{\frac{1}{\alpha}}}$. To read more about conformable quantum mechanics see ref [34, 40]. The conformable Schrödinger equation in spherical coordinates can be written as

$$\left(\nabla^{2\alpha} - \frac{2m^\alpha}{\hbar_\alpha^{2\alpha}}(V_\alpha(r^\alpha) - E^\alpha)\right)\psi_\alpha(r^\alpha, \theta^\alpha, \varphi^\alpha) = 0. \tag{9}$$

where

$$\nabla^{2\alpha} = \frac{1}{r^{2\alpha}} D_r^\alpha [r^{2\alpha} D_r^\alpha] + \frac{1}{r^{2\alpha}\sin(\theta^\alpha)} D_\theta^\alpha [\sin(\theta^\alpha) D_\theta^\alpha] + \frac{1}{r^{2\alpha}\sin^2(\theta^\alpha)} D_\varphi^{2\alpha}. \tag{10}$$

After substituting in eq.(9), we obtain

$$\begin{aligned}&\frac{1}{R_\alpha} D_r^\alpha [r^{2\alpha} D_r^\alpha R_\alpha] + \frac{1}{Y_\alpha \sin(\theta^\alpha)} D_\theta^\alpha [\sin(\theta^\alpha) D_\theta^\alpha Y_\alpha] + \frac{1}{Y_\alpha \sin^2(\theta^\alpha)} D_\varphi^{2\alpha} Y_\alpha \\ &- \frac{2m^\alpha r^{2\alpha}}{\hbar_\alpha^{2\alpha}}(V_\alpha(r^\alpha) - E^\alpha) = 0.\end{aligned} \tag{11}$$

The first part of this equation that depends on r^α and equal to a constant is given as

$$\frac{1}{R_\alpha} D_r^\alpha [r^{2\alpha} D_r^\alpha R_\alpha] - \frac{2m^\alpha r^{2\alpha}}{\hbar_\alpha^{2\alpha}}(V_\alpha(r^\alpha) - E^\alpha) = \alpha^2 \ell(\ell+1). \tag{12}$$

This equation is the conformable radial equation and the solution of this equation depends on the potential $V_\alpha(r^\alpha)$. The second part of equation (11) reads as

$$\frac{1}{Y_\alpha \sin(\theta^\alpha)} D_\theta^\alpha [\sin(\theta^\alpha) D_\theta^\alpha Y_\alpha] + \frac{1}{Y_\alpha \sin^2(\theta^\alpha)} D_\varphi^{2\alpha} Y_\alpha = -\alpha^2 \ell(\ell+1). \tag{13}$$

Using separation of variable $Y_\alpha(\theta^\alpha, \varphi^\alpha) = \Theta_\alpha(\theta^\alpha)\Phi_\alpha(\varphi^\alpha)$, we obtain

$$\frac{1}{\Theta_\alpha \sin(\theta^\alpha)} D_\theta^\alpha [\sin(\theta^\alpha) D_\theta^\alpha \Theta_\alpha] + \frac{1}{\Phi_\alpha \sin^2(\theta^\alpha)} D_\varphi^{2\alpha} \Phi_\alpha = -\alpha^2 \ell(\ell+1), \tag{14}$$

after multiplied this equation by $\sin^2(\theta^\alpha)$, we obtain

$$\frac{\sin(\theta^\alpha)}{\Theta_\alpha} D_\theta^\alpha [\sin(\theta^\alpha) D_\theta^\alpha \Theta_\alpha] + \alpha^2 \ell(\ell+1)\sin^2(\theta^\alpha) + \frac{1}{\Phi_\alpha} D_\varphi^{2\alpha} \Phi_\alpha = 0. \tag{15}$$

The part of this equation that depends on φ^α and equal to a constant is given as

$$\frac{1}{\Phi_\alpha} D_\varphi^{2\alpha} \Phi_\alpha = -\alpha^2 m^2, \tag{16}$$

thus, the solution of this equation reads as,

$$\Phi_\alpha(\varphi^\alpha) = Ae^{im\varphi^\alpha} + Be^{-im\varphi^\alpha}. \tag{17}$$

We will adopt the part $Ae^{im\varphi^\alpha}$ because Φ_α is a single-valued function where m is an integer. Thus,

$$\Phi_\alpha(\varphi^\alpha) = Ae^{im\varphi^\alpha}. \tag{18}$$

The part of eq.(15) that depends on θ^α and equal to a constant is given as

$$\frac{\sin(\theta^\alpha)}{\Theta_\alpha} D_\theta^\alpha [\sin(\theta^\alpha) D_\theta^\alpha \Theta_\alpha] + \alpha^2 \ell(\ell+1)\sin^2(\theta^\alpha) = \alpha^2 m^2. \tag{19}$$

Multiplying this equation by Θ_α, we obtain

$$\sin(\theta^\alpha)D^\alpha_\theta[\sin(\theta^\alpha)D^\alpha_\theta\Theta_\alpha]+\alpha^2\left[\ell(\ell+1)\sin^2(\theta^\alpha)-m^2\right]\Theta_\alpha=0. \tag{20}$$

let $\Theta_\alpha(\theta^\alpha)=X_\alpha(x^\alpha), x^\alpha=\cos(\theta^\alpha)\to \alpha x^{\alpha-1}dx=-\alpha\theta^{\alpha-1}\sin(\theta^\alpha)\to D^\alpha_\theta=-\sin(\theta^\alpha)D^\alpha_x$, After substituting in this eqution, we obtain

$$-(1-x^{2\alpha})D^\alpha_x[-(1-x^{2\alpha})D^\alpha_x X_\alpha]+\alpha^2\left[\ell(\ell+1)(1-x^{2\alpha})-m^2\right]X_\alpha=0. \tag{21}$$

after multiplied this equation by $\frac{1}{(1-x^{2\alpha})}$, we obtain

$$(1-x^{2\alpha})D^\alpha_x D^\alpha_x X_\alpha-2\alpha x^\alpha D^\alpha_x X_\alpha+\alpha^2\left[\ell(\ell+1)-\frac{m^2}{(1-x^{2\alpha})}\right]X_\alpha=0. \tag{22}$$

This equation is called conformable associated Legendre differential equation and its solution is given by [41]

$$X_\alpha=P^{m\alpha}_{\ell\alpha}=\frac{(-1)^m(1-x^{2\alpha})^{\frac{m}{2}}}{\alpha^\ell 2^\ell \ell!}D^{(\ell+m)\alpha}(x^{2\alpha}-1)^\ell. \tag{23}$$

