Applied Mathematics & Information Sciences
An International Journal

Some issues on quasi-arithmetic intuitionistic fuzzy OWA operators

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Received: 10 Oct. 2012, Revised: 7 Dec. 2012, Accepted: 19 Dec. 2012
Published online: 1 May 2013

Abstract: This paper presents a critical analysis of conventional operations on intuitionistic fuzzy values (IFVs), and some limitations of the conventional aggregation operator on IFVs are analyzed in detail. In order to overcome these limitations, according to new operation laws on IFVs based on Achimedean t-norm and t-conorm, we propose a new quasi-intuitionistic fuzzy OWA (QIFOWA) operator. Some corresponding specific QIFOWA operators are given to show a consistent result. Our revisions will undoubtedly result in the improvement of both the theoretic development and the practical application in aggregation operator for decision making problems.

Keywords: aggregation operator, intuitionistic fuzzy set, Achimedean t-norm, OWA operator.

1. Introduction

In some decision making processes, the decision maker cannot assess with crisp numbers because of the vague or imprecise knowledge. As an extension of the Zadehs fuzzy set [19], Intuitionistic fuzzy set (IFS) [1] is characterized by their capacity to assign each element in the universe of discourse a membership degree and a non-membership degree in describing imprecise and uncertain phenomena. IFS is used more extensively due to its capability in modeling subjective and imprecise human decision behaviors. Tremendous efforts have been spent and significant advances have been made in solving various uncertain decision making problems. Xu [12] developed the intuitionistic fuzzy weighed averaging operator, the intuitionistic fuzzy ordered weighted averaging operator. Xu and Yager [15] gave some other aggregation operators combining the geometric mean. Zhao et al. [20] and Li [8] introduced the generalized intuitionistic fuzzy aggregation operator. Tan and Chen [9] developed intuitionistic fuzzy Choquet integral operator for multi-criteria decision making. Yager [17] introduced another intuitionistic fuzzy OWA operator. Based on Archimedean t-conorm and t-norm [7], Beliakov et al. [2] provided generalizing averaging operators for IFS. Further Xia et al. [11] introduced intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm. Moreover, a lot of work [10,13,14,16] has been done about the weight vector of the intuitionistic fuzzy aggregation operators. Very recently, Yang and Chen [18] extended quasi-arithmetic OWA operator to different intuitionistic fuzzy situations, and proposed the quasi-intuitionistic fuzzy ordered weighted averaging operator, the quasi-intuitionistic fuzzy aggregation operator based on the Choquet integral and the DempsterCShafer belief structure. Since different definitions of operations on IFSs and their aggregation were proposed in the literature (see, for example [2]), the aim of this paper is to analyze their merits and drawbacks and extract those of them that provide the consistent results of operations on IFSs.

For these reasons, the rest of this paper is set out as follows. Section 2 presents the basic definition of IFS, the commonly used arithmetical operations on IFVs and some aggregation operators on IFVs are introduced. In Section 3, we provide a critical analysis of the operations presented in Section 2 to elicit their disadvantages. Section 4 we give new operation on IFSs and propose a new aggregation operator on IFVs to overcome these faults and obtain some consistent conclusions. Finally, the concluding section summaries the paper.

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2. The basic definitions of intuitionistic fuzzy set

Let X be an ordinary finite non-empty set. An intuitionistic fuzzy set [1] in X is an expression A given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$$

where $\mu_A : X \to [0, 1]$, $\nu_A : X \to [0, 1]$ with the condition: $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all x in X. The numbers $\mu_A(x)$ and $\nu_A(x)$ denote, respectively, the degree of membership and the degree of non-membership of the element x in the set A. $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the intuitionistic index of an element x in the intuitionistic fuzzy set A. Obviously, $0 \leq \pi_A(x) \leq 1$.

In this paper, the pair $(\mu_A(x), \nu_A(x))$ is called an intuitionistic fuzzy value (IFV). Let Ω be the set of all intuitionistic fuzzy values on X.

