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On Fractional Inequalities for General Fractional Operators

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Abstract: The Leibniz and the power law rules of differentiation's do not hold for fractional derivatives with non-local kernels. Instead, infinite series representations were derived for the Leibniz and the power law rules of fractional derivatives. These rules produce heavy calculations, and they are not practical in implementations. In this paper, we derive certain inequalities of the general fractional operators of Caputo type. These inequalities are similar to the Leibniz and the power law rules for integer derivatives, where we replace the equality's by inequalities. Because the general fractional operators involve many fractional operators as particular cases, the current study will involve several types of fractional differential and integral operators and include some recent studies in the literature as particular cases. We present several examples to illustrate the applicability of the obtained results.

Keywords: General fractional operators; fractional differential inequalities; differentiation rules.

1 Introduction

We divide the introduction section into two parts. The first part is devoted to the general fractional operators and the second part is devoted to fractional inequalities.

1.1 General fractional operators

Because the basic theory of many fractional operators and their related fractional differential equations is similar, general fractional operators (GFOs) have been introduced and discussed by several authors recently. The idea is, if certain facts, identities or theories related to fractional operators are satisfied by different types of operators, then it is not proper to study them as individual cases while we can study them in the general case. Therefore, recent studies were devoted to make general classifications of fractional operators [14,21,22] and study related theories. Recently, the so-called general fractional operators and their associated general fractional differential equations have attracted the research interests of many researchers. The idea of involving the integro-differential operators on fractional calculus was initiated by Kuchubei [25], where he introduced the generalized fractional operators of Sonine kernels and studied related fractional differential equations. However, there are several singular and non-singular kernels in the literature which don't satisfy the Sonine conditions. For instance the kernels of the fractional operators with non-singular kernels [8,13] do not satisfy the Sonine conditions. Therefore, the GFOs have been investigated which involve the Sonine kernels as particular cases. Recently, in a series of papers [29,30,31,32] Luchko has established the basic theory of GFOs with several types of kernels including the Sonine kernels. In [10,28] certain maximum principles to fractional differential inequalities with the GFOs were derived and implemented to analyse the solutions of related fractional differential models. In [37,38], Tarasov implemented the GFOs for formulation of a general fractional dynamics and a general non-Markovian quantum dynamics. We refer the reader to [28] for a survey on rececent results about the fractional differential equations with GFOs.

The general fractional derivatives (GFDs) of the Riemann-Liouville and Caputo types are defined by

$$({}_a\mathbb{D}_k^{RL} f)(t) = \frac{d}{dt}\int_a^t k(t-\tau)f(\tau)\,d\tau,\ t>a, \tag{1}$$

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$$({}_a\mathbb{D}_k^C f)(t) = \int_a^t k(t-\tau)f'(\tau)\,d\tau,\ t>a, \tag{2}$$

whereas the general fractional integral (GFI) of Riemann-Liouville type is defined by

$$({}_a\mathbb{I}_\psi f)(t) = \int_a^t \psi(t-s)f(s)ds,\ t>a.$$

We assume the kernels $k,\psi:(0,+\infty)\to\mathbb{R}$, $k,\psi\in C^1(0,+\infty)\cap L_1^{\text{loc}}(\mathbb{R}_+)$, where $L_1^{\text{loc}}(\mathbb{R}_+)$ denotes the space of functions that are locally integrable on the positive real semi-axis. We also assume that k,ψ satisfy the following natural conditions:

$$k(t),\psi(t)\geq 0,\ k'(t),\psi'(t)\leq 0,\ t>0, \tag{3}$$

and they have integrable singularity at the origin, i.e, $tk(t), t\psi(t)\to 0$, as, $t\to 0$. To ensure existence of the GFDs and GFI we consider the space of functions

$$CW^1[a,b] = \{f\in C[a,b]:\ f'\in C^1(a,b]\cap L^1(a,b)\}.$$

Here we consider a general class of kernels that involves many known fractional operators as particular cases. Among these kernels are the ones that satisfy the Sonine conditions [36] which have been studied extensively in recent years [25, 29].