Thus, the conformable spherical harmonic solution for eq.(13) is given as

$$Y^{m\alpha}_{\ell\alpha}(\theta^\alpha,\varphi^\alpha)=N^{m\alpha}_{\ell\alpha}e^{im\varphi^\alpha}P^{m\alpha}_{\ell\alpha}(\cos(\theta^\alpha)), \tag{24}$$

where $N^{m\alpha}_{\ell\alpha}$ is normalization constant, can be calculated using normalization condition

$$\int|Y^{m\alpha}_{\ell\alpha}|^2d^\alpha\Omega=|N^{m\alpha}_{\ell\alpha}|^2\int P^{m\alpha}_{\ell'\alpha}(\cos(\theta^\alpha))P^{m\alpha}_{\ell\alpha}(\cos(\theta^\alpha))d^\alpha\Omega \tag{25}$$

where $d^\alpha\Omega=\sin(\theta^\alpha)d^\alpha\theta d^\alpha\varphi$.
Using the orthogonality of conformable associated Legendre functions [41], we get

$$\int|Y^{m\alpha}_{\ell\alpha}|^2d^\alpha\Omega=|N^{m\alpha}_{\ell\alpha}|^2\frac{(2\pi)^\alpha}{\alpha}\frac{\alpha^{2m-1}2(\ell+m)!}{(2\ell+1)(\ell-m)!}=1,$$

then, the normalization constant read as $N^{m\alpha}_{\ell\alpha}=\sqrt{\frac{(2\ell+1)(\ell-m)!}{\alpha^{2m-2}2(\ell+m)!(2\pi)^\alpha}}$. Thus the orthonormal conformable spherical harmonic functions

$$Y^{m\alpha}_{\ell\alpha}=\sqrt{\frac{(2\ell+1)(\ell-m)!}{\alpha^{2m-2}2(\ell+m)!(2\pi)^\alpha}}e^{im\varphi^\alpha}P^{m\alpha}_{\ell\alpha}(\cos(\theta^\alpha)). \tag{26}$$

2.2 The relation between $Y^{m\alpha}_{\ell\alpha}$ and $Y^{-m\alpha}_{\ell\alpha}$

The relation between $Y^{m\alpha}_{\ell\alpha}$ and $Y^{-m\alpha}_{\ell\alpha}$ is given by

$$Y^{-m\alpha}_{\ell\alpha}=(-1)^m Y^{m^*\alpha}_{\ell\alpha} \tag{27}$$

Proof. in the first step we will prove the relation between $P^{m\alpha}_{\ell\alpha}$ and $P^{-m\alpha}_{\ell\alpha}$, Define $P^{-m\alpha}_{\ell\alpha}$ using eq.(23) as,

$$P^{-m\alpha}_{\ell\alpha}=\frac{(-1)^m(1-x^{2\alpha})^{-\frac{m}{2}}}{\alpha^\ell 2^\ell \ell!}D^{(\ell-m)\alpha}(x^{2\alpha}-1)^\ell \tag{28}$$

But, $D^{(\ell+m)\alpha}(x^{2\alpha}-1)^\ell=D^{(\ell+m)\alpha}(x^\alpha-1)^\ell(x^\alpha+1)^\ell$, now let $f=x^\alpha-1, g=x^\alpha+1$. Then,

$$D^{(\ell+m)\alpha}(fg)^\ell=D^{(\ell+m)\alpha}(f)^\ell(g)^\ell=D^{(\ell+m)\alpha}(f^{\frac{1}{\alpha}})^{\alpha\ell}(g^{\frac{1}{\alpha}})^{\alpha\ell}$$

Where $w = f^{\frac{1}{\alpha}}, z = g^{\frac{1}{\alpha}} \to D^{(\ell+m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}]$.
Using Leibniz rule, we obtain

$$\begin{aligned} D^{(\ell+m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] &= \sum_{k=0}^{\ell+m} \binom{\ell+m}{k} D^{(\ell+m-k)\alpha}(w)^{\alpha\ell} D^{k\alpha}(z)^{\alpha\ell} \\ &= \sum_{k=m}^{\ell} \binom{\ell+m}{k} D^{(\ell+m-k)\alpha}(w)^{\alpha\ell} D^{k\alpha}(z)^{\alpha\ell} \end{aligned}$$

where $D^{k\alpha}(z)^{\alpha\ell} = \frac{\alpha^k \ell!}{(\ell-k)!}(z)^{(\ell-k)\alpha}, D^{(\ell+m-k)\alpha}(w)^{\alpha\ell} = \frac{\alpha^{\ell+m-k}\ell!}{(k-m)!}(w)^{(k-m)\alpha}$. Then,

$$\begin{aligned} D^{(\ell+m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] &= \sum_{k=m}^{\ell} \binom{\ell+m}{k} \frac{\alpha^k \ell!}{(\ell-k)!}(z)^{(\ell-k)\alpha} \frac{\alpha^{\ell+m-k}\ell!}{(k-m)!}(w)^{(k-m)\alpha} \\ &= \sum_{k=m}^{\ell} \binom{\ell+m}{k} \frac{\alpha^{\ell+m}(\ell!)^2}{(\ell-k)!(k-m)!}(w)^{(k-m)\alpha}(z)^{(\ell-k)\alpha} \\ &= \sum_{k=m}^{\ell} \frac{\alpha^{\ell+m}(\ell!)^2(\ell+m)!}{k!(\ell+m-k)!(\ell-k)!(k-m)!}(w)^{(k-m)\alpha}(z)^{(\ell-k)\alpha} \end{aligned} \tag{29}$$

In the same way

$$\begin{aligned} D^{(\ell-m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] &= \sum_{r=0}^{\ell-m} \binom{\ell-m}{r} D^{(\ell-m-r)\alpha}(w)^{\alpha\ell} D^{r\alpha}(z)^{\alpha\ell} \\ &= \sum_{r=0}^{\ell-m} \binom{\ell-m}{r} \frac{\alpha^{\ell-m-r}\ell!}{(r+m)!}(w)^{(r+m)\alpha} \frac{\alpha^r \ell!}{(\ell-r)!}(z)^{(\ell-r)\alpha} \\ &= \sum_{r=0}^{\ell-m} \frac{(\ell-m)!\alpha^{\ell-m}(\ell!)^2}{r!(\ell-m-r)!(r+m)!(\ell-r)!}(w)^{(r+m)\alpha}(z)^{(\ell-r)\alpha} \end{aligned} \tag{30}$$

