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Stochastic Optimal Control Models for the Insurance Company with Bankruptcy Return

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Abstract: In this paper we consider the optimal control problem for a insurance company. Our objective is to maximize the expectation of discounted dividends and its terminal value which represents the company liquidation value upon the time of bankruptcy. The surplus of the insurance company is governed by the Brownian motion with a constant drift and a diffusion term. The company can manage its risk exposure simultaneously through proportional reinsurance. Apart from the proportional reinsurance, the insurance company also pays out dividends with bounded dividends rate. With the help of the stochastic dynamic programming approach, we solve the control problem of maximizing the expectation of discounted dividends and the terminal value. We first construct a solution to the HJB equation and then verify that the solution of the HJB equation is indeed the optimal value function for our problem. We also give explicit expressions of the optimal strategies.

Keywords: proportional reinsurance, terminal value, the diffusion model, value function, HJB equation, the verification theorem

1. Introduction

With the development of economy and finance theory, mathematical instruments get increasingly more attention in both theoretical and practice fields(see [1]). Among this mathematical instruments, recently, stochastic optimal control theory is widely used in asset pricing and actuary calculation(see [2]). In this paper, an optimal risk exposure and dividends distribution for the insurance company, whose surplus process is assumed to be controlled by diffusion model, will be studied.
The research on optimal dividend problem can be traced back to [3], who proposes the solution of maximizing the expectation of the discounted dividends in discrete time settings. [4,5] develops De Finetti's contribution and makes it sensible to economists. [6,7] discuss the optimal dividends-payment strategies for the classical risk model in continuous time settings.
[8,9] consider the optimal dividend problem with a bounded dividend rate. [10] study the proportional reinsurance policies and dividends pay-out for a diffusion model. The diffusion model with excess-of-loss reinsurance and dividends pay-out is considered by [11]. [12] take the influence of bankruptcy value into account when investigating diffusion models with proportional reinsurance. They incorporate the model by assigning a value P to the bankruptcy of the company, and make a detailed study of how to maximize the total discounted value of the company wealth and the liquidation value P of the company. When P is negative, the company is fined for going into bankruptcy, and when P is positive, it is interpreted as the value has accrued from the sale of non-liquid assets.
In this paper, we consider the problem of maximizing the expectation of the sum of the discounted dividends and the bankruptcy return P, in the framework of diffusion approximation of the classic risk model with proportional reinsurance. We introduce the insurance risk model and present our control problem in Section 2. In Section 3, we give the corresponding HJB equation and construct a solution of the HJB equation. In Section 4, we present the verification theorem to show that the optimal strategy is reasonable. In the last part of this paper, we calculate some numerical examples and draw some plots to

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elaborate the optimal return function in different conditions.

2. The model

The risk process or the surplus process of an insurance company without reinsurance and dividend payments can be modeled by the classical Cramer-Lundberg model

$$R(t) = x + pt - \sum_{i=1}^{N_t} X_i,$$

where x is the initial capital of an insurance company, $p > 0$ is the premium rate, $\{N(t)\}$ is a Poisson process with intensity $\beta > 0$ and $X_i(i = 1, 2, \cdots)$ is the size of the i-th claim. The claims, independent with $N(t)$, are i.i.d random variables with the continuous distribution F and finite first and second moments m, s^2 respectively. We suppose that the premium rate p is calculated by the expected value principle, i. e.

$$p = (1+\eta)\beta m,$$

where $\eta > 0$ is the relative safety loading.

Assume that the insurance company arranges a proportional reinsurance strategy. In other words, the insurance company covers the fraction a of each claim, and the reinsurance company covers the rest. Thus the corresponding surplus process for this insurance company becomes

$$R(t) = x + apt - a\sum_{i=1}^{N_t} X_i.$$

This process can be approximated by a Brownian motion with the drift $a\mu$ and diffusion $a\sigma$ where $\mu = p - \beta m$ and $\sigma^2 = \beta s^2$ (see [13]). Without loss of generality, we assume $\beta = 1$. Therefore, the controlled reserve process $\{R(t)\}$ with initial value x is given by

$$dR(t) = \mu a_\pi(t)dt + \sigma a_\pi(t)dw_t,$$

where $w(t)$ is a standard Brownian motion.

An admissible control policy π is a two dimensional stochastic process $(a_\pi(t), l_\pi(t))$, where $0 \le a_\pi(t) \le 1$ corresponds to the risk exposure and $0 \le l_t \le M$ is the dividend rate at time t. Denote by Π the set of all admissible policies. Thus under the policy $\pi \in \Pi$, the dynamics of the controlled process R are formulated by

$$\begin{cases} dR^\pi(t) = (\mu a_\pi(t) - l_\pi(t))dt + \sigma a_\pi(t)dw_t \\ R^\pi(t) = x \end{cases}$$

where x corresponds to the strictly positive initial capital of the risk process.

The bankruptcy or the ruin time is defined as the first hitting time when the surplus process arrives zero, i.e.

$$\tau_\pi = \inf\{t : R^\pi(t) \le 0\}.$$

For each strategy π, the performance functional (value function) is given by

$$V_x(\pi) = E\left(\int_0^{\tau_\pi} e^{-ct} l_\pi(t)dt + e^{-c\tau_\pi}P\right),$$

where c is a given discount factor as mentioned as the previous section, P can be viewed as a salvage or recovery value of the company at the time of bankruptcy. The main task of this paper is to provide the optimal policy π^* for the optimal value function

$$V(x) = \sup_{\pi\varepsilon\Pi} V_\pi(x), \qquad (1)$$

such that $V(x) = V_{\pi^*}(x)$.

Next section, we use the dynamic programming approach to solve the above stochastic control problem.

3. HJB Equation and Its Solution

Firstly, in the spirit of [10], we present the concavity of $V(x)$ by the following lemma.