1.2 Inequalities in fractional calculus

Inequalities are important tool in studying various types of integral and differential equations, and without them the theory of integro-differential equations will not be at the current stage. Certain inequalities of the fractional derivative of a function at its extreme point for various types of fractional derivatives were developed and implemented to derive maximum principles of related fractional differential inequalities [4,5,6,7,9,16,24,27]. The derivatives considered are the Riemann-Liouvile derivative [4,6], Caputo-Fabrizio derivative [5], derivative with Mittag-Leffler kernel [7], Prabhakar derivative [9], Atangana-Baleanu derivative [16], sequential Caputo derivative [24], and the general fractional derivative [27]. Lyapunove type inequalities to fractional models were derived in [1] for the Atangana-Baleanue derivative, in [23] for the Caputo derivative and in [33] for the Hadamard derivative. Certain fractional-differential inequalities were derived in [34] and implemented to study the qualitative properties of related systems of fractional differential equations. Existence and uniqueness results as well as priori bounds estimates of solutions were established. New integral inequalities with Atangana-Baleanu derivative were established in [35] for s-convex functions, and as particular cases, various Hermite-Hadamrd inequalities were derived. We refer the reader to [12] for a survey on useful fractional inequalities and their applications. The well-known Leibniz and power law rules of integer derivatives do not hold true for fractional derivatives with nonlocal kernels. They hold only for the conformable fractional derivative which is considered as a local derivative. There are infinite sum representations of these rules for the well-known Riemann-Liouville and Caputo derivatives. These representations require further conditions on the function (it has be analytic in the domain) and they are difficult for implementations and calculations. Therefore, related fractional inequalities have been developed. In the following we summarize some of these inequalities in the literature.

Lemma 1.*[11,17,20] Let $u:[a,\infty)\to\mathbb{R}$ be continuously differentiable, then for $t>a$, it holds that*

$$({}^C D_a^\alpha u^2)(t)\leq 2u(t)({}^C D_a^\alpha u)(t),\ \ \textit{for all}\ 0<\alpha<1, \tag{4}$$

where ${}^C D_a^\alpha$ denotes the Caputo fractional derivative of order α.

A generalization of the above result was derived in [11,19].

Lemma 2.*[11,19] Let $u:[a,\infty)\to\mathbb{R}$ be continuously differentiable, and $\mu=\frac{m}{n}\geq 1$, where $m>0$, is an even number and $n\in\mathbb{N}$, then for $t>a$, it holds that*

$$({}^C D_a^\alpha u^\mu)(t)\leq \mu u^{\mu-1}(t)({}^C D_a^\alpha u)(t),\ \ \textit{for all}\ 0<\alpha<1. \tag{5}$$

Another generalization was obtained in [26].

Lemma 3.[26] *Let $u : [a,\infty) \to \mathbb{R}$ be continuously differentiable, which is non-negative and monotonically decreasing and $x : [a,\infty) \to \mathbb{R}$ is continuously differentiable, then for $t > a$, it holds that*

$$({}^{C}D_a^{\alpha} u\, x^2)(t) \leq 2u(t)x(t)({}^{C}D_a^{\alpha} x)(t), \ \textit{for all}\ \ 0 < \alpha < 1. \tag{6}$$

The above result is implemented to establish Lyapunov stability analysis for related systems of fractional order differential equations. While for the Riemann-Liouville fractional integral operator we have the following inequality.

Lemma 4.[15] *Let u and v be two synchronous functions on $[0,\infty)$, then for all $t > 0$ it holds that*

$$({}_0I^{\alpha}(uv))(t) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}}({}_0I^{\alpha}u)(t)({}_0I^{\alpha}v)(t), \ \ \alpha \in \mathbb{R}^{+}, \tag{7}$$

where $u, v \in L^1[0,\infty)$, and synchronous functions are defined in the next section.
In this paper, we extend these inequalities to the GFIs and GFDs of Caputo type. Section 2 is devoted to the new fractional inequalities, and some illustrative examples and discussion are highlighted in Section 3.

2 Fractional inequalities of GFOs

We start with inequalities related to the GFIs and then present inequalities related to the GFDs.