Since the omitted terms in the sum vanish $D^{k\alpha}(f)^r = 0$ if $k > r$, and change the summation variable to $k = r+m$ and substituting in eq.(30), we get

$$D^{(\ell-m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] = \sum_{k=m}^{\ell} \frac{(\ell-m)!\alpha^{\ell-m}(\ell!)^2(w)^{(k)\alpha}(z)^{(\ell+m-k)\alpha}}{(k-m)!(\ell-k)!(k)!(\ell+m-k)!} \tag{31}$$

Multiply eq.(31) by $\frac{\alpha^{2m}(\ell+m)!(w)^{m\alpha}(z)^{m\alpha}}{\alpha^{2m}(\ell+m)!(w)^{m\alpha}(z)^{m\alpha}}$, we have

$$D^{(\ell-m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] = \frac{(\ell-m)!(w)^{m\alpha}(z)^{m\alpha}}{(\ell+m)!\alpha^{2m}} \sum_{k=m}^{\ell} \frac{(\ell+m)!\alpha^{\ell+m}(\ell!)^2(w)^{(k-m)\alpha}(z)^{(\ell-k)\alpha}}{(k-m)!(\ell-k)!(k)!(\ell+m-k)!} \tag{32}$$

Making use of eq.(29) , we obtain

$$D^{(\ell-m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] = \frac{(\ell-m)!(w)^{m\alpha}(z)^{m\alpha}}{(\ell+m)!\alpha^{2m}} D^{(\ell+m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] \tag{33}$$

After substitutions, we have

$$D^{(\ell-m)\alpha}[(x^{2\alpha}-1)^{\ell}] = \frac{(\ell-m)!(x^{2\alpha}-1)^m}{(\ell+m)!\alpha^{2m}} D^{(\ell+m)\alpha}[(x^{2\alpha}-1)^{\ell}] \tag{34}$$

Now substituting in eq.(28) , we have

$$P_{\ell\alpha}^{-m\alpha} = \frac{(-1)^m(\ell-m)!}{(\ell+m)!\alpha^{2m}} \frac{(-1)^m(1-x^{2\alpha})^{\frac{m}{2}}}{\alpha^{\ell}2^{\ell}\ell!} D^{(\ell+m)\alpha}[(x^{2\alpha}-1)^{\ell}] \tag{35}$$

Using eq.(23), we obtain

$$P_{\ell\alpha}^{-m\alpha} = \frac{(-1)^m(\ell-m)!}{\alpha^{2m}(\ell+m)!} P_{\ell\alpha}^{m\alpha}. \tag{36}$$

In the second step We define $Y_{\ell\alpha}^{-m\alpha}$ using eq.(26)

$$Y_{\ell\alpha}^{-m\alpha} = \sqrt{\frac{(2\ell+1)(\ell+m)!}{\alpha^{-2m-2}2(\ell-m)!(2\pi)^{\alpha}}} e^{-im\varphi^{\alpha}} P_{\ell\alpha}^{-m\alpha}(\cos(\theta^{\alpha})). \tag{37}$$

Substituting $P_{\ell\alpha}^{-m\alpha}$ from eq.(36), we obtain

$$\begin{aligned} Y_{\ell\alpha}^{-m\alpha} &= \sqrt{\frac{(2\ell+1)(\ell+m)!}{\alpha^{-2m-2}2(\ell-m)!(2\pi)^{\alpha}}} e^{-im\varphi^{\alpha}} \frac{(-1)^m(\ell-m)!}{\alpha^{2m}(\ell+m)!} P_{\ell\alpha}^{m\alpha}(\cos(\theta^{\alpha})) \\ &= (-1)^m \sqrt{\frac{(2\ell+1)(\ell-m)!}{\alpha^{2m-2}2(\ell+m)!(2\pi)^{\alpha}}} e^{-im\varphi^{\alpha}} P_{\ell\alpha}^{m\alpha}(\cos(\theta^{\alpha})) \\ &= (-1)^m Y_{\ell\alpha}^{m^*\alpha}. \end{aligned} \tag{38}$$

Some of the low-lying conformable spherical harmonic functions are enumerated in the following table

Table 1: The first nine conformable spherical harmonics $Y_{\ell\alpha}^{m\alpha}$.

ℓ	m	$Y_{\ell\alpha}^{m\alpha}$
0	0	$\sqrt{\frac{\alpha^2}{2(2\pi)^{\alpha}}}$
1	-1	$\alpha\sqrt{\frac{3}{4(2\pi)^{\alpha}}} e^{-i\varphi^{\alpha}} \sin(\theta^{\alpha})$
	0	$\sqrt{\frac{3\alpha^2}{2(2\pi)^{\alpha}}} \cos(\theta^{\alpha})$
	1	$-\alpha\sqrt{\frac{3}{4(2\pi)^{\alpha}}} e^{i\varphi^{\alpha}} \sin(\theta^{\alpha})$
2	-2	$\sqrt{\frac{15\alpha^2}{16(2\pi)^{\alpha}}} \sin^2(\theta^{\alpha}) e^{-i2\varphi^{\alpha}}$
	-1	$\alpha\sqrt{\frac{15}{4(2\pi)^{\alpha}}} e^{-i\varphi^{\alpha}} \cos(\theta^{\alpha}) \sin(\theta^{\alpha})$
	0	$\sqrt{\frac{5\alpha^2}{8(2\pi)^{\alpha}}} (3\cos^2(\theta^{\alpha})-1)$
	1	$-\alpha\sqrt{\frac{15}{4(2\pi)^{\alpha}}} e^{i\varphi^{\alpha}} \cos(\theta^{\alpha}) \sin(\theta^{\alpha})$
	2	$\sqrt{\frac{15\alpha^2}{16(2\pi)^{\alpha}}} \sin^2(\theta^{\alpha}) e^{i2\varphi^{\alpha}}$

The conformable spherical harmonic density for $Y_{2\alpha}^{1\alpha}$ and for different values of α are plotted in 3D and 2D using Mathematica as follows,