For every two intuitionistic fuzzy values A and B the following operations and relations are valid:

$$\begin{array}{l} 1)\ A = B \text{ if and only if } \mu_A(x) = \mu_B(x) \text{ and} \\ \quad \nu_A(x) = \nu_B(x) \text{ for all } x \in X; \\ 2)\ A \leq B \text{ if and only if } \mu_A(x) \leq \mu_B(x) \text{ and} \\ \quad \nu_A(x) \geq \nu_B(x) \text{ for all } x \in X. \end{array} \tag{1}$$

However, for some intuitionistic fuzzy values, Eq.(1) is not always satisfied in some situation. In order to rank intuitionistic fuzzy values, in the following, we use a score function and an accuracy function of intuitionistic fuzzy values for the comparison between two intuitionistic fuzzy values [12].

Definition 1. Let $A = (\mu_A(x), \nu_A(x))$ and $B = (\mu_B(x), \nu_B(x))$ be two intuitionistic fuzzy values in the set X, $S(A) = \mu_A(x) - \nu_A(x)$ and $S(B) = \mu_B(x) - \nu_B(x)$ be the score functions of A and B, respectively, and let $H(A) = \mu_A(x) + \nu_A(x)$ and $H(B) = \mu_B(x) + \nu_B(x)$ be the accuracy functions of A and B, respectively. If $S(A) < S(B)$, then A is smaller than B, denoted by $A < B$; if $S(A) = S(B)$, then

1) If $H(A) < H(B)$, then A is smaller than B, denoted by $A < B$;

2) If $H(A) = H(B)$, then A and B represent the same information, denoted by $A = B$.

Definition 2 [12]. Let $A = (\mu_A(x), \nu_A(x))$ and $B = (\mu_B(x), \nu_B(x))$ be two intuitionistic fuzzy sets in the set X and $\lambda > 0$, then

1) $A + B = (\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x))$;
2) $AB = (\mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x))$;
3) $\lambda A = (1 - (1 - \mu_A(x))^\lambda, (\nu_A(x))^\lambda)$;
4) $A^\lambda = ((\mu_A(x))^\lambda, 1 - (1 - \nu_A(x))^\lambda)$.

The operations in Definition 2 are used to define the following intuitionistic ordered weighted averaging (IOWA) operator with respect to a weighting vector w with $\sum_{i=1}^{n} w_i = 1$ to aggregate local criteria for solving MCDM problems in the intuitionistic fuzzy setting

$$\begin{aligned} \text{IOWA} &= w_1 A_1 + w_2 A_2 + \cdots + w_n A_n \\ &= (1 - \prod_{i=1}^{n} (1 - \mu_{A_{(i)}})^{w_i}, \prod_{i=1}^{n} (\nu_{A_{(i)}})^{w_i}), \end{aligned} \tag{2}$$

where $A_1, \ldots, A_n$ be IFVs representing the values of local criteria and $() : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a permutation such that $A_{(1)} \geq A_{(2)} \geq \ldots \geq A_{(1)}$ according to the total order in Definition 1.

Definition 3 [3]. For a given strictly monotone and continuous function $f : [0, 1] \to [-\infty, +\infty]$, called a generating function or generator, and a weighting vector $w = (w_1, w_2, \ldots, w_n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, the weighted quasi-arithmetic mean is the function

$$\text{WQAM}_{w,f}(x) = f^{-1}(\sum_{i=1}^{n} w_i f(x_i)). \tag{3}$$

It is nature to consider the quasi-OWA operator

$$\text{QOWA}_{w,f}(x) = f^{-1}(\sum_{i=1}^{n} w_i f(x_{(i)})), \tag{4}$$

where $x_{(1)} \geq \ldots \geq x_{(n)}$.

For convenience, an intuitionistic fuzzy value would be simply denoted as $\alpha_i = (u_i, v_i)$ in the following, where $u_i \in [0, 1]$, $v_i \in [0, 1]$, and $u_i + v_i \leq 1$.

According to operations in Definition 2 and ordering relation in Definition 1, Yang and Chen [16] generalize the QOWA to the intuitionistic fuzzy setting to derive the quasi-intuitionistic fuzzy ordered weighted averaging (QIFOWA) operator as follows.

Definition 4. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $\alpha_i = (u_i, v_i)$ $(i = 1, 2, \ldots, n)$ be a collection of IFVs on X. A QIFOWA operator of dimension n is a mapping from Ω^n to Ω that has an associated weighting vector $w = (w_1, w_2, \ldots, w_n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, such that:

$$\text{QIFOWA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = f^{-1}(\sum_{j-1}^{n} w_j f(x_{(j)})), \tag{5}$$

where $() : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a permutation such that $\alpha_{(1)} \geq \alpha_{(2)} \geq \ldots \geq \alpha_{(n)}$ and $f(x)$ is a strictly monotonic continuous function.