Lemma 1. *The function $V(x)$ defined in (1) is concave.*

Proof: This lemma can be proved by making the following modifications in the proof of Proposition 1.1 of [10]. Replace (1.6) in [10] with $l_{\pi_\xi}(t) = \lambda l_{\pi_{x_1}}(t) + (1-\lambda)l_{\pi_{x_2}}(t)$, and substitute $\tau_{\pi_\xi} = \tau_{\pi_{x_1}} \vee \tau_{\pi_{x_2}}$ by $\tau_{\pi_\xi} = \tau_{\pi_{x_1}} \wedge \tau_{\pi_{x_2}}$ when $P > 0$ and by $\tau_{\pi_\xi} = \tau_{\pi_{x_1}} \vee \tau_{\pi_{x_2}}$ when $P < 0$.

Now we shall adopt the dynamic programming approach to solve the maximization problem of (1). For an excellent account on the dynamic programming approach, interested readers may refer to [14,15]. From some standard arguments in [15], we have the following theorem.

Theorem 1. *Assume V defined by (1) is twice-continuous differentiable on $(0,\infty)$. Then V satisfies the following Hamilton-Jacobi-Bellman (hereforth HJB, for more details, see [15]) equation*

$$\sup_{a\in[0,1], l\in[0,M]}\left[\tfrac{1}{2}\sigma^2a^2V''(x) + (\mu a - l)V'(x) - cV(x) + l\right] = 0 \qquad (2)$$

with initial condition $V(0) = P$.

Lemma 1 and Theorem 1 imply that, in order to find the optimal value function $V(x)$, we need to find a concave solution of the HJB equation (2). Furthermore, from the definition of the optimal value function, we have

$$V(x) \le E\left[\frac{M}{c}(1-e^{-c\tau}) + e^{-c\tau}P\right] = \frac{M}{c} + E\left[\left(P - \frac{M}{c}\right)e^{-c\tau}\right] \le \max\left\{P, \frac{M}{c}\right\}.$$

Next we are going to construct a bounded concave solution of the HJB equation (2). Without loss of generality, we consider the condition $P \ge 0$. Throughout

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this section, we let $l(x)$ and $a(x)$ denote the functions that for all x maximize the left hand side of (2) in which V is replaced by f.

Let $u_1 = \inf\{u \geq 0 : f'(u) = 1\}$, then by the concavity of $f(x)$ we have,

$$l(x) = \begin{cases} 0, & x < u_1; \\ M, & x > u_1. \end{cases}$$

Therefore for any $x < u_1$, the HJB equation (2) is transforms to

$$\sup_{a\varepsilon[0,1]} [\frac{1}{2}\sigma^2 a^2 f''(x) + \mu a f'(x) - cf(x)] = 0. \tag{3}$$

Differentiating the inside terms of the maximum in (3) with respect to a yields the following first order condition for the candidate maximum point $a(x)$:

$$a(x) = \frac{-\mu f'(x)}{\sigma^2 f''(x)}.$$

Substituting this into (3) yields

$$\frac{-\mu^2 (f'(x))^2}{2\sigma^2 f''(x)} - cf(x) = 0. \tag{4}$$

It is easy to verify that the function $f_1(x)$ defined by

$$f_1(x) = (c_1 x + P^{\frac{1}{\gamma}})^\gamma,$$

with $\gamma = \frac{c}{\frac{\mu^2}{2\sigma^2} + c}$ and an unknown constant c_1 is a solution of the equation (4). The unknown constant c_1 will be determined later. Thus the maximizer $a(x)$ is given by

$$a(x) = \frac{\mu}{\sigma^2(1-\gamma)}(x + \frac{1}{c_1}P^{\frac{1}{\gamma}}).$$

Note that the maximizer $a(x)$ should lie in $[0,1]$, therefore, the above form solution only for the case of

$$\frac{\mu}{\sigma^2(1-\gamma)}(x + \frac{1}{c_1}P^{\frac{1}{\gamma}}) \in [0,1],$$

that is

$$x \leq \frac{\sigma^2}{\mu}(1-\gamma) - \frac{1}{c_1}P^{\frac{1}{\gamma}} := u_0.$$

Comparing u_0 and u_1, we have two cases to consider: $u_0 < u_1$ and $u_0 \geq u_1$. Later, we shall give the necessity and sufficient conditions for $u_0 < u_1$ and $u_0 \geq u_1$.

3.1. The case of $u_0 < u_1$

Since we try to find a solution of HJB equation defined on $x \geq 0$, we still need to consider the following two different cases: $0 \leq u_0$ and $u_0 < 0$.

We first consider the case of $0 \leq u_0$. In this case, note that the supremum part of equation (3) is a second-order polynomial in a, therefore, when $x \in [u_0, u_1)$, equation (3) reaches its supremum at $a = 1$. Thus substituting $a = 1$ into equation (3) yields

$$\frac{1}{2}\sigma^2 f''(x) + \mu f'(x) - cf(x) = 0. \tag{5}$$

The solution of (5) is

$$f_2(x) = c_2 e^{d_1 x} + c_3 e^{d_2 x},$$

where d_1 and d_2 are the positive and negative solutions of the characteristic equation

$$\frac{1}{2}\sigma^2 s^2 + \mu s - c = 0.$$

When $x \geq u_1$, we can obtain that the HJB equation (2) shall reach its supremum at $a = 1, l = M$ and therefore $f(x)$ should satisfy the following differential equation

$$\frac{1}{2}\sigma^2 f''(x) + (\mu - M)f'(x) - cf(x) + M = 0,\ x \geq u_1. \tag{6}$$