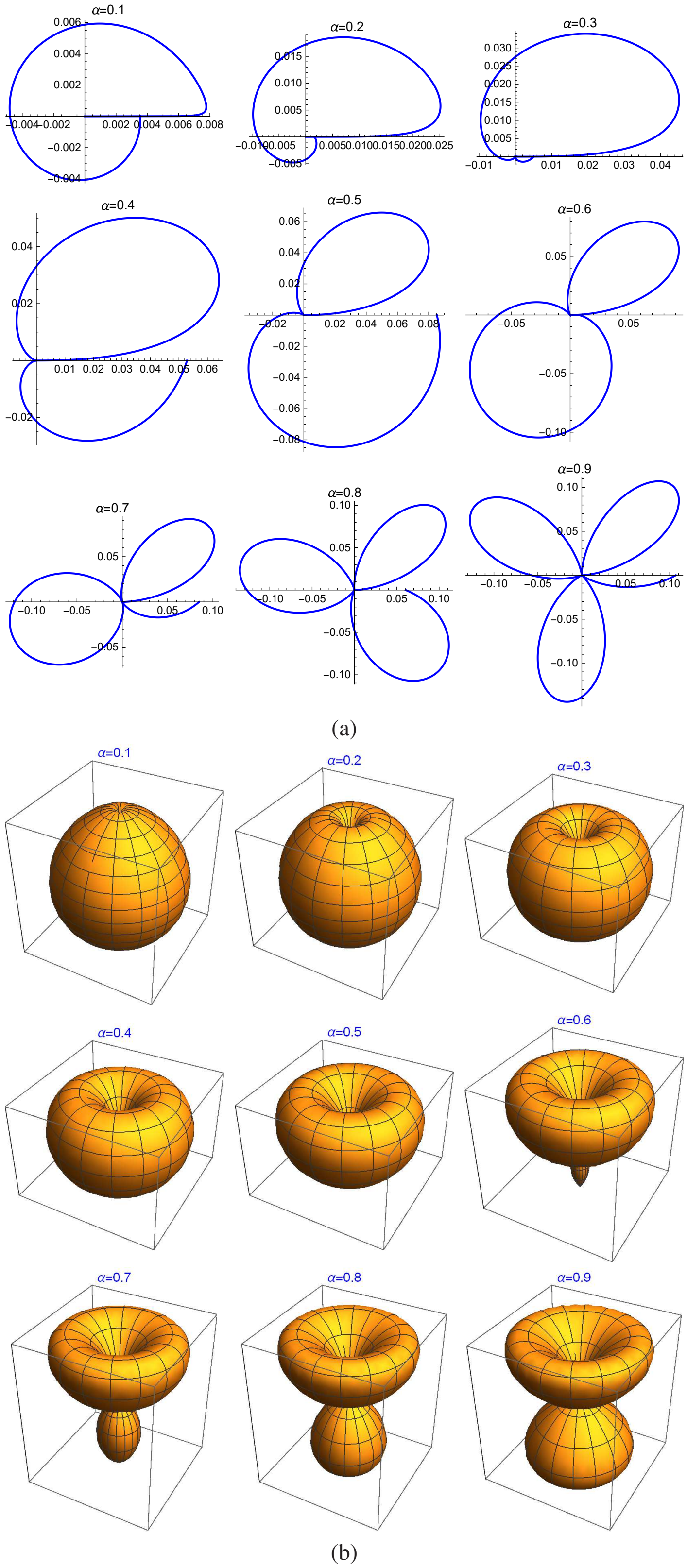
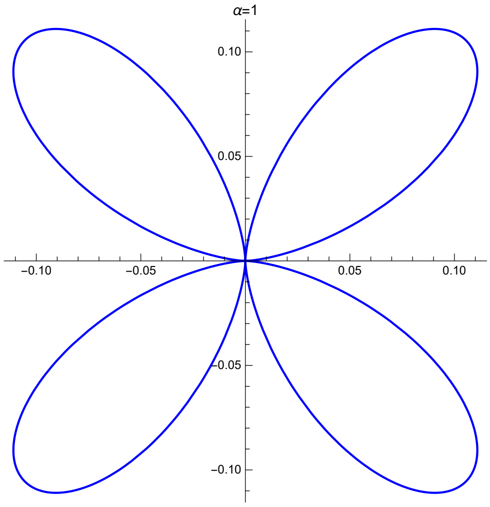
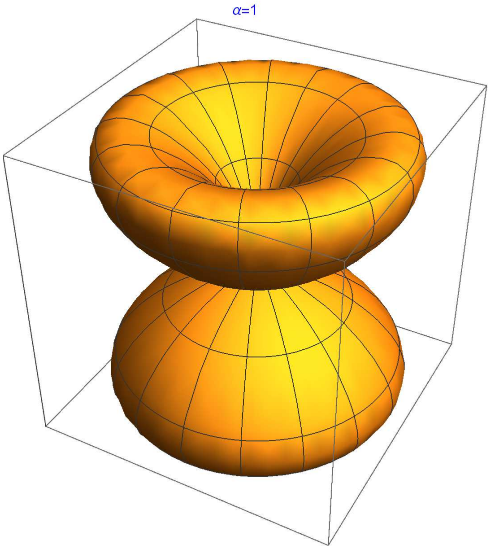


Fig. 1: (a) $|Y_{2\alpha}^{1\alpha}|^2$ at different values of α from 0.1 to 0.9 in polar coordinates.; (b) $|Y_{2\alpha}^{1\alpha}|^2$ with different values of α from 0.1 to 0.9 in 3D.



(a)



(b)

Fig. 2: (a) $|Y_{2\alpha}^{1\alpha}|^2$ when $\alpha = 1$ in polar coordinate.; (b) $|Y_{2\alpha}^{1\alpha}|^2$ when $\alpha = 1$ in 3D.

2.3 Application: Conformable Coulomb's Potential

The conformable Coulomb's potential reads as [37]

$$V_\alpha(r^\alpha) = -\frac{a^\alpha}{r^\alpha}, \tag{39}$$

where $a = \frac{e^2}{4\pi\varepsilon_0}$. Inserting this potential in eq.(12), we obtain

$$D_r^\alpha[r^{2\alpha} D_r^\alpha R_\alpha] + \left[\frac{2m^\alpha r^{2\alpha}}{\hbar_\alpha^{2\alpha}}(E^\alpha + \frac{a^\alpha}{r^\alpha}) - \alpha^2\ell(\ell+1)\right] R_\alpha = 0. \tag{40}$$

Making the substitution $R_\alpha(r^\alpha) = \frac{u_\alpha(r^\alpha)}{r^\alpha}$, we arrive to the following equation

$$D_r^\alpha D_r^\alpha u_\alpha + \left[-k^2 + \frac{2m^\alpha}{\hbar_\alpha^{2\alpha}}\frac{a^\alpha}{r^\alpha} - \frac{\alpha^2\ell(\ell+1)}{r^{2\alpha}}\right] u_\alpha = 0. \tag{41}$$

where $k^2 = -\frac{2m^\alpha E^\alpha}{\hbar_\alpha^{2\alpha}}$. Make the change of variable $r^\alpha = \frac{\rho^\alpha}{2k}$, then $D_r^\alpha = 2kD_\rho^\alpha$, so, we obtain

$$4k^2 D_\rho^\alpha D_\rho^\alpha u_\alpha + \left[-k^2 + \frac{2m^\alpha}{\hbar_\alpha^{2\alpha}}\frac{a^\alpha 2k}{\rho^\alpha} - \frac{4k^2\alpha^2\ell(\ell+1)}{\rho^{2\alpha}}\right]u_\alpha = 0. \tag{42}$$