Further, Yang and Chen [18] consider some QIFOWA operators corresponding to specific forms of $f(x)$ as follows.

(1) If $f(x) = x^\lambda$, the QIFOWA operator becomes the generalized intuitionistic fuzzy ordered weighted averaging (GIFOWA) operator given in [20]:

$$\text{QIFOWA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\sum_{j=1}^{n} w_j \alpha_{(j)}^\lambda)^{1/\lambda}. \tag{6}$$

(2) If $f(x) = x$, the QIFOWA operator becomes the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator proposed in [13]:

$$\mathrm{QIFOWA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \sum_{j=1}^{n} w_j \alpha_{(j)}. \tag{7}$$

(3) If $f(x) = -\log_a x$, the QIFOWA operator becomes the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator as given in [15]:

$$\mathrm{QIFOWA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \prod_{j=1}^{n} \alpha_{(j)}^{w_j}. \tag{8}$$

(4) If $f(x) = \sin(\frac{\pi}{2}x), \cos(\frac{\pi}{2}x)$ or $\tan(\frac{\pi}{2}x)$, then the QIFOWA operator becomes the following trigonometric intuitionistic fuzzy ordered weighted averaging operators:

$$\begin{aligned}
&\mathrm{QIFOWA}_{\sin(\frac{\pi}{2}x)}(\alpha_1, \alpha_2, \cdots, \alpha_n)\\
&= \frac{\pi}{2}\arcsin(\sum_{j=1}^{n} w_j \sin(\frac{\pi}{2}\alpha_{(j)}))\\
&= \Big(\frac{\pi}{2}\arcsin(\sum_{j=1}^{n} w_j \sin(\frac{\pi}{2}u_{(j)})),\\
&\qquad \frac{\pi}{2}\arcsin(\sum_{j=1}^{n} w_j \sin(\frac{\pi}{2}v_{(j)}))\Big).
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\mathrm{QIFOWA}_{\cos(\frac{\pi}{2}x)}(\alpha_1, \alpha_2, \cdots, \alpha_n)\\
&= \frac{\pi}{2}\arccos(\sum_{j=1}^{n} w_j \cos(\frac{\pi}{2}\alpha_{(j)}))\\
&= \Big(\frac{\pi}{2}\arccos(\sum_{j=1}^{n} w_j \cos(\frac{\pi}{2}u_{(j)})),\\
&\qquad \frac{\pi}{2}\arccos(\sum_{j=1}^{n} w_j \cos(\frac{\pi}{2}v_{(j)}))\Big).
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\mathrm{QIFOWA}_{\tan(\frac{\pi}{2}x)}(\alpha_1, \alpha_2, \cdots, \alpha_n)\\
&= \frac{\pi}{2}\arctan(\sum_{j=1}^{n} w_j \tan(\frac{\pi}{2}\alpha_{(j)}))\\
&= \Big(\frac{\pi}{2}\arctan(\sum_{j=1}^{n} w_j \tan(\frac{\pi}{2}u_{(j)})),\\
&\qquad \frac{\pi}{2}\arctan(\sum_{j=1}^{n} w_j \tan(\frac{\pi}{2}v_{(j)}))\Big).
\end{aligned} \tag{11}$$

(5) If $f(x) = \gamma^x$, $\gamma \neq 1$, then the QIFOWA operator becomes the exponential intuitionistic fuzzy ordered weighted averaging operator:

$$\begin{aligned}
&\mathrm{QIFOWA}_{\gamma^x}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \log_\gamma(\textstyle\sum_{j=1}^{n} w_j \gamma^{\alpha_{(j)}})\\
&= (\log_\gamma(\textstyle\sum_{j=1}^{n} w_j \gamma^{u_{(j)}}), \log_\gamma(\textstyle\sum_{j=1}^{n} w_j \gamma^{v_{(j)}})).
\end{aligned} \tag{12}$$

Similarly, Yang and Chen [18] generalize the QOWA to the quasi-interval-valued intuitionistic fuzzy ordered wei
-ighted averaging operator, the quasi-intuitionistic fuzzy Choquet ordered averaging operator and the quasi-intuition
-istic fuzzy ordered weighted averaging operator based on the Dempster-Shafer belief structure.