Definition 1.*We say f and g are synchronous on $[a,b]$, if it holds that*

$$(f(x)-f(y))(g(x)-g(y)) \geq 0, \ \textit{for all}\ x,y \in [a,b].$$

They are asynchronous on $[a,b]$, if they satisfy the revised inequality

$$(f(x)-f(y))(g(x)-g(y)) \leq 0, \ \textit{for all}\ x,y \in [a,b].$$

We remark here that if f and g are both non-decreasing or both non-increasing on $[a,b]$, then they are synchronous functions. If one is non-decreasing and the other one is non-increasing then they are asynchronous functions.

2.1 Fractional inequalities of GFIs

Lemma 5.*Let $f, g \in CW^1[a,b]$ be two synchronous functions on $[a,b]$, then it holds that*

$$({}_a\mathbb{I}_{\psi} fg)(t) \geq [({}_a\mathbb{I}_{\psi}\ 1)(t)]^{-1}({}_a\mathbb{I}_{\psi} f)(t)({}_a\mathbb{I}_{\psi} g)(t), \ \ a < t \leq b. \tag{8}$$

*Proof.*Because f and g are synchronous on $[a,b]$, then

$$f(x)g(x)+f(y)g(y) \geq f(x)g(y)+f(y)g(x) \ \text{ for all } x,y \in [a,b],$$

and thus,

$$\int_a^t f(x)g(x)\psi(t-x)dx + f(y)g(y)\int_a^t \psi(t-x)dx \geq g(y)\int_a^t \psi(t-x)f(x)dx + f(y)\int_a^t \psi(t-x)g(x)dx.$$

The above equation yields

$$({}_a\mathbb{I}_{\psi}\ fg)(t) + f(y)g(y)(\mathbb{I}_{\psi}\ 1)(t) \geq g(y)({}_a\mathbb{I}_{\psi} f)(t) + f(y)({}_a\mathbb{I}_{\psi} g)(t). \tag{9}$$

Multiplying the above equation by $\psi(t-y)$ and integrating over y, we have

$$({}_a\mathbb{I}_{\psi}\ fg)(t)\int_a^t \psi(t-y)dy + ({}_a\mathbb{I}_{\psi}\ 1)(t)\int_a^t \psi(t-y)f(y)g(y)dy \geq$$
$$({}_a\mathbb{I}_{\psi} f)(t)\int_a^t \psi(t-y)g(y)dy + ({}_a\mathbb{I}_{\psi} g)(t)\int_a^t \psi(t-y)f(y)dy,$$

or

$$2({}_a\mathbb{I}_{\psi}\ fg)(t)({}_a\mathbb{I}_{\psi}\ 1)(t) \geq 2({}_a\mathbb{I}_{\psi} f)(t)({}_a\mathbb{I}_{\psi} g)(t),$$

which proves the result.

Corollary 1.*Let $f, g \in CW^1[a,b]$ be two asynchronous functions on $[a,b]$, then it holds that*

$$({}_a\mathbb{I}_\psi fg)(t) \le [({}_a\mathbb{I}_\psi\ 1)(t)]^{-1}({}_a\mathbb{I}_\psi f)(t)({}_a\mathbb{I}_\psi g)(t),\ \ a < t \le b. \tag{10}$$

*Proof.*We have f and $h = -g$ are synchronous, and the result follows from Lemma 5.

Corollary 2.*For $u \in CW^1[a,b]$, it holds that*

$$({}_a\mathbb{I}_\psi u^2)(t) \ge [({}_a\mathbb{I}_\psi\ 1)(t)]^{-1}[({}_a\mathbb{I}_\psi u)(t)]^2. \tag{11}$$

By considering $f = g = u$, we have f and g are synchronous functions, and the results follow from Lemma 5.