Thus,

$$D_\rho^\alpha D_\rho^\alpha u_\alpha + \left[-\frac{1}{4} + \frac{\lambda_\alpha}{\rho^\alpha} - \frac{\alpha^2\ell(\ell+1)}{\rho^{2\alpha}}\right]u_\alpha = 0. \tag{43}$$

where $\lambda_\alpha = \frac{m^\alpha a^\alpha}{\hbar_\alpha^{2\alpha} k}$. This equation can be solved in two cases.
The first case: $\rho^\alpha \to 0$, we obtain

$$\rho^{2\alpha} D_\rho^\alpha D_\rho^\alpha u_\alpha - \alpha^2\ell(\ell+1)u_\alpha = 0. \tag{44}$$

Thus, the solution for this equation is given by

$$u_\alpha(\rho^\alpha) = A\rho^{\alpha(\ell+1)}. \tag{45}$$

The Second case: $\rho^\alpha \to \infty$, which leads to the following equation

$$D_\rho^\alpha D_\rho^\alpha u_\alpha - \frac{1}{4}u_\alpha = 0. \tag{46}$$

So, the solution for this equation is given by

$$u_\alpha(\rho^\alpha) = C\exp\left(-\frac{\rho^\alpha}{2\alpha}\right). \tag{47}$$

Thus, we assume a general solution of the form

$$u_\alpha(\rho^\alpha) = A\rho^{\alpha(\ell+1)}\exp\left(-\frac{\rho^\alpha}{2\alpha}\right)v_\alpha(\rho^\alpha). \tag{48}$$

Where $v_\alpha(\rho^\alpha)$ is the analytical function. After substituting in eq.(43), we obtain

$$\rho^\alpha D_\rho^\alpha D_\rho^\alpha v_\alpha + [2\alpha\ell + 2\alpha - \rho^\alpha]D_\rho^\alpha v_\alpha + [\lambda_\alpha - \alpha(\ell+1)]v_\alpha = 0, \tag{49}$$

This equation is the conformable associated Laguerre equation. Its solution is given in terms of the conformable associated Laguerre functions

$$v_\alpha = L_{s\alpha}^m\left(\frac{x^\alpha}{\alpha}\right) = \frac{x^{-m\alpha}\exp\left(\frac{x^\alpha}{\alpha}\right)}{\alpha^s s!}D^{s\alpha}\left[x^{(s+m)\alpha}\exp\left(-\frac{x^\alpha}{\alpha}\right)\right]. \tag{50}$$

where $m = 2\ell+1, \quad \lambda_\alpha = n\alpha, \quad s = n-\ell-1$.
Thus, the general solution eq.(48) can be written in the following form

$$u_\alpha(\rho^\alpha) = A\rho^{\alpha(\ell+1)}\exp\left(-\frac{\rho^\alpha}{2\alpha}\right)L_{(n-\ell-1)\alpha}^{2\ell+1}\left(\frac{\rho^\alpha}{\alpha}\right). \tag{51}$$

One may calculate the constant A using conformable normalization condition [34]

$$\begin{aligned}\int_0^\infty |u_\alpha(\rho^\alpha)|^2\frac{d^\alpha\rho}{2k} =& \frac{|A|^2}{2k}\int_0^\infty \rho^{2\alpha(\ell+1)}\exp\left(-\frac{\rho^\alpha}{\alpha}\right)\left[L_{(n-\ell-1)\alpha}^{2\ell+1}\left(\frac{\rho^\alpha}{\alpha}\right)\right]^2 d^\alpha\rho,\\ =&1.\end{aligned} \tag{52}$$

Making use of the relation given by

$$\int_0^\infty e^{-\frac{x^\alpha}{\alpha}}x^{m\alpha+\alpha}L_{s\alpha}^m(\frac{x^\alpha}{\alpha})L_{k\alpha}^m(\frac{x^\alpha}{\alpha})d^\alpha x = \frac{\alpha^{m+1}(m+s)!}{s!}[2s+m+1]. \tag{53}$$

The constant A is then given as

$$A = \sqrt{\frac{k(n-\ell-1)!}{n\alpha^{2\ell+2}(n+\ell)!}}. \tag{54}$$

Thus, $\frac{k}{n} = \frac{1}{\alpha r_b^{\alpha} n^2}$, where r_b^{α} called $\alpha-$ Bohr radius it is equal $r_b^{\alpha} = \frac{(4\pi\varepsilon_0)^{\alpha}\hbar_{\alpha}^{2\alpha}}{m^{\alpha}e^{2\alpha}}$. So, the radial wave function (51) becomes

$$u_{\alpha}(\rho^{\alpha}) = \sqrt{\frac{(n-\ell-1)!}{r_b^{\alpha} n^2 \alpha^{2\ell+3}(n+\ell)!}}\rho^{\alpha(\ell+1)} \exp\left(-\frac{\rho^{\alpha}}{2\alpha}\right) L_{(n-\ell-1)\alpha}^{2\ell+1}\left(\frac{\rho^{\alpha}}{\alpha}\right). \tag{55}$$

As a result, the conformable radial wave function is

$$\begin{aligned} R_{\alpha}(\rho^{\alpha}) =& 2k\frac{u_{\alpha}(\rho^{\alpha})}{\rho^{\alpha}}, \\ =& \sqrt{\left(\frac{2}{\alpha n r_b^{\alpha}}\right)^3 \frac{(n-\ell-1)!}{2n\alpha^{2\ell+2}(n+\ell)!}}\rho^{\alpha\ell} \exp\left(-\frac{\rho^{\alpha}}{2\alpha}\right) L_{(n-\ell-1)\alpha}^{2\ell+1}\left(\frac{\rho^{\alpha}}{\alpha}\right). \end{aligned} \tag{56}$$