3. The limitations of the quasi-arithmetic intuitionistic fuzzy OWA operator

In the following, we provide a critical analysis of the quasi-arithmetic intuitionistic fuzzy operations presented in Section 2.

Some limitations of conventional operations on IFVs were analyzed in [2]. On the one hand, the addition in Definition 2 is not an addition invariant operation. On the other hand, another problem with the aggregation operation (2) is that it is not monotone with respect to the ordering in Definition 1. Obviously, QIFOWA operator has the same limitations of IWAM operator (2).

For quasi-OWA operator or quasi-intuitionistic fuzzy ordered weighted averaging operator, generating function plays a very key role. Different generating function can derive different quasi-OWA or QIFOWA operators. For quasi-OWA operator or weighted quasi-arithmetic mean in Definition 3, $x_i \in [0, 1]$ can be regarded as a fuzzy set. And the generating function $f : [0, 1] \to [-\infty, +\infty]$ is a continuous strictly monotone function. However, unlike fuzzy sets, intuitionistic fuzzy set A is represented by two-comp-
onent variant $(\mu_A(x), \nu_A(x))$. If we extend quasi-OWA operator to intuitionistic fuzzy set, the generating function of quasi-intuitionistic fuzzy OWA is totally different generating function of quasi-OWA. If two kinds of generating functions are same, intuitionistic fuzzy OWA should have different expression form which is different from that of [18]. In fact, Yang and Chen [18] do not differentiate them, and do not give a clear definition to generating function. They only point that generating function is a strictly monotonic continuous function. In this case, the forms of Eq. (6)-(8) are different from that of Eq. (9)-(12). Let us analyze the above some special form of QIFOWA operators Eq. (6)-(12) to discuss what is generating function.

From Eq. (6)-(7), it is easy to know that their generating function $f : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ is a continuous strictly monotone two-component function, which is different from generating function of quasi-OWA operator. It is well-known that GIFOWA operator in [20] and IFOWA operator in [12] are derived from intuitionistic fuzzy operation laws in Definition 2. According to Definition 2, it is easy to obtain their inverse function of generating function in Eq. (6)-(7) and have the forms Eq. (6)-(7). However, until now logarithm operation law for intuitionistic fuzzy sets is not well defined so that we do not know what can $\log_a(u_A(x), u_A(x))$ be represented. So we do not know how to represent the inverse function of generating function in Eq. (8). We do not obtain Eq.(8) for $f(x) = -\log_a x$ at all.

From Eq. (9)-(12), it is easy to know that their generating function $f : [0,1] \to [0,1]$ is a continuous strictly monotone one-component function, which is different from the generating function in Eq. (6)-(7). If generating function in Eq. (9)-(12) is same as that in Eq. (6)-(7), similar to Eq.(8), we dont obtain these explicit formula for Eq. (9)-(12) at all because the operation laws in Eq. (9)-(12) for intuitionistic fuzzy sets are not defined. In fact, in order to obtain the Eq. (8)-(12) Yang and Chen [18] unconsciously assume the supposition that

$$\text{IFOWA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\sum_{j=1}^{n} w_j u_{(j)}, \sum_{j=1}^{n} w_j v_{(j)}), \quad (13)$$

and there are same generating function defined on $u_{(j)}$ and $v_{(j)}$, that is, the generating function $f : [0,1] \to [0,1]$ is a continuous strictly monotone function. However, Yang and Chen do not give enough reason to explain why the above supposition is right. If the supposition of Eq. (13) is correct, following Eq. (13) and Eq. (4), it is easy to know that Eq. (6) and Eq. (7) are not correct.

From the above analysis, because Yang and Chen dont clearly define the generating function in Definition 4, we can not derive consistent results between Eq. (6)-(8) and Eq. (9)-(12). In some cases, the generating function in Definition 4 is given by function $f : [0,1] \times [0,1] \to [0,1] \times [0,1]$. In other cases, the generating function in Definition 4 is given by function $f : [0,1] \to [0,1]$.