The solution of (6) is

$$f_3(x) = \frac{M}{c} + c_4 e^{d_3 x} + c_5 e^{d_4 x}$$

where d_3 and d_4 are the positive and negative solutions of the characteristic equation

$$\frac{1}{2}\sigma^2 s^2 + (\mu - M)s - c = 0.$$

By the bounded property of $f(x)$, we must have $c_4 = 0$. Thus above analysis yields the following representation of $f(x)$:

$$f(x) = \begin{cases} (c_1 x + P^{\frac{1}{\gamma}})^\gamma, & 0 \leq x < u_0; \\ c_2 e^{d_1 x} + c_3 e^{d_2 x}; & u_0 \leq x < u_1; \\ \frac{M}{c} + c_5 e^{d_4 x}; & x \geq u_1, \end{cases} \tag{7}$$

where $c_1, c_2, c_3, c_5, u_0, u_1$ are unknown constants and will be determined later. The corresponding maximizer of the HJB equation (2) are given by

$$a(x) = \begin{cases} (\frac{\mu}{\sigma^2(1-\gamma)}(x + \frac{1}{c_1}P^{\frac{1}{\gamma}}), & 0 \leq x < u_0; \\ 1; & x > u_0. \end{cases} \tag{8}$$

$$l(x) = \begin{cases} 0, & 0 \leq x < u_1; \\ M; & x \geq u_1. \end{cases} \tag{9}$$

Now we are going to determine the necessary and sufficient condition for $u_0 < u_1$ and the unknown constants. We choose these unknown constants in such a way that f and its first derivative are continuous at the points u_0 and u_1. Thus we have the following equations:

$$c_1^\gamma (\frac{\sigma^2(1-\gamma)}{\mu})^\gamma = c_2 e^{d_1 u_0} + c_3 e^{d_2 u_0}, \tag{10}$$

$$c_1^\gamma \gamma (\frac{\sigma^2(1-\gamma)}{\mu})^{\gamma-1} = c_2 d_1 e^{d_1 u_0} + c_3 d_2 e^{d_2 u_0} \tag{11}$$

$$\frac{M}{c} + c_5 e^{d_4 u_1} = c_2 e^{d_1 u_1} + c_3 e^{d_2 u_1}, \tag{12}$$

$$c_2 d_1 e^{d_1 u_1} + c_3 d_2 e^{d_2 u_1} = 1, \tag{13}$$

$$c_5 d_4 e^{d_4 u_1} = 1. \tag{14}$$

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From (10) and (11), we have

$$c_2 e^{d_1 u_0} + c_3 e^{d_2 u_0} = \frac{\sigma^2(1-\gamma)}{\gamma\mu}(c_2 d_1 e^{d_1 u_0} + c_3 d_2 e^{d_2 u_0}).$$

Therefore

$$\frac{c_3}{c_2} e^{(d_2-d_1)u_0} = \frac{\frac{d_1\sigma^2(1-\gamma)}{\gamma\mu} - 1}{1 - \frac{d_2\sigma^2(1-\gamma)}{\gamma\mu}}. \quad (15)$$

From (12)-(14), we have

$$(\frac{M}{c} + \frac{1}{d_4})(c_2 d_1 e^{d_1 u_1} + c_3 d_2 e^{d_2 u_1}) = c_2 e^{d_1 u_1} + c_3 e^{d_2 u_1}.$$

After simple calculations, we can rewrite above equation as follows,

$$\frac{c_3}{c_2} e^{(d_2-d_1)u_1} = \frac{\frac{d_1 M}{c} + \frac{d_1}{d_4} - 1}{1 - \frac{d_2 M}{c} - \frac{d_2}{d_4}}. \quad (16)$$

To show that $f(x)$ given by (7) solves the HJB equation (2), we need to ensure that $u_0 \le u_1$. (15)/(16) gives

$$e^{(d_2-d_1)(u_0-u_1)} = \frac{(\frac{d_1\sigma^2(1-\gamma)}{\gamma\mu} - 1)(1 - \frac{d_2 M}{c} - \frac{d_2}{d_4})}{(1 - \frac{d_2\sigma^2(1-\gamma)}{\gamma\mu})(\frac{d_1 M}{c} + \frac{d_1}{d_4} - 1)}. \quad (17)$$

Since

$$\frac{d_1\sigma^2(1-\gamma)}{\gamma\mu} - 1 < 0, \qquad 1 - \frac{d_2 M}{c} - \frac{d_2}{d_4} > 0,$$
$$1 - \frac{d_2\sigma^2(1-\gamma)}{\gamma\mu} > 0, \qquad \frac{d_1 M}{c} + \frac{d_1}{d_4} - 1 < 0,$$

we obtain

$$\begin{aligned} u_0 \le u_1 \iff & (\frac{d_1\sigma^2(1-\gamma)}{\gamma\mu} - 1)(1 - \frac{d_2 M}{c} - \frac{d_2}{d_4}) \\ & \le (1 - \frac{d_2\sigma^2(1-\gamma)}{\gamma\mu})(\frac{d_1 M}{c} + \frac{d_1}{d_4} - 1) \\ \iff & \frac{\sigma^2(1-\gamma)}{\gamma\mu} \le \frac{M}{c} + \frac{1}{d_4} \\ \iff & M \ge \frac{\mu}{2} - \frac{c}{d_4} \\ \iff & M \ge \frac{\mu}{2} + \frac{\sigma^2 c}{\mu}. \end{aligned} \quad (18)$$

Next, we shall try to find out the unknown constants c_1, c_2, c_3, c_5, u_0, u_1. Let

$$A = \frac{\frac{d_1\sigma^2(1-\gamma)}{\gamma\mu} - 1}{1 - \frac{d_2\sigma^2(1-\gamma)}{\gamma\mu}}, \quad B = \frac{\frac{d_1 M}{c} + \frac{d_1}{d_4} - 1}{1 - \frac{d_2 M}{c} - \frac{d_2}{d_4}}. \quad (19)$$

Then by equation (17), (15) and (16), we have

$$u_0 - u_1 = \frac{1}{d_2 - d_1}\ln\frac{A}{B}, \quad (20)$$

$$c_3 e^{d_2 u_0} = A c_2 e^{d_1 u_0}, \; c_3 e^{d_2 u_1} = B c_2 e^{d_1 u_1}. \quad (21)$$