After re-substituting $\rho^{\alpha} = \frac{2r^{\alpha}}{\alpha r_b^{\alpha} n}$, we obtain

$$R_{\alpha}(r^{\alpha}) = \sqrt{\left(\frac{2}{\alpha n r_b^{\alpha}}\right)^3 \frac{(n-\ell-1)!}{2n\alpha^{2\ell+2}(n+\ell)!}} \left[\frac{2r^{\alpha}}{\alpha r_b^{\alpha} n}\right]^{\ell} \exp\left(-\frac{r^{\alpha}}{\alpha^2 r_b^{\alpha} n}\right) L_{(n-\ell-1)\alpha}^{2\ell+1}\left(\frac{2r^{\alpha}}{\alpha^2 r_b^{\alpha} n}\right). \tag{57}$$

Table 2: The conformable spherical harmonics $R_{\alpha}(r^{\alpha})$ for different values.

n	ℓ	$R_{\alpha}(r^{\alpha})$
1	0	$\sqrt{\frac{4}{\alpha^5 r_b^{3\alpha}}} \exp\left(-\frac{r^{\alpha}}{\alpha^2 r_b^{\alpha}}\right)$
2	0	$\sqrt{\frac{1}{8\alpha^5 r_b^{3\alpha}}} \left[2 - \frac{r^{\alpha}}{\alpha^2 r_b^{\alpha}}\right] \exp\left(-\frac{r^{\alpha}}{2\alpha^2 r_b^{\alpha}}\right)$
	1	$\sqrt{\frac{1}{6\alpha^9 r_b^{5\alpha}}} \frac{r^{\alpha}}{2} \exp\left(-\frac{r^{\alpha}}{2\alpha^2 r_b^{\alpha}}\right)$
3	0	$\sqrt{\frac{4}{3^3\alpha^5 r_b^{3\alpha}}} \left[\frac{2}{3}\left(\frac{r^{\alpha}}{3\alpha^2 r_b^{\alpha}}\right)^2 - \frac{2r^{\alpha}}{3\alpha^2 r_b^{\alpha}} + 1\right] \exp\left(-\frac{r^{\alpha}}{3\alpha^2 r_b^{\alpha}}\right)$
	1	$\sqrt{\frac{8}{3^7\alpha^9 r_b^{5\alpha}}} r^{\alpha} \left[2 - \frac{r^{\alpha}}{3\alpha^2 r_b^{\alpha}}\right] \exp\left(-\frac{r^{\alpha}}{3\alpha^2 r_b^{\alpha}}\right)$
	2	$\sqrt{\frac{1}{10\alpha^9 3^5 r_b^{3\alpha}}} \left[\frac{2r^{\alpha}}{\alpha r_b^{\alpha} 3}\right]^2 \exp\left(-\frac{r^{\alpha}}{3\alpha^2 r_b^{\alpha}}\right)$

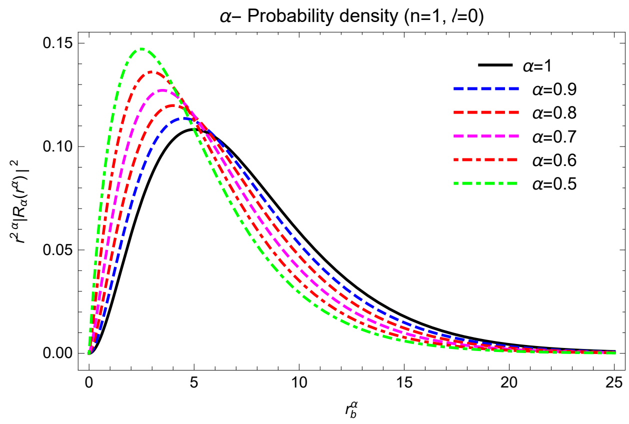


Fig. 3: The α−probability density $r^{2\alpha}|R_\alpha(r^\alpha)|^2$ at different values of α.

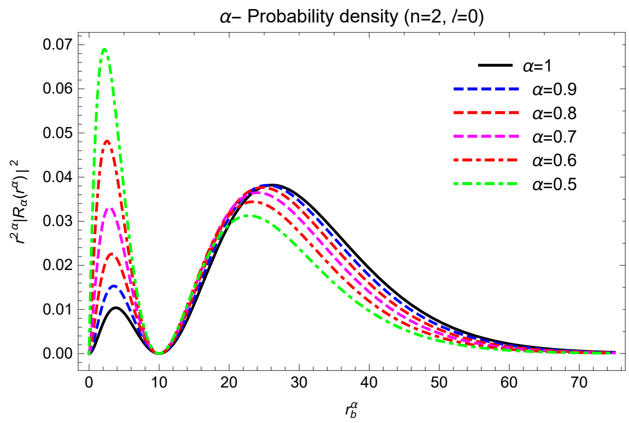


Fig. 4: The α−probability density $r^{2\alpha}|R_\alpha(r^\alpha)|^2$ at different values of α.

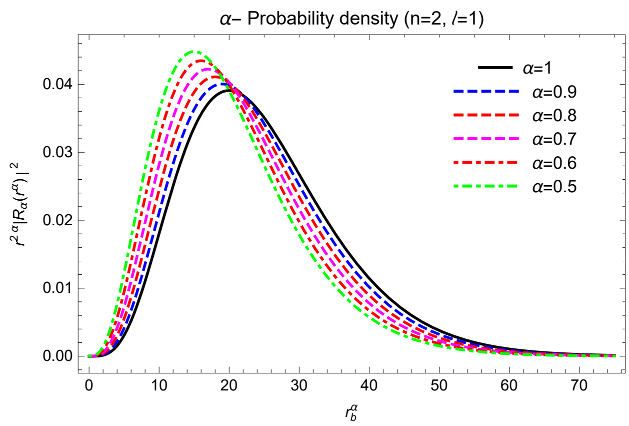


Fig. 5: The α−probability density $r^{2\alpha}|R_\alpha(r^\alpha)|^2$ at different values of α.

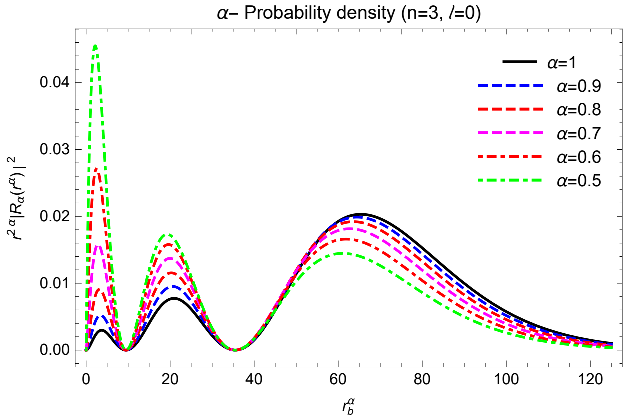


Fig. 6: The $\alpha-$probability density $r^{2\alpha}|R_{\alpha}(r^{\alpha})|^2$ at different values of α.