2.2 Fractional inequalities of GFDs

Lemma 6.*Let $f, g \in CW^1[a,b]$, then it holds that*

$$\begin{aligned}({}_a\mathbb{D}_k fg)(t) &= f(t)({}_a\mathbb{D}_k g)(t) + g(t)({}_a\mathbb{D}_k f)(t) - (f(a)-f(t))(g(a)-g(t))k(t)\\ &\quad - \int_0^t (f(s)-f(t))(g(s)-g(t))\frac{d}{ds}k(t-s)ds.\end{aligned} \tag{12}$$

*Proof.*Integration by parts of $I = \int_a^t (f(s)-f(t))(g(s)-g(t))\frac{d}{ds}k(t-s)ds$, yields

$$\begin{aligned}I &= \Big((f(s)-f(t))(g(s)-g(t)\Big)k(t-s)\Big|_a^t\\ &\quad - \int_a^t k(t-s)\Big((f(s)-f(t))g'(s) + f'(s)(g(s)-g(t))\Big)ds\\ &= -(f(a)-f(t))(g(a)-g(t))k(t) - \int_a^t k(t-s)\Big(f(s)g'(s) + f'(s)g(s)\Big)ds\\ &\quad + f(t)\int_a^t k(t-s)g'(s)ds + g(t)\int_a^t k(t-s)f'(s)ds\\ &= -(f(a)-f(t))(g(a)-g(t))k(t) - ({}_a\mathbb{D}_k\, fg)(t) + f(t)({}_a\mathbb{D}_k\, g)(t) + g(t)({}_a\mathbb{D}_k\, f)(t),\end{aligned}$$

which proves the result.

We remark here the kernel $k(t)$ has integrable singularity at $t = 0$, and thus

$$\lim_{s\to t}(f(s)-f(t))(g(s)-g(t))k(t-s) = 0.$$

Because $k(t) \ge 0$ and $\frac{d}{ds}k(t-s) \ge 0$, and as a direct consequence of the above result we have

Corollary 3.*(i) If $f, g \in CW^1[a,b]$ are two synchronous functions then it holds that*

$$({}_a\mathbb{D}_k fg)(t) \le f(t)({}_a\mathbb{D}_k g)(t) + g(t)({}_a\mathbb{D}_k f)(t). \tag{13}$$

(ii) If $f, g \in CW^1[a,b]$ are two asynchronous functions, then it holds that

$$({}_a\mathbb{D}_k fg)(t) \ge f(t)({}_a\mathbb{D}_k g)(t) + g(t)({}_a\mathbb{D}_k f)(t). \tag{14}$$

Corollary 4.*For $u \in CW^1[a,b]$, it holds that*

$$({}_a\mathbb{D}_k u^2)(t) \le 2u(t)({}_a\mathbb{D}_k u)(t). \tag{15}$$

*Proof.*By substituting $f = g = u$ in Eq. (12) we have

$$({}_a\mathbb{D}_k u^2)(t) = 2u(t)({}_a\mathbb{D}_k u)(t) - (u(a)-u(t))^2k(t) - \int_a^t (u(s)-u(t))^2k'(t-s)ds.$$

The result follows as $k(t) \ge 0$, and$\frac{d}{ds}(k(t-s)) \ge 0$.

The result in Eq. (15) was presented by several authors for the particular cases of Riemann-Liouville and Caputo fractional derivatives [17,19]. In general for $\mu \in \mathbb{R}^+$, and by writing $u^\mu = u^{\mu/2} u^{\mu/2}$ one can easily deduce that

$$({}_a\mathbb{D}_k u^\mu)(t) \le 2u^{\frac{\mu}{2}}(t)({}_a\mathbb{D}_k u^{\frac{\mu}{2}})(t), \tag{16}$$

provided that $u^{\frac{\mu}{2}}$ is well defined. Applying induction arguments one can verify that

$$({}_a\mathbb{D}_k u^\mu)(t) \le 2^\mu u^{\mu-1}(t)({}_a\mathbb{D}_k u)(t), \tag{17}$$

where $\mu = 2^k, k$ is a positive integer. In the above result the factor 2^μ is weird, a stronger result is presented in the following result.

Lemma 7. *Let $u \in CW^1[a,b]$, then it holds that*

$$({}_a\mathbb{D}_k u^\mu)(t) \le \mu u^{\mu-1}(t)({}_a\mathbb{D}_k u)(t), \tag{18}$$

where $\mu = \frac{m}{n}, m,n \in \mathbb{N}$, and m is even number.