Thus from (13), we get

$$c_2(Bd_2 + d_1)e^{d_1 u_1} = 1. \quad (22)$$

Noted that $u_0 = \frac{\sigma^2(1-\gamma)}{\mu} - \frac{1}{c_1}P^{\frac{1}{\gamma}}$, we have

$$c_1 = \frac{P^{\frac{1}{\gamma}}}{\frac{\sigma^2(1-\gamma)}{\mu} - u_0}. \quad (23)$$

Putting it into (10), we get

$$(\frac{P^{\frac{1}{\gamma}}}{\frac{\sigma^2(1-\gamma)}{\mu} - u_0})^{\gamma}(\frac{\sigma^2(1-\gamma)}{\mu})^{\gamma} = (A+1)c_2 e^{d_1 u_0}.$$

By (20), we have

$$\begin{aligned} P(\frac{\frac{\sigma^2(1-\gamma)}{\mu}}{\frac{\sigma^2(1-\gamma)}{\mu} - u_1 - \frac{1}{d_2-d_1}\ln\frac{A}{B}})^{\gamma} &= (A+1)c_2 e^{d_1(u_1 + \frac{1}{d_2-d_1}\ln\frac{A}{B})} \\ &= \frac{A+1}{Bd_2 + d_1} e^{\frac{d_1}{d_2-d_1}\ln\frac{A}{B}}, \end{aligned} \quad (24)$$

which gives

$$u_1 = \frac{\sigma^2(1-\gamma)}{\mu} - \frac{1}{d_2-d_1}\ln\frac{A}{B} - \frac{\sigma^2(1-\gamma)}{\mu} (\frac{A+1}{Bd_2+d_1} e^{\frac{d_1}{d_2-d_1}\ln\frac{A}{B}} \frac{1}{P})^{-\frac{1}{\gamma}}. \quad (25)$$

By (20) and (25), we obtain

$$u_0 = \frac{\sigma^2(1-\gamma)}{\mu} - \frac{\sigma^2(1-\gamma)}{\mu} \cdot (\frac{A+1}{Bd_2+d_1} e^{\frac{d_1}{d_2-d_1}\ln\frac{A}{B}} \frac{1}{P})^{-\frac{1}{\gamma}}. \quad (26)$$

Plugging (26) into (23), we get

$$c_1 = \frac{P^{\frac{1}{\gamma}}}{\frac{\sigma^2(1-\gamma)}{\mu}(\frac{A+1}{Bd_2+d_1} e^{\frac{d_1}{d_2-d_1}\ln\frac{A}{B}} \frac{1}{P})^{-\frac{1}{\gamma}}}. \quad (27)$$

Inserting (25) into (22), we have

$$c_2 = \frac{1}{Bd_2+d_1}\exp\{-d_1(\frac{\sigma^2(1-\gamma)}{\mu} - \frac{1}{d_2-d_1}\ln\frac{A}{B} - \frac{\sigma^2(1-\gamma)}{\mu}(\frac{A+1}{Bd_2+d_1} e^{\frac{d_1}{d_2-d_1}\ln\frac{A}{B}} \frac{1}{P})^{-\frac{1}{\gamma}})\}. \quad (28)$$

Combining (21) and (26) yields

$$c_3 = \frac{A}{Bd_2+d_1}\exp\{-d_2\frac{\sigma^2(1-\gamma)}{\mu} + \frac{d_1}{d_2-d_1}\ln\frac{A}{B} + d_2\frac{\sigma^2(1-\gamma)}{\mu}(\frac{A+1}{Bd_2+d_1} e^{\frac{d_1}{d_2-d_1}\ln\frac{A}{B}} \frac{1}{P})^{-\frac{1}{\gamma}}\}. \quad (29)$$

Combining (14) and (25) yields

$$c_5 = \frac{1}{d_4}\exp\{-d_4(\frac{\sigma^2(1-\gamma)}{\mu} - \frac{1}{d_2-d_1}\ln\frac{A}{B} - \frac{\sigma^2(1-\gamma)}{\mu}(\frac{A+1}{Bd_2+d_1} e^{\frac{d_1}{d_2-d_1}\ln\frac{A}{B}} \frac{1}{P})^{-\frac{1}{\gamma}})\}. \quad (30)$$

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Thus in the case of $u_0 \geq 0$, the following function

$$f(x) = \begin{cases} (c_1x + P^{\frac{1}{\gamma}})^\gamma, & 0 \leq x < u_0; \\ c_2e^{d_1x} + c_3e^{d_2x}; & u_0 < x < u_1; \\ \frac{M}{c} + c_5e^{d_4x}; & x \geq u_1. \end{cases} \tag{31}$$

with u_1, u_0, c_1, c_2, c_3, c_5 given by (25), (26), (27), (28), (29) and (30) respectively is a solution of the HJB equation (2). The necessity and sufficient condition for $u_0 \geq 0$ is given by

$$PB(Bd_2 + d_1) \geq A(A+1)e^{\frac{d_1}{d_2 - d_1}}. \tag{32}$$

Now we consider the case of $u_0 < 0$ i.e. $PB(Bd_2 + d_1) < A(A+1)e^{\frac{d_1}{d_2-d_1}}$. In this case, we shall derive the optimal value function in the case of $u_1 \geq 0$ and $u_1 < 0$ respectively, whose necessity and sufficient condition will be given in (35).