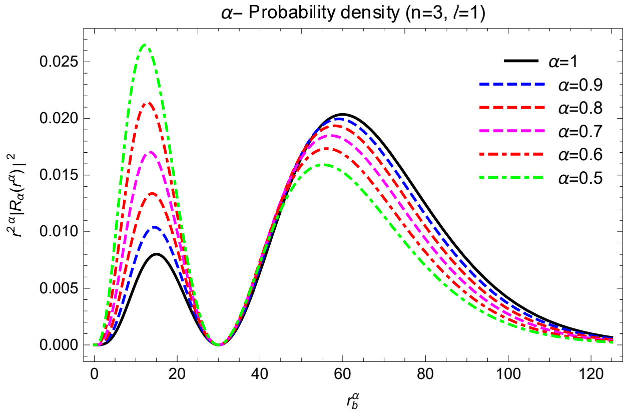


Fig. 7: The $\alpha-$probability density $r^{2\alpha}|R_{\alpha}(r^{\alpha})|^2$ at different values of α.

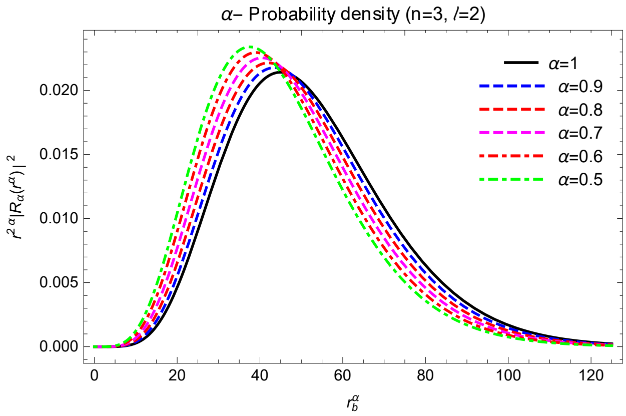


Fig. 8: The $\alpha-$probability density $r^{2\alpha}|R_{\alpha}(r^{\alpha})|^2$ at different values of α.

Thus, the $\alpha-$ energy levels are taken from this formula

$$E^{\alpha}=-\frac{(13.6\,eV)^{\alpha}}{2^{1-\alpha}\alpha^{2}n^{2}}, \tag{58}$$

where $\frac{m^{\alpha}}{2^{\alpha}\hbar_{\alpha}^{2\alpha}}\left(\frac{e^{2}}{4\pi\varepsilon_{0}}\right)^{2\alpha}=(13.6\,eV)^{\alpha}$.

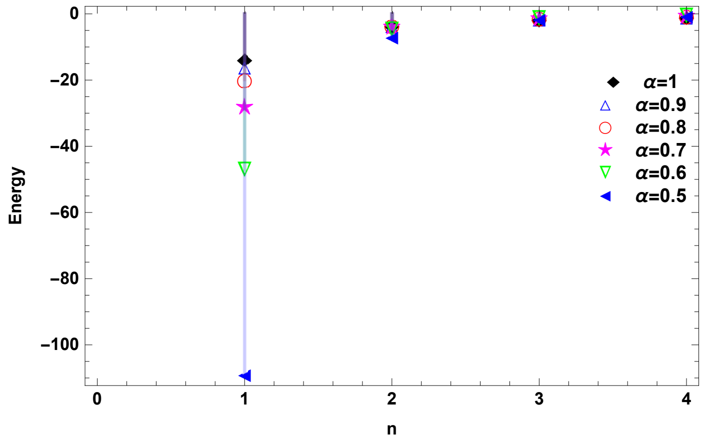


Fig. 9: The $\alpha-$energy levels E^{α} as a function of quantum number n at different values of α.

It is noted that the energy of an excited state varies with α gradually, which could indicate the presence of fractional levels with the highest sub-level corresponding to $\alpha=1$.

Thus, the solution for the conformable Schrödinger equation for the Hydrogen atom is given by

$$\begin{aligned}\psi_{n\ell m\alpha}(r^{\alpha},\theta^{\alpha},\varphi^{\alpha})=&R_{n\ell\alpha}(r^{\alpha})Y_{\ell\alpha}^{m\alpha},\\ =&\sqrt{\left(\frac{1}{\alpha n r_{b}^{\alpha}}\right)^{3}\frac{2(n-\ell-1)!(2\ell+1)(\ell-m)!}{n\alpha^{2\ell+2m}(n+\ell)!(\ell+m)!(2\pi)^{\alpha}}}\left[\frac{2r^{\alpha}}{\alpha r_{b}^{\alpha}n}\right]^{\ell},\\ &\exp\left(-\frac{r^{\alpha}}{\alpha^{2}r_{b}^{\alpha}n}\right)L_{(n-\ell-1)\alpha}^{2\ell+1}\left(\frac{2r^{\alpha}}{\alpha^{2}r_{b}^{\alpha}n}\right)e^{im\varphi^{\alpha}}P_{\ell\alpha}^{m\alpha}(\cos(\theta^{\alpha})).\end{aligned} \tag{59}$$

Table 3: The conformable wave function $\psi_{n\ell m\alpha}(r^{\alpha},\theta^{\alpha},\varphi^{\alpha})$ for different values of quantum number.

n	ℓ	m	$\psi_{n\ell m\alpha}(r^{\alpha},\theta^{\alpha},\varphi^{\alpha})$
1	0	0	$\sqrt{\frac{2}{\alpha^{3}r_{b}^{3\alpha}(2\pi)^{\alpha}}}\exp\left(-\frac{r^{\alpha}}{\alpha^{2}r_{b}^{\alpha}}\right)$
2	0	0	$\sqrt{\frac{1}{16(2\pi)^{\alpha}\alpha^{3}r_{b}^{3\alpha}}}\left[2-\frac{r^{\alpha}}{\alpha^{2}r_{b}^{\alpha}}\right]\exp\left(-\frac{r^{\alpha}}{2\alpha^{2}r_{b}^{\alpha}}\right)$
	1	0	$\sqrt{\frac{1}{4\alpha^{7}r_{b}^{5\alpha}(2\pi)^{\alpha}}}\frac{r^{\alpha}}{2}\exp\left(-\frac{r^{\alpha}}{2\alpha^{2}r_{b}^{\alpha}}\right)\cos(\theta^{\alpha})$
	1	1	$-\sqrt{\frac{1}{8\alpha^{7}r_{b}^{5\alpha}(2\pi)^{\alpha}}}\frac{r^{\alpha}}{2}\exp\left(-\frac{r^{\alpha}}{2\alpha^{2}r_{b}^{\alpha}}\right)e^{i\varphi^{\alpha}}\sin(\theta^{\alpha})$

