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Hamiltonian Analysis Formulation of Lee-Wick Field Using Riemann-Liouville Fractional Derivatives

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Abstract: In this paper, we generalized the Hamilton formulation for continuous systems with third order derivatives and applied it to Lee-Wick generalized electrodynamics. A combined Riemann–Liouville functional fractional derivative operator was built, and a fractional variational principle was established under this formulation. The fractional Euler-Lagrange equations and fractional Hamilton's equations were created using functional fractional derivatives. We found that the Euler-Lagrange equation and the Hamiltonian equation resulted in the same outcome. We looked at one example in an effort to explain the formalism.

Keywords: Fractional derivatives, Lee-Wick generalized electrodynamics, Hamiltonian formulation, Euler -Lagrange equations.

1 Introduction

Fractional derivatives, or derivatives of arbitrary orders, are expansions of classical calculus that have been effectively applied in a variety of scientific and engineering domains. Many researchers have recently been interested in fractional conformable integrals. Various remarkable inequalities, properties, and applications for fractional conformable integrals appear in the literature [1-3]. Fractional derivatives have played significant roles in physics, engineering, and applied mathematics [4-10]. Hilfer fractional integro-differential equations were proposed due to the presence of nonlocal conditions [11]. The authors used the Monch theory and noncompactness approaches. The presence of an integral solution to the model under consideration is guaranteed by the results of fractional calculus theory, the measure of noncompactness, and the Monch fixed point theorem. Another work on the analytical solutions of a fractional form of the Bogoyavlensky–Konopelchenko equation is described in [12], in which the Atangana–Baleanu-Riemann derivative was utilized to convert the conventional form of the model into a nonlinear fractional PDE with an integer order. The main idea behind these tactics is to insert a new variable into the equation to transform it into a linear equation with an ordinary derivative.
During the past decades, new studies of systems with higher-order fractional derivatives have been discussed [13-16], and the path integral quantization for both conservative and non-conservative systems has been recovered. The measure of non-compactness (MNC), fixed point theorems, and k-set contraction were utilized to solve non-autonomous fractional differential equations with integral impulse condition [17]. An example supports the obtained results. In this study, the treatment for systems with third-order derivatives has been presented; the researchers have generalised the Hamiltonian formulation for continuous systems with third-order derivatives and applied this formulation to Lee-Wick generalised electrodynamics. Kottakkaran et al. presented a novel generalization of the Struve function called the generalized Galué type Struve functional (GTSF), which was defined and used in the kernel with integral operators like Appell's functions or Horn's function. The Fox–Wright function was utilized to express the acquired results. It is designed to provide solutions to certain generic families of fractional kinetic equations related with the Galué type generalization of Struve function [18] as a novel application of the newly built generalized GTSF. Recent symposia proceedings and monographs have underscored the application of fractional calculus in continuum mechanics, physics, electromagnetics and signal processing. Here, we specify some of these applications.
The first, Atangana-Baleanu fractional derivative, was used to deduce formulas for velocity and temperature fields generated directly using the Laplace transform approach. The results of the inverse of equation findings of the Nusselt number have been obtained in tabular form using Zakian's explicit formula. According to the findings, increasing the fraction of nanoparticle volume led in heat transfer enrichment [19]. The use of fractional derivatives in the fractional diffusion process

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for an electro-analog model is advantageous since it allows for simulation of heat dissipation in the circuit board. Furthermore, the established employing current operators of fractional calculus via the governing differential equation were used to simulate the issue. The comparison of both fractional differentiation approaches clearly disclosed the heat transfer of the circuit board and the diffusion process with dissipation [20]. In conjunction with the Laplace transform method, applying fractional calculus to the viscous-diffusion equation within a semi-infinite space has exhibited explicit analytical (fractional) solutions for fluid speed and shear stress anywhere throughout the domain.
After comparing the fractional results for boundary shear-stress and fluid speed with the existing analytical results for the first and second Stokes problems, the fractional methodology has shown to be much simpler and more powerful than existing techniques after validation [21].The functional derivatives method has been succesfully used to solve various types of integer and fractional equations. In addition to being used to calculate theinteraction energy between a stationary point charge and a conducting plate, this method can alsobe developed for examining the underlying effects of fractional order derivatives on energy ineraction physical phenomena in Lee–Wick electrodynamics.
These functions have prompted us to adopt the functional derivatives method in order toreformulate the fractional Lee–Wick electrodynamics equation and derive equations of motion. This work aims to a generalization of the aforementioned work on Hamilton's equation for Lee-Wick Field using functional fractional derivatives.
The remaining of this paper is organized as follows: In Section 2, the definitions of fractional derivatives are discussed briefly. In Section 3 the fractional form of the Euler-Lagrangian equation is presented. In Section 4, the fractional form of the Euler-Lagrangian equation in terms of functional derivative for the Lee- Wick field is constructed. In Section 5 the fractional form of the Euler-Lagrange equation in terms of momentum density of the third order for Lee -Wick field is investigated. After that, in Section 6, the equations of motion for Lee–Wick field in terms of Hamiltonian density in fractional form are obtained. Then, in Section 7, we derived the Hamilton's equation in terms$\left(\boldsymbol{\phi} \text{ and } \boldsymbol{A_i} \; and \; \boldsymbol{A_j}\right)$. One example is described in Section 8. In Section 9, we obtain the fractional Lee- Wick equations using the Euler-Lagrange equations. The work closes with some concluding remarks.