We first consider the case of $u_0 < 0$, $u_1 \geq 0$. In this case, the function f is given by

$$f(x) = \begin{cases} c_2e^{d_1x} + c_3e^{d_2x}; & 0 \leq x < u_1; \\ \frac{M}{c} + c_5e^{d_4x}; & x \geq u_1. \end{cases} \tag{33}$$

From the boundary condition, we have

$$f(0) = c_2 + c_3 = P.$$

To ensure the function f and its derivatives are continuous at the point u_1, we need the function and its first and second derivatives to be continuous at u_1. Thus equations (12), (13) and (14) must be satisfied. After some calculations, we have

$$Be^{-d_2u_1} + e^{-d_1u_1} = \frac{(B+1)P}{\frac{M}{c} + \frac{1}{d_4}}. \tag{34}$$

Note that the left side of the last equation is digressive and so we have

$$\begin{cases} u_1 \geq 0 \Leftrightarrow B+1 \geq \frac{(B+1)P}{\frac{M}{c}+\frac{1}{d_4}}; \\ u_1 < 0 \Leftrightarrow B+1 < \frac{(B+1)P}{\frac{M}{c}+\frac{1}{d_4}}. \end{cases} \tag{35}$$

Although the explicit expression for u_1 is hard to derive from (34), it is easily confirmed in numerical calculations. Now we can give explicit expressions of c_2, c_3, c_5 through u_1,

$$\begin{cases} c_2 = \frac{P}{Be^{(d_1-d_2)u_1}+1}; \\ c_3 = \frac{PBe^{(d_1-d_2)u_1}}{Be^{(d_1-d_2)u_1}+1}; \\ c_5 = \frac{1}{d_4}e^{d_4u_1}. \end{cases} \tag{36}$$

Now we consider the second case of $u_0 \leq u_1 < 0$. In this case, the function f is given by

$$f(x) = \frac{M}{c} + c_5e^{d_4x}. \tag{37}$$

Also from the boundary condition, we have

$$f(0) = \frac{M}{c} + c_5 = P. \tag{38}$$

Therefore,

$$c_5 = P - \frac{M}{c}. \tag{39}$$

Summarize above analysis yields the following theorem.

Theorem 2. *Suppose A and B are defined by (19) and $u_0 \leq u_1$, i.e. $M \geq \frac{\mu}{2} + \frac{\sigma^2 c}{\mu}$, then we have*

1.If $u_0 \geq 0$, i.e. $PB(Bd_2+d_1) \geq A(A+1)e^{\frac{d_1}{d_2-d_1}}$, then the function $f(x)$ defined by (31) with c_1, c_2, c_3, c_5 given by (27)- (30) is a solution of the HJB equation (2);
2.If $u_0 < 0, u_1 \geq 0$, i.e.,$PB(Bd_2+d_1) < A(A+1)e^{\frac{d_1}{d_2-d_1}}$,$B+1 \geq \frac{(B+1)P}{\frac{M}{c}+\frac{1}{d_4}}$, then the function defined by (33) with c_2, c_3, c_5 given by (36) is a solution of the HJB equation (2);
3.If $u_0 < 0, u_1 < 0$, i.e.,$PB(Bd_2+d_1) < A(A+1)e^{\frac{d_1}{d_2-d_1}}$,$B+1 < \frac{(B+1)P}{\frac{M}{c}+\frac{1}{d_4}}$, then the function defined by (37) with c_5 given by (39) is a solution of the HJB equation (2).

3.2. The case of $u_0 > u_1$

Now, we consider the case of $u_0 > u_1$, i.e. $M < \frac{\mu}{2} + \frac{\sigma^2 c}{\mu}$. We shall first show that $u_0 = \infty$ in this case. If not, from the definition of u_0, u_1 in the previous sections, the solution of HJB equation (2) should satisfy equation (6) when $x \geq u_0$. Through a similar method used in the previous section, we can construct the following solution of the HJB equation for $x > u_0$,

$$f_3(x) = \frac{M}{c} + c_5e^{d_4x}.$$

For $u_1 < x < u_0$, the HJB equation (2) changes to the following equation

$$\max_{a\varepsilon[0,1]}[\frac{1}{2}\sigma^2a^2f''(x) + (\mu a - M)f'(x) - cf(x) + M] = 0. \tag{40}$$

Therefore the point u_0 is determined by

$$-\frac{\mu f_2'(u_0)}{\sigma^2 f_2''(u_0)} = 1. \tag{41}$$

However in this case the "smooth fit" property fails since

$$-\frac{\mu f_3'(u_0)}{\sigma^2 f_3''(u_0)} = \frac{-\mu}{\sigma^2 d_4} < 1 = -\frac{\mu f_2'(u_0)}{\sigma^2 f_2''(u_0)}. \tag{42}$$

Thus the above equation suggests that no such u_0 exists and therefore $u_0 = \infty$. Now we only have one switching point u_1 and we can construct a solution of HJB equation as follows.

We first consider the case of $u_1 > 0$. For $x < u_1$, we solve the equation (3) to obtain a solution of the HJB equation (2). Through a similar method used in the

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previous section we obtain the following solution of equation (3):

$$f_1(x) = (c_1 x + P^{\frac{1}{\gamma}})^\gamma,$$

where $\gamma = \frac{c}{\frac{\mu^2}{2\sigma^2}+c}$ and c_1 is an unknown constant.