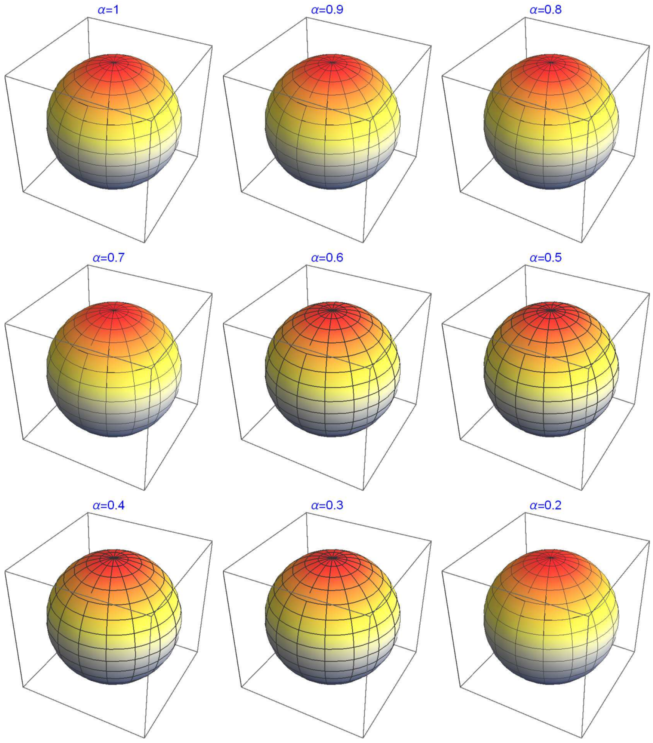


Fig. 10: Plot $|\psi_{200\alpha}|^2$ with different value of α from 0.2 to 1 in 3D

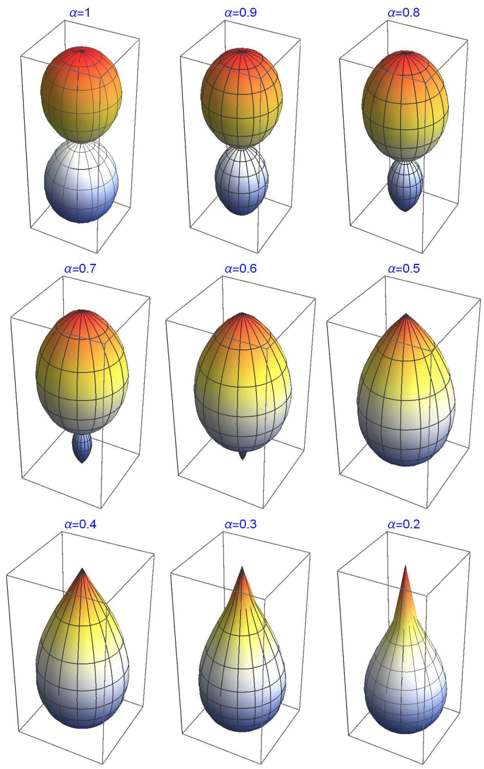


Fig. 11: Plot $|\psi_{210\alpha}|^2$ with different value of α from 0.2 to 1 in 3D

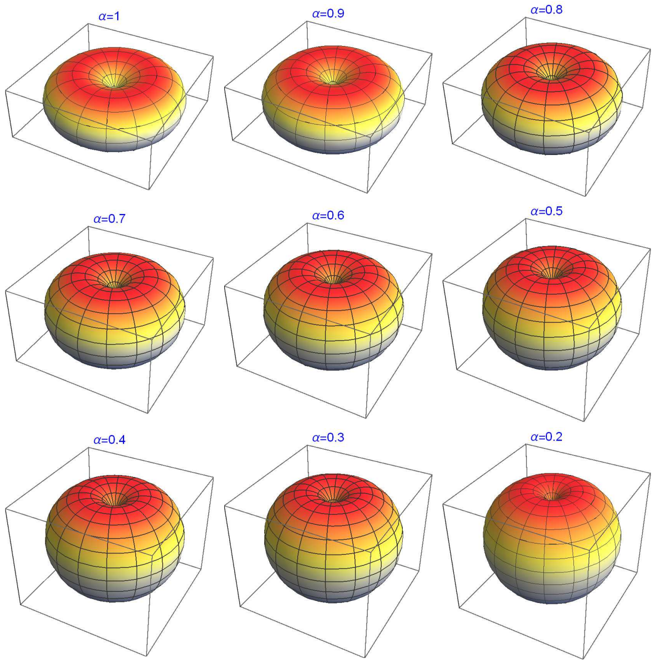


Fig. 12: Plot $|\psi_{211\alpha}|^2$ with different value of α from 0.2 to 1 in 3D

3 Conclusion

We have solved the conformable Schrödinger equation for the conformable Coulomb's potential. We obtained the conformable spherical harmonic function as the solution of the angular part and the conformable radial wave function in terms of the conformable associated Laguerre function. We observed that the conformable spherical harmonics goes to the traditional spherical harmonic function when α goes to 1 as well as the radial wave function. To illustrate our calculation we have drawn the conformable spherical harmonic function for $\ell = 2$ and $m = 1$ in 3D and 2D, with different values of α. We observed that in Figure 1 the density function gradually converts to the traditional density function given in Figure 2. Also, the same thing has been seen for the density function in the polar plot. In addition, we plot the $\alpha-$ probability density of radial function $(n = 1, \ell = 0)$ of different values of α and the same for $(n = 2, \ell = 0); (n = 2, \ell = 1); (n = 3, \ell = 0); (n = 3, \ell = 1)$ and $(n = 3, \ell = 2)$. (See figures 3 to 8). The conformable Schrödinger equation in 3D-spherical coordinates is solved and wave functions and energy levels for different values of α are obtained. The conformable wave functions for $n = 1$ and $n = 2$ are calculated. It is observed that the traditional wave function can be recovered when $\alpha = 1$.Besides, the α- probability density for $n = 1, \ell = 0$, $n = 2, \ell = 1$ and $n = 3, \ell = 0, 1, 2$ are drown for different values of α. It is concluded that the α- probability density gradually converts to the traditional case.

Note

This paper comprises two manuscripts that were previously submitted and announced by ArXiv: The first manuscript of this link https://arxiv.org/abs/2203.11615 and the second manuscript of this link https://arxiv.org/abs/2209.02699.

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