There are same questionable results for quasi-interval-valued intuitionistic fuzzy ordered weighted averaging operator, the quasi-intuitionistic fuzzy Choquet ordered averaging operator and the quasi-intuitionistic fuzzy ordered weighted averaging operator based on the Dempster-Shafer belief structure.

4. A new definition of quasi-arithmetic intuitionistic fuzzy OWA operator

In the following, according to new operation laws on IFVs, a new quasi-intuitionistic fuzzy ordered weighted averaging (QIFOWA) operator is proposed to overcome these limitations.

Until now, there is no definition and investigate for two-component generating function for intuitionistic fuzzy sets and its inverse function. In order to overcome this fault, in this section, according to Archimedean t-norms and t-conorms [6] we give an alternative definition of quasi-arithmetic intuitionistic fuzzy aggregation operator to obtain consistent results.

Fuzzy sets are considered to be functions $\mu :\to [0,1]$ [19]. In the 1980s, Klement combined t-norms with fuzzy sets [4,5], and operations on fuzzy sets are performed using triangular norms. A triangular norm T is defined to be a binary function: $T : [0,1] \times [0,1] \to [0,1]$ which is commutative, associative, monotone in each component and satisfies the boundary condition $T(x,1) = x$. The corresponding t-conorm of T (or the dual of T) is the function $S : [0,1] \times [0,1] \to [0,1]$ defined by $S(x,y) = 1 - T(1-x, 1-y)$. Here we can use any pair of dual t-norms and t-conorms.

A t-norm $T(x,y)$ is called Archimedean t-norm if it is continuous and $T(x,y) < x$ for all $x \in [0,1]$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $x, y \in [0,1]$ [6]. We first concentrate on continuous Archimedean t-norms and t-conorms, and in particular on the product t-norm, because it was used in many existing definitions.

It is well known [7] that a strict Archimedean t-norm is expressed via its additive generator g as $T(x,y) = g^{-1}(g(x) + g(y))$, and the same applies to its dual t-conorm, $S(x,y) = h^{-1}(h(x) + h(y))$ with $h(t) = g(1-t)$. We notice that an additive generator of a continuous Archimedean t-norm is a strictly decreasing function $g : [0,1] \to [0,\infty]$ such that $g(1) = 0$.

Based on Archimedean t-norm and t-conorm, Beliakov et al. [2] defined the sum operation on two intuitionistic fuzzy values $\alpha_i = (u_i, v_i)$ $(i = 1,2)$ as $\alpha_1 + \alpha_2 = (S(u_1,u_2), T(v_1,v_2))$, which can be expressed by the following:

$$\begin{aligned}\alpha_1 + \alpha_2 &= (S(u_1,u_2), T(v_1,v_2)) \\ &= (h^{-1}(h(u_1) + h(u_2)), g^{-1}(g(v_1) + g(v_2))).\end{aligned}$$

And let $\lambda\alpha_i = (u_{\lambda_i}, v_{\lambda_i})$ $(\lambda > 0)$, then $g(v_{\lambda_i}) = \lambda g(v_i)$, $h(u_{\lambda_i}) = \lambda h(u_i)$, $\lambda\alpha_i = (h^{-1}(\lambda h(u_i)), g^{-1}(\lambda g(v_i)))$.

According to the above operational laws on Archimedean t-norm and t-conorm, it can be deduced to obtain the following conclusion [2,11]:

Proposition 1. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $\alpha_i = (u_i, v_i)$ $(i = 1,2,\ldots,n)$ be a collection of IFVs on X. Intuitionistic Ordered Weighted Averaging based on an Archimedean t-norm T and t-conorm S with respect to an associated weighting vector $w = (w_1, w_2, \ldots, w_n)$ with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, AST-IOWA$_w$, can be expressed by:

$$\begin{aligned}&\text{AST-IOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ &= (h^{-1}(\textstyle\sum_{j=1}^{n} w_j h(u_{(j)})), g^{-1}(\textstyle\sum_{j=1}^{n} w_j g(v_{(j)}))),\end{aligned} \quad (14)$$

where $() : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ is a permutation such that $\alpha_{(1)} \geq \alpha_{(2)} \geq \ldots \geq \alpha_{(n)}$ according to the total order of Definition 3. And $g$is an additive generator of Archimedean t-norm T, and h is an additive generator of Archimedean t-conorm S.