For $x > u_1$, We need only to solve the equation (40) for finding a solution of the HJB equation (2). Let $f^*(x)$ be a solution of the following differential equation

$$\max_{a\varepsilon[0,1]} [\frac{1}{2}\sigma^2 a^2 f''(x) + (\mu a - M) f'(x) - c f(x)] = 0. \tag{43}$$

Then it is not difficult to express the solution of the equation (40) through the solution of the equation (43) as follows

$$f(x) = f^*(x) + \frac{M}{c}. \tag{44}$$

Now we try to construct a solution of the equation (43). Differential with respect to a and setting to 0 yields the maximum point of the equation (43)

$$a(x) = -\frac{\mu f'(x)}{\sigma^2 f''(x)}.$$

Substituting above equation into (43) yields the following equation

$$\frac{-\mu^2 f'(x)^2}{2\sigma^2 f''(x)} - M f'(x) - c f(x) = 0. \tag{45}$$

To solve the above equation, we try to fit a solution of the form

$$f(x) = \alpha e^{\beta x} + c_2 \ , \ x > u_1. \tag{46}$$

Plug (46) into (45), we obtain that

$$-\frac{\mu^2}{2\sigma^2}\alpha e^{\beta x} - M\alpha\beta e^{\beta x} - c\alpha e^{\beta x} - c c_2 = 0. \tag{47}$$

It is easy to know that

$$\begin{cases} c_2 = 0, \\ \beta = -\frac{\frac{\mu^2}{2\sigma^2}+c}{M}. \end{cases}$$

Setting $\eta = \frac{2\sigma^2}{\mu^2}$, we can give the following representation of β

$$\beta = -\frac{1+c\eta}{M\eta}.$$

Thus the solution of eqaution (45) has the following form

$$f^*(x) = \alpha e^{-\frac{1+c\eta}{M\eta}x} \ , \ x > u_1.$$

So the solution to (40) is

$$f(x) = \alpha e^{-\frac{1+c\eta}{M\eta}x} + \frac{M}{c} \ , \ x > u_1.$$

Finally, we get

$$f(x) = \begin{cases} (c_1 x + P^{\frac{1}{\gamma}})^\gamma, & x < u_1; \\ \alpha e^{-\frac{1+c\eta}{M\eta}x} + \frac{M}{c}, & x > u_1. \end{cases} \tag{48}$$

The maximizing function $a(x)$ is then given by

$$a(x) = \begin{cases} \frac{\mu}{\sigma^2(1-\gamma)}(x + \frac{1}{c_1}P^{\frac{1}{\gamma}}), & x < u_1; \\ \frac{\mu M\eta}{\sigma^2(1+c\eta)}, & x > u_1. \end{cases} \tag{49}$$

where u_1, c_1 and α are unknown constants derived below. By the twice order continuous differentiability at u_1, we get the following equations

$$(c_1 u_1 + P^{\frac{1}{\gamma}})^\gamma = \alpha e^{-\frac{1+c\eta}{M\eta}u_1} + \frac{M}{c} \tag{50}$$

$$c_1\gamma(c_1 u_1 + P^{\frac{1}{\gamma}})^{\gamma-1} = 1 \tag{51}$$

$$-\alpha\frac{1+c\eta}{M\eta}e^{-\frac{1+c\eta}{M\eta}u_1} = 1. \tag{52}$$

From (50) and (52), we obtain

$$c_1 u_1 + P^{\frac{1}{\gamma}} = (\frac{M}{c} - \frac{M\eta}{1+c\eta})^{\frac{1}{\gamma}}. \tag{53}$$

Combing equation (53) and (51), we obtain

$$\frac{c_1 u_1 + P^{\frac{1}{\gamma}}}{c_1\gamma} = \frac{M}{c} - \frac{M\eta}{1+c\eta}. \tag{54}$$

Substituting (53) into (54), we have

$$c_1 = \frac{(\frac{M}{c} - \frac{M\eta}{1+c\eta})^{\frac{1-\gamma}{\gamma}}}{\gamma} \tag{55}$$

$$u_1 = \frac{(\frac{M}{c} - \frac{M\eta}{1+c\eta})^{\frac{1}{\gamma}} - P^{\frac{1}{\gamma}}}{c_1} = \gamma\frac{(\frac{M}{c} - \frac{M\eta}{1+c\eta})^{\frac{1}{\gamma}} - P^{\frac{1}{\gamma}}}{(\frac{M}{c} - \frac{M\eta}{1+c\eta})^{\frac{1-\gamma}{\gamma}}} \tag{56}$$

$$\alpha = -\frac{M\eta}{1+c\eta}e^{\frac{1+c\eta}{M\eta}u_1} = -\frac{M\eta}{1+c\eta}e^{\gamma\frac{1+c\eta}{M\eta}\frac{(\frac{M}{c} - \frac{M\eta}{1+c\eta})^{\frac{1}{\gamma}} - P^{\frac{1}{\gamma}}}{(\frac{M}{c} - \frac{M\eta}{1+c\eta})^{\frac{1-\gamma}{\gamma}}}}. \tag{57}$$

Thus we have the following expression of the solution of the HJB equation (40).

$$f(x) = \begin{cases} (c_1 x + P^{\frac{1}{\gamma}})^\gamma, & 0 \le x < u_1; \\ \alpha e^{-\frac{1+c\eta}{M\eta}x} + \frac{M}{c}, & x > u_1. \end{cases} \tag{58}$$

where c_1, u_1, α is given by (55) - (57).
Because $0 < \gamma < 1$ and $\eta = \frac{2\sigma^2}{\mu^2} > 0$,

$$u_1 \ge 0 \Longleftrightarrow \frac{M}{c} - \frac{M\eta}{1+M\eta} \ge P \tag{59}$$

The maximizing function $a(x)$ is then given by

$$a(x) = \begin{cases} \frac{\mu}{\sigma^2(1-\gamma)}(x + \frac{1}{c_1}P^{\frac{1}{\gamma}}), & 0 \le x < u_1; \\ \frac{\mu M\eta}{\sigma^2(1+c\eta)}, & x > u_1. \end{cases} \tag{60}$$

Next we shall consider the case of $u_1 < 0$, i.e. $\frac{M}{c} - \frac{M\eta}{1+M\eta} < P$. The boundary condition $f(0) = P$ yields

$$f(0) = \alpha + \frac{M}{C} = P.$$

So the return function is given by

$$f(x) = (P - \frac{M}{c})e^{-\frac{1+c\eta}{M\eta}x} + \frac{M}{c}. \tag{61}$$

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The maximizing function $a(x)$ is then given by

$$a(x) = \frac{\mu M\eta}{\sigma^2(1+c\eta)}. \tag{62}$$

Theorem 3. *Suppose $u_0 > u_1$, i.e. $M < \frac{\mu}{2} + \frac{\sigma^2 c}{\mu}$, then*

1. *If $u_1 \geq 0$, i.e. $\frac{M}{c} - \frac{M\eta}{1+M\eta} \geq P$, then the function $f(x)$ defined by (58) with c_1, u_1, α given by (55) - (57) is a solution of the HJB equation (2);*
2. *If $u_1 < 0$, i.e. $\frac{M}{c} - \frac{M\eta}{1+M\eta} < P$, then the function $f(x)$ defined by (61) is a solution of the HJB equation (2);*

4. The Verification Theorem

In this section, we shall show that the solution of the HJB equation constructed in the last section is indeed the optimal value function.