Proof. Let $h(s) = u^\mu(s) - \mu u^{\mu-1}(t)u(s) + (\mu-1)u^\mu(t)$, then it holds that $h(s) \ge 0$. The proof of $h(s) \ge 0$, was presented in Eq. (23) of [19] using a generalized Young inequality [18]. Because $\frac{d}{ds}k(t-s) \ge 0$, and using integration by parts we have

$$\begin{aligned}
0 \le I &= \int_a^t h(s)\frac{d}{ds}k(t-s)ds \\
&= h(s)k(t-s)|_a^t - \int_a^t h'(s)k(t-s)ds \\
&= -h(a)k(t) - \int_a^t (\mu u^{\mu-1}(s)u'(s) - \mu u^{\mu-1}(t)u'(s))k(t-s)ds \\
&= -h(a)k(t) - \int_a^t (\mu u^{\mu-1}(s)u'(s))k(t-s)ds + \mu u^{\mu-1}(t)\int_a^t u'(s)k(t-s)ds \\
&= -h(a)k(t) - ({}_a\mathbb{D}_k u^\mu)(t) + \mu u^{\mu-1}(t)({}_a\mathbb{D}_k u)(t).
\end{aligned}$$

The last inequality yields

$$({}_a\mathbb{D}_k u^\mu)(t) \le \mu u^{\mu-1}(t)({}_a\mathbb{D}_k u)(t) - h(a)k(a),$$

which proves the result in Eq. (18) as $h(a)k(t) \ge 0$.

Remark. If $u(t) \ge 0,\ t \in [a,b]$, then the result in Eq. (18) holds true for any $\mu = \frac{m}{n}, m,n \in \mathbb{N}$, where m is not necessary even number. The condition on m to be an even number is added to guarantee that $u^\mu(t),$ and $u^{\mu-1}(t),\ t \in [a,b]$ are well-defined.

3 Illustrative examples and discussion

The GFOs involve many known fractional operators in the literature as particular cases. Out of many in the literature, in the following we present two examples. The power law kernels, see [2,3]

$$\psi(t) = h_\alpha(t),\ k(t) = h_{1-\alpha}(t),\ 0 < \alpha < 1,\ h_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \tag{19}$$

is the standard example of GFOs which yields the well-known Riemann-Liouville and Caputo fractional operators. The inequalities related to this particular case were discussed by several authors. For this particular case we have

$$({}_a\mathbb{I}\ 1)(t) = \frac{1}{\Gamma(\alpha+1)}(t-a)^\alpha,\ t > a.$$

If f and g are synchronous functions then using Inequality (8) we have

$$({}_a\mathbb{I}_\psi fg)(t) \ge \left[\frac{1}{\Gamma(\alpha+1)}(t-a)^\alpha\right]^{-1}({}_a\mathbb{I}_\psi f)(t)({}_a\mathbb{I}_\psi g)(t),$$

which reduces to the Ineq. (7) as $a=0$. It is well-known that

$$({}_0\mathbb{D}_k E_\alpha(\lambda t^\alpha))(t) = \lambda E_\alpha(\lambda t^\alpha)(t),\ \lambda \in \mathbb{R},$$

where $E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+\alpha)}$ denotes the Mittag-Leffler function of one parameter. Using Ineq. (18) we arrive at the following inequality for the Mittag-Leffler function

$$\left({}_0\mathbb{D}_k \left(E_\alpha(\lambda t^\alpha)\right)^\mu\right)(t) \le \lambda\mu\left(E_\alpha(\lambda t^\alpha)\right)^\mu,\ \mu>0.$$

One more interesting particular case is the following Sonine kernels that was first deduced by Sonine [36]

$$\psi(t) = (\sqrt{t})^{\alpha-1} J_{\alpha-1}(2\sqrt{t}),\ \ k(t) = (\sqrt{t})^{-\alpha} I_{-\alpha}(2\sqrt{t}),\ \ 0<\alpha<1,$$

where

$$J_\mu(t) = \sum_{n=0}^{\infty} \frac{(-1)^n (t/2)^{2n+\mu}}{n!\Gamma(n+\mu+1)},\ I_\nu(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n+\nu}}{n!\Gamma(n+\nu)},$$

denotes the Bessel and the modified Bessel functions, respectively. For this particular case we have

$$({}_a\mathbb{D}_k f)(t) = \int_a^t (\sqrt{t-s})^{-\alpha} I_{-\alpha}(2\sqrt{t-s}) f'(s)ds, \tag{20}$$

$$({}_a\mathbb{I}_\psi f)(t) = \int_a^t (\sqrt{t-s})^{\alpha-1} J_{\alpha-1}(2\sqrt{t-s}) f(s)ds. \tag{21}$$