If corresponding t-norms and t-conorms are applied to define the intuitionistic ordered weighted averaging operator, it is shown that AST-IOWA operators are consistent with aggregation operation on the ordinary fuzzy sets. And it is noted that these operators are also monotone with respect to the order defined in Definition 1. Compared with operational laws in Definition 2, for these advantages of intuitionistic fuzzy operational laws on Archimedean t-norm and t-conorm, we can refer to [2].

Because there are some positive properties of these aggregation operators defined by corresponding t-norms and t-conorms, in the following, According to corresponding t-norms and t-conorms, we generalize the QOWA to the intuitionistic fuzzy setting to derive another quasi-intuitionistic fuzzy ordered weighted averaging (QIFOWA) operator to overcome the limitations of quasi-arithmetic operations on intuitionistic fuzzy values in Section 2.

Definition 5. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $\alpha_i = (u_i, v_i)$ $(i = 1, 2, \ldots, n)$ be a collection of IFVs on X. A QIFOWA operator of dimension n based on an Archimedean t-conorm T and t-norm S is a mapping from Ω^n to Ω that has an associated weighting vector $w = (w_1, w_2, \ldots, w_n)$ with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, such that:

$$\begin{aligned}&\mathrm{AST-QIFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n)\\ &= \Big(h^{-1}(f^{-1}(\textstyle\sum_{j=1}^{n} w_j f(h(u_{(j)})))),\\ &\qquad g^{-1}(f^{-1}(\textstyle\sum_{j=1}^{n} w_j f(g(v_{(j)}))))\Big),\end{aligned} \tag{15}$$

where $() : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a permutation such that $\alpha_{(1)} \geq \alpha_{(2)} \geq \ldots \geq \alpha_{(n)}$ according to the total order of Definition 1, generator $f : [0,1] \to [0,1]$ is a strictly monotonic continuous function, and g is an additive generator of Archimedean t-norm T, and h is an additive generator of Archimedean t-norm S.

In the following, we consider some AST-QIFOWA operators corresponding to specific forms of $h(x), g(x)$ and $f(x)$.

(1) Taking into account that $h^{-1} = h = Id$ and $g^{-1} = g = 1 - Id$ on $[0,1]$, and that $\sum_{i=1}^{n} w_i = 1$ ensures the argument of g^{-1} is in $[0,1]$.

a) If $f(x) = x$, the QIFOWA operator reduces the following:

$$\mathrm{QIFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\sum_{j=1}^{n} w_j u_{(j)}, \sum_{j=1}^{n} w_j v_{(j)}),$$

called by intuitionistic fuzzy ordered weighted averaging operator, which is different from that in [12]. That is to say, in this case, (15) is reduced to (13).

b) $f(x) = x^\lambda$, the QIFOWA operator reduces to the following:

$$\begin{aligned}&\mathrm{QIFOWA}_\lambda(\alpha_1, \alpha_2, \cdots, \alpha_n)\\ &= ((\textstyle\sum_{j=1}^{n} w_j u_{(j)}^\lambda)^{1/\lambda}, (1 - \textstyle\sum_{j=1}^{n} w_j v_{(j)}^\lambda)^{1/\lambda}),\end{aligned}$$

called by generalized intuitionistic fuzzy ordered weighted averaging operator, which is different from that in [20].

c) If $f(x) = -\log_a x$, the QIFOWA operator reduces to the following:

$$\mathrm{QIFOWA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\textstyle\prod_{j=1}^{n} u_{(j)}^{w_j}, 1 - \textstyle\prod_{j=1}^{n} (1 - v_{(j)})^{w_j}),$$

called by intuitionistic fuzzy ordered weighted geometric operator, which is different from that in [15].