Let $f(x)$ be given by

$$f(x) = \begin{cases} (31),\ u_0 \leq u_1,\ u_0 > 0; \\ (33),\ u_0 \leq u_1,\ u_0 < 0,\ u_1 \geq 0; \\ (37),\ u_0 \leq u_1,\ u_1 < 0; \\ (58),\ u_0 > u_1,\ u_1 \geq 0; \\ (61),\ u_0 > u_1,\ u_1 < 0 \end{cases} \tag{63}$$

and define the admissible policy π^* for $t < \tau_{\pi^*}$ as

$$a_{\pi^*}(t) = a(R_t^{\pi^*}),\ \ l_{\pi^*}(t) = l(R_t^{\pi^*}), \tag{64}$$

where

$$a(x) = \begin{cases} (8),\ \ u_0 \leq u_1,\ u_0 \geq 0; \\ 1,\ \ \ \ u_0 \leq u_1,\ u_0 < 0; \\ (49),\ u_0 > u_1,\ u_1 \geq 0; \\ (60),\ u_0 \geq u_1,\ u_1 < 0 \end{cases} \tag{65}$$

and

$$l(x) = \begin{cases} M,\ x \geq u_1; \\ 0,\ \ x < u_1. \end{cases} \tag{66}$$

Here u_0 and u_1 are determined as follows: if $M \geq \frac{\mu}{2} + \frac{\sigma^2 c}{\mu}$ and $PB(Bd_2+d_1) \geq A(A+1)e^{\frac{d_1}{d_2-d_1}}$, then u_0 and u_1 are given by (26) and (25) respectively; if $M \geq \frac{\mu}{2} + \frac{\sigma^2 c}{\mu}$, $PB(Bd_2+d_1) < A(A+1)e^{\frac{d_1}{d_2-d_1}}$ and $B+1 \geq \frac{(B+1)P}{\frac{M}{c}+\frac{1}{d_4}}$, then u_1 can be derived from (34). If $M < \frac{\mu}{2} + \frac{\sigma^2 c}{\mu}$, then $u_0 = \infty$, and u_1 is given by (56) when $\frac{M}{c} - \frac{M\eta}{1+M\eta} \geq P$.

Theorem 4. *Let $V(x)$ be the optimal return function given by (1), $f(x)$ be the function defined by (63) and π^* be given by (64), then*

$$V(x) = f(x) = V_{\pi^*}(x).$$

The proof of this theorem is similar to Theorem 2.3 of [16] and so we omit here.

5. Numerical Calculations

In this section we calculate the optimal return function $V(x)$ for $P = 2$, $\mu = 1$ and the arbitrarily selected M, c and σ (in Fig 1, Fig 2, Fig 3, Fig 4 and Fig 5 shown below, we employ the symbol "sigma" to represent σ). We category $M < \frac{\mu}{2} + \frac{\sigma^2 c}{\mu}$ as Case I, and $M \geq \frac{\mu}{2} + \frac{\sigma^2 c}{\mu}$ as Case II.

Fig 1. Fig 3. Fig 5. Fig 7. Fig 9 belong to Case I, and Fig 2. Fig 4. Fig 6. Fig 8. Fig 10 belong to Case II.

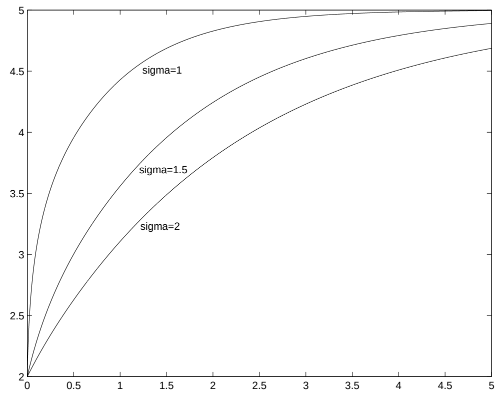

Figure 1 V(x) when c=0.1, M=0.5, P=2, $\mu = 1$

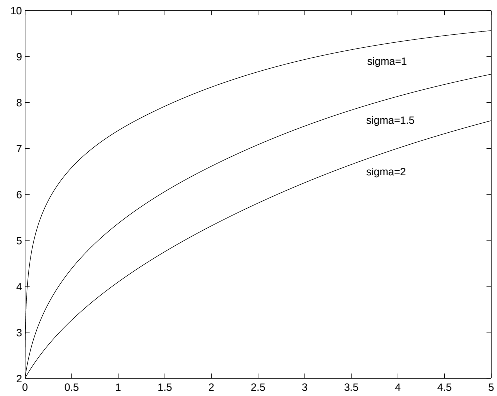

Figure 2 V(x) when c=0.1, M=1, P=2, $\mu = 1$

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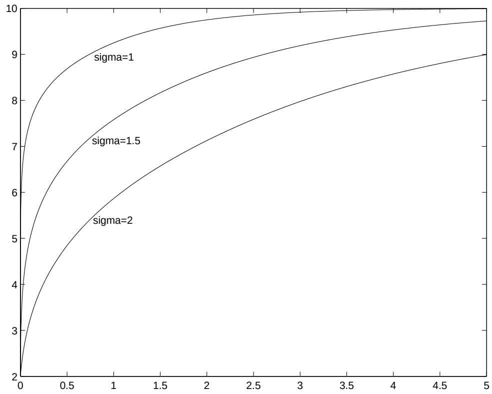