We have

$$\begin{aligned}({}_0\mathbb{I}_\psi\ 1)(t) &= \int_0^t (\sqrt{t-s})^{\alpha-1} J_{\alpha-1}(2\sqrt{t-s})ds = \int_0^t \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha)}(t-s)^{n+\alpha-1}ds \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} t^{n+\alpha} = (\sqrt{t})^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)}(\sqrt{t})^{2n+\alpha} = (\sqrt{t})^\alpha J_\alpha(2\sqrt{t}).\end{aligned}$$

So, if f and g are synchronous functions then using Inequality (8) we have

$$({}_0\mathbb{I}_\psi fg)(t) \ge \left[(\sqrt{t})^\alpha J_\alpha(2\sqrt{t})\right]^{-1} ({}_0\mathbb{I}_\psi f)(t)({}_0\mathbb{I}_\psi g)(t),$$

which generates the new inequality

$$\begin{aligned}\int_0^t (\sqrt{t-s})^{\alpha-1} J_{\alpha-1}(2\sqrt{t-s}) f(s)g(s)ds \ge \left[(\sqrt{t})^\alpha J_\alpha(2\sqrt{t})\right]^{-1} &\int_0^t (\sqrt{t-s})^{\alpha-1} J_{\alpha-1}(2\sqrt{t-s}) f(s)ds\, * \\ &\int_0^t (\sqrt{t-s})^{\alpha-1} J_{\alpha-1}(2\sqrt{t-s}) g(s)ds.\end{aligned} \tag{22}$$

Now, let us consider $f(t)=t$, we have

$$\begin{aligned}({}_0\mathbb{D}_k\ t)(t) &= \int_0^t (\sqrt{t-s})^{-\alpha} I_{-\alpha}(2\sqrt{t-s})ds = \int_0^t (\sqrt{t-s})^{-\alpha} \sum_{n=0}^{\infty} \frac{(\sqrt{t-s})^{2n-\alpha}}{n!\Gamma(n-\alpha)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n-\alpha+1)} t^{n-\alpha+1} = (\sqrt{t})^{1-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n-\alpha+1)}(\sqrt{t})^{2n-\alpha+1} \\ &= (\sqrt{t})^{1-\alpha} I_{1-\alpha}(2\sqrt{t}).\end{aligned} \tag{23}$$

Because $f(t)=t$, is increasing function then for increasing function g, Ineq. (13) yields the inequality

$$({}_0\mathbb{D}_k tg)(t) \le t({}_0\mathbb{D}_k g)(t) + g(t)({}_0\mathbb{D}_k t)(t) = t({}_0\mathbb{D}_k g)(t) + g(t)(\sqrt{t})^{1-\alpha} I_{1-\alpha}(2\sqrt{t}). \tag{24}$$

While for decreasing function g, Ineq. (14) yields the inequality

$$({}_0\mathbb{D}_k tg)(t) \ge t({}_0\mathbb{D}_k g)(t) + g(t)({}_0\mathbb{D}_k t)(t) = t({}_0\mathbb{D}_k g)(t) + g(t)(\sqrt{t})^{1-\alpha} I_{1-\alpha}(2\sqrt{t}). \tag{25}$$

We refer the reader to [31] for more examples of GFOs. The main results presented in this paper are new inequalities for the fractional derivatives and fractional integrals of functions with the general fractional operators. The proofs of these results were presented in elegant way and they are easier to follow comparing with some particular cases presented in the literature. The results include wide classes of fractional differential and integral operators as particular cases and many new inequalities can be deduced. The results presented in this paper can be extended to the general fractional operators with the non-singular kernels. Another research direction is to extend the Lyapunove inequalities of fractional boundary problems with GFDs. These tasks are left for a future work.

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