d) If $f(x) = \sin(\frac{\pi}{2}x), \cos(\frac{\pi}{2}x)$ or $\tan(\frac{\pi}{2}x)$, the QIFOWA operator reduces to the following, respectively:

$$\begin{aligned}&\mathrm{QIFOWA}_{\sin(\frac{\pi}{2}x)}(\alpha_1, \alpha_2, \cdots, \alpha_n)\\ &= \Big(\frac{\pi}{2}\arcsin(\sum_{j=1}^{n} w_j \sin(\frac{\pi}{2}u_{(j)})),\\ &\qquad 1 - \frac{\pi}{2}\arcsin(\sum_{j=1}^{n} w_j \sin(\frac{\pi}{2}(1 - v_{(j)})))\Big),\end{aligned}$$

$$\begin{aligned}&\mathrm{QIFOWA}_{\cos(\frac{\pi}{2}x)}(\alpha_1, \alpha_2, \cdots, \alpha_n)\\ &= \Big(\frac{\pi}{2}\arccos(\sum_{j=1}^{n} w_j \cos(\frac{\pi}{2}u_{(j)})),\\ &\qquad 1 - \frac{\pi}{2}\arccos(\sum_{j=1}^{n} w_j \cos(\frac{\pi}{2}(1 - v_{(j)})))\Big),\end{aligned}$$

$$\begin{aligned}&\mathrm{QIFOWA}_{\tan(\frac{\pi}{2}x)}(\alpha_1, \alpha_2, \cdots, \alpha_n)\\ &= \Big(\frac{\pi}{2}\arctan(\sum_{j=1}^{n} w_j \tan(\frac{\pi}{2}u_{(j)})),\\ &\qquad 1 - \frac{\pi}{2}\arctan(\sum_{j=1}^{n} w_j \tan(\frac{\pi}{2}(1 - v_{(j)})))\Big),\end{aligned}$$

which is the trigonometric intuitionistic fuzzy ordered weighted averaging operators defined by Yang and Chen [18].

e) If $f(x) = \gamma^x, \gamma \neq 1$, the QIFOWA operator reduces to the following:

$$\begin{aligned}&\mathrm{QIFOWA}_{\gamma^x}(\alpha_1, \alpha_2, \cdots, \alpha_n)\\ &= (\log_\gamma(\textstyle\sum_{j=1}^{n} w_j \gamma^{u_{(j)}}), 1 - \log_\gamma(\textstyle\sum_{j=1}^{n} w_j \gamma^{1 - v_{(j)}})),\end{aligned}$$

which is the exponential intuitionistic fuzzy ordered weighted averaging operator defined by Yang and Chen [18].

Remark 1. From the above analysis, according to Definition 5, we obtain some consistent specific QIFOWA operators forms where some specific QIFOWA operators are different from these in [18]. The consistent results can be not derived from Definition 4.

(2) Taking into account that $f = Id$, and that $\sum_{i=1}^{n} w_i = 1$ ensures the argument of h^{-1} and g^{-1} are in $[0,1]$. In this case, it is obvious that the QIFOWA operator reduces to IOWA operator in Proposition 1.

a) If $g(t) = -\log(t)$, then $h(t) = -\log(1-t)$, $g^{-1}(t) = e^{-t}, h^{-1}(t) = 1 - e^{-t}$, the QIFOWA operator reduces to the following:

$$\mathrm{QIFOWA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (1 - \prod_{j=1}^{n}(1-u_{(j)})^{w_j}, \prod_{j=1}^{n} v_{(j)}^{w_j}),$$

which is the intuitionistic fuzzy ordered weighted averaging operator defined by Xu [12]:

b) If $g(t) = \log(\frac{2-t}{t})$, then $h(t) = \log(\frac{2-(1-t)}{1-t})$, $g^{-1}(t) = \frac{2}{e^t+1}$, $h^{-1}(t) = 1 - \frac{2}{e^t+1}$, the QIFOWA operator reduces to the following:

$$\mathrm{QIFOWA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\frac{\prod_{j=1}^{n}(1+u_{(j)})^{w_j} - \prod_{j=1}^{n}(1-u_{(j)})^{w_j}}{\prod_{j=1}^{n}(1+u_{(j)})^{w_j} + \prod_{j=1}^{n}(1-u_{(j)})^{w_j}}, \frac{2\prod_{j=1}^{n} v_{(j)}^{w_j}}{\prod_{j=1}^{n}(2-v_{(j)})^{w_j} + \prod_{j=1}^{n} v_{(j)}^{w_j}}\right)$$

which is called the Einstein intuitionistic fuzzy ordered weighted averaging (EIFOWA) operator.