Figure 3 V(x) when c=0.05, M=0.5, P=2, $\mu = 1$

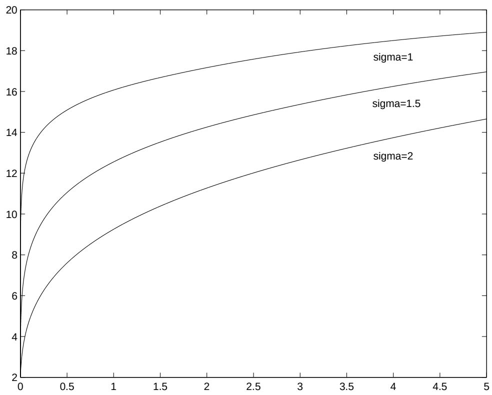

Figure 4 V(x) when c=0.05, M=1, P=2, $\mu = 1$

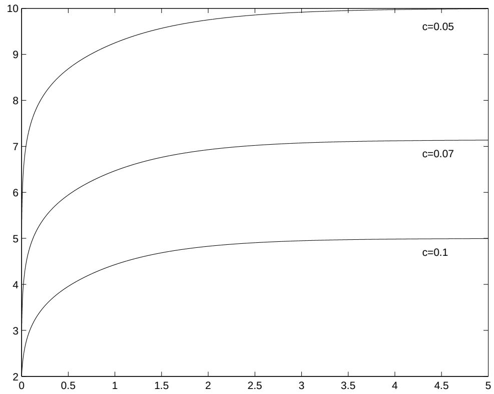

Figure 5 V(x) when $\sigma = 1$, M=0.5, P=2, $\mu = 1$

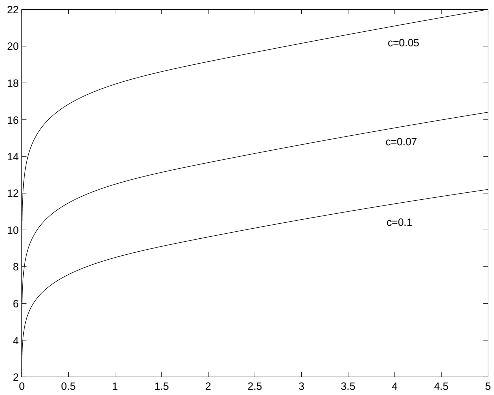

Figure 6 V(x) when $\sigma = 1$, M=1, P=2, $\mu = 1$

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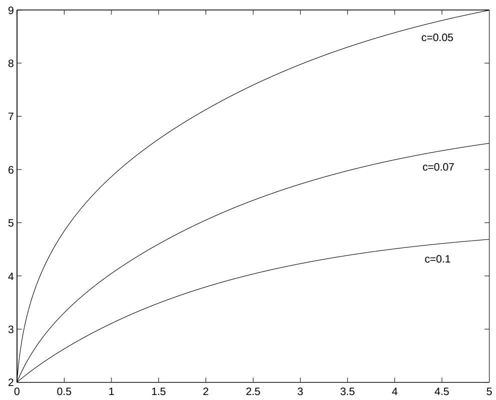

Figure 7 V(x) when $\sigma = 2$, M=0.5, P=2, $\mu = 1$

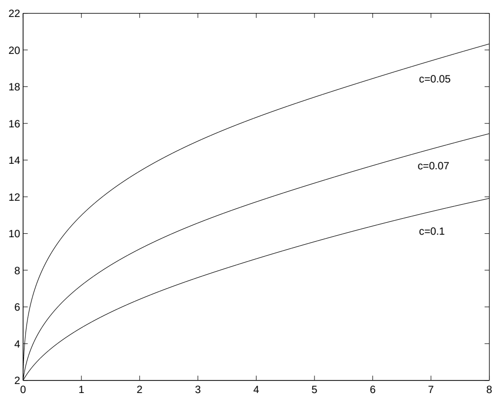

Figure 8 V(x) when $\sigma = 2$, M=1, P=2, $\mu = 1$

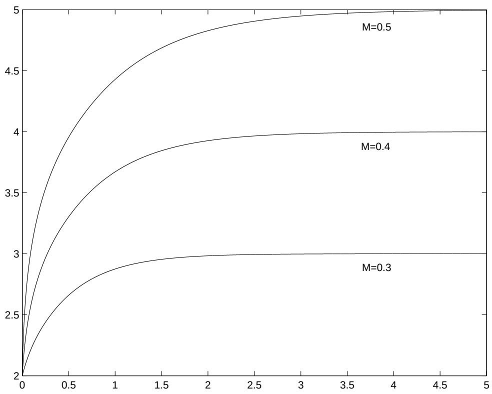

Figure 9 V(x) when c=0.1, $\sigma = 1$,P=2, $\mu = 1$

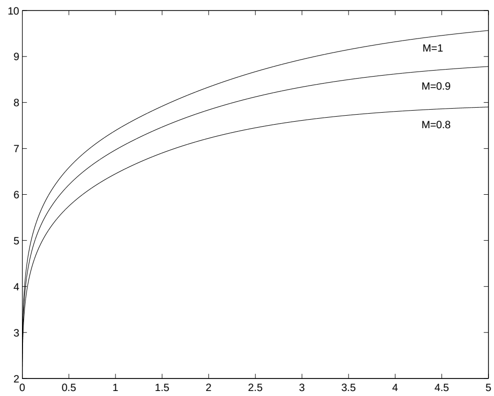

Figure 10 V(x) when c=0.1, $\sigma = 1$, P=2, $\mu = 1$

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