c) If $g(t) = \log(\frac{\gamma+(1-\gamma)t}{t})$, $\lambda > 0$, then $h(t) = \log(\frac{\gamma+(1-\gamma)(1-t)}{1-t})$, $g^{-1}(t) = \frac{\gamma}{e^t+\gamma-1}$, $h^{-1}(t) = 1 - \frac{\gamma}{e^t+\gamma-1}$, the QIFOWA operator reduces to the following:

$$\mathrm{QIFOWA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\frac{\prod_{j=1}^{n}(1+(\gamma-1)u_{(j)})^{w_j} - \prod_{j=1}^{n}(1-u_{(j)})^{w_j}}{\prod_{j=1}^{n}(1+(\gamma-1)u_{(j)})^{w_j} + (\gamma-1)\prod_{j=1}^{n}(1-u_{(j)})^{w_j}}, \frac{\gamma\prod_{j=1}^{n} v_{(j)}^{w_j}}{\prod_{j=1}^{n}(1+(\gamma-1)(1-v_{(j)}))^{w_j} + (\gamma-1)\prod_{j=1}^{n} v_{(j)}^{w_j}}\right)$$

which is called the Hammer intuitionistic fuzzy ordered weighted averaging (HIFOWA) operator. Especially, if $\gamma = 1$, then the HIFOWA operator reduces to the IFOWA operator defined by Xu [12]; if $\gamma = 2$, then the HIFOWA operator reduces to the EIFOWA operator.

d) If $g(t) = \log(\frac{\gamma-1}{\gamma^t-1})$, $\lambda > 0$, then $h(t) = \log(\frac{\gamma-1}{\gamma^{1-t}-1})$, $g^{-1}(t) = \frac{\log(\frac{\gamma-1+e^{\lambda}}{e^{\gamma}})}{\log\gamma}$, $h^{-1}(t) = 1 - \frac{\log(\frac{\gamma-1+e^{\lambda}}{e^{\gamma}})}{\log\gamma}$, the QIFOWA operator reduces to the following:

$$\mathrm{QIFOWA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(1 - \log_\gamma(1 + \frac{\prod_{j=1}^{n}(\gamma^{1-u_{(j)}}-1)^{w_j}}{\gamma-1}), \log_\gamma(1 + \frac{\prod_{j=1}^{n}(\gamma^{v_{(j)}}-1)^{w_j}}{\gamma-1})\right)$$

which is called the Frank intuitionistic fuzzy weighted ordered averaging (FIFOWA) operator. Especially, if $\gamma \to 1$, then the FIFOWA operator reduces to the IFOWA operator defined by Xu [12].

Remark 2. For generator function $f = Id$, we have some specific QIFOWA operators which are same as these in [11].

(2) According to Definition 5, it is easy to generalize the QOWA to the quasi-interval-valued intuitionistic fuzzy ordered weighted averaging operator, the quasi-intuitionistic fuzzy Choquet ordered averaging operator and the quasi-intuitionistic fuzzy ordered weighted averaging operator based on the DempsterCShafer belief structure. And we can also obtain some corresponding quasi-arithmetic intuitionistic fuzzy OWA operators for corresponding specific forms of $h(x), g(x)$ and f(x).

5. Conclusion

In this paper, we analyze some limitations of aggregation operations defined by traditional operation laws on intuitionistic fuzzy values in detail. Further, some new operation laws are defined by Archimedean t-conorm and t-norm. In order to overcome these limitations, we give a new definition of quasi-arithmetic intuitionistic fuzzy aggregation operator according to the new operation laws. Some special forms of QIFOWA operator are given to illustrate the alternative definition different from that of Yang and Chen [18], and to show consistent result each other. Our revisions will undoubtedly result in the improvement of both the theoretic development and the practical application in aggregation operator for decision making problems.

Acknowledgement

This work is supported by the Funds for Creative Research Groups of China (No. 71221061), the projects of major international cooperation NSFC (No. 71210003), the National Natural Science Foundation of China (No. 70801064, 70801012, 71271217), the Program for New Century Excellent Talents in University of China (No. NCET-12-0541), Hunan Provincial Natural Science Foundation of China (No.11JJ3088), and the Central South University 985 Project "Resource-conserving and Environment-friendly Society" Research Center of China (No. ZNLX1106),the Fundamental Research Funds for the Central Universities (Project No. N110406004).

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