2 Basic definitions

In this part of study, we briefly present some fundamental definitions used in this work. The left and right Riemann-Liouville fractional derivatives are defined as follows:
The left Riemann- Liouville fractional derivative [22, 23].

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha+1} f(\tau) d\tau \, . \tag{1}$$

The right Riemann- Liouville fractional derivative [22, 23].

$$ {}_xD_b^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_a^x (\tau-x)^{n-\alpha+1} f(\tau) d\tau. \tag{2}$$

Here α is the order of the derivative such that $n-1 \le \alpha \le n$ and is not equal to zero. If α is an integer, these derivatives are defined in the usual sense ,i.e.,

$$ f(x) = \left(\frac{d}{dx}\right)^n f(x), \tag{3}$$

$$ {}_aD_t^{\alpha} f(x) = \left(\frac{d}{dx}\right)^n f(t) \qquad \alpha = 1,2,.. \tag{4}$$

3 Fractional Euler-Lagrange equations for Lee-Wick

A covariant form of the action would involve a Lagrangian density L via $S = \int \mathcal{L} \; d^4x = \int \delta\mathcal{L} \; d^3x dt$, Where

$$ \mathcal{L} = \mathcal{L}\begin{bmatrix} \psi_\mu, \; {}_aD_{x_\mu}^{\alpha} \psi_\rho(x,t), {}_{x_\mu}D_b^{\beta} \psi_\rho(x,t), \\ {}_aD_{x_\mu}^{\alpha} \; {}_aD_{x_\sigma}^{\alpha} \; {}_aD_{x_\varepsilon}^{\alpha} \psi_\rho(x,t), {}_{x_\mu}D_b^{\beta} \; {}_{x_\sigma}D_b^{\beta} \; {}_{x_\varepsilon}D_b^{\beta} \psi_\rho(x,t) \end{bmatrix} $$

and with $L = \int \mathcal{L}\ d^4x$ The corresponding covariant Euler-Lagrange equations are:

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \psi_\rho} - {}_aD^{\alpha}_{x_\mu} \frac{\partial \mathcal{L}}{\partial\, {}_aD^{\alpha}_{x_\mu}\psi_\rho(x,t)} - {}_{x_\mu}D^{\beta}_{b} \frac{\partial \mathcal{L}}{\partial\, {}_{x_\mu}D^{\beta}_{b}\psi_\rho(x,t)} \\ - {}_aD^{\alpha}_{x_\mu}\, {}_aD^{\alpha}_{x_\sigma}\, {}_aD^{\alpha}_{x_\varepsilon} \frac{\partial \mathcal{L}}{\partial\, {}_aD^{\alpha}_{x_\mu}\, {}_aD^{\alpha}_{x_\sigma}\, {}_aD^{\alpha}_{x_\varepsilon}\psi_\rho(x,t)} \\ - {}_{x_\mu}D^{\beta}_{b}\, {}_{x_\sigma}D^{\beta}_{b}\, {}_{x_\varepsilon}D^{\beta}_{b} \frac{\partial \mathcal{L}}{\partial\, {}_{x_\mu}D^{\beta}_{b}\, {}_{x_\sigma}D^{\beta}_{b}\, {}_{x_\varepsilon}D^{\beta}_{b}\psi_\rho(x,t)} \end{bmatrix} = 0 \quad (5)$$

where ψ is the field variable, eq. (5) , stands for the Euler Lagrange equation, a math equation which contains both the combined Riemann–Liouville fractional derivative with third order derivatives.,α=β=1, ${}_aD^{\alpha}_{x_\mu} = \partial_\mu$, ${}_{x_\mu}D^{\beta}_{b} = -\partial_\mu$ *and* ${}_aD^{\alpha}_{x_\mu}\, {}_aD^{\alpha}_{x_\sigma}\, {}_aD^{\alpha}_{x_\varepsilon} = \partial_\mu\, \partial_\sigma\, \partial_\varepsilon$, ${}_{x_\mu}D^{\beta}_{b}\, {}_{x_\sigma}D^{\beta}_{b}\, {}_{x_\varepsilon}D^{\beta}_{b} = -\partial_\mu\, \partial_\sigma\, \partial_\varepsilon$.Using, α=β =1, we can rewrite Eq (5), which becomes the following :

$$\frac{\partial L}{\partial \psi_\rho} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \psi_\rho)} - \partial_\mu\, \partial_\sigma\, \partial_\varepsilon \frac{\partial L}{\partial\left(\partial_\mu\, \partial_\sigma\, \partial_\varepsilon \psi_\rho\right)} = 0\ . \quad (6)$$

4 The fractional form of the Euler- Lagrangian equation for the Lee- Wick field in terms of functional derivative for the Lee- Wick field

Using Eq (11),we can write the variation in L using ordinary derivatives:

$$\iint \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \psi_\mu}\delta\psi_\rho + \frac{\partial \mathcal{L}}{\partial\, {}_{x_\mu}D^{\beta}_{b}\psi(x,t)_\rho}\delta\, {}_{x_\mu}D^{\beta}_{b}\psi_\rho + \frac{\partial \mathcal{L}}{\partial\, {}_aD^{\alpha}_{x_\mu}\psi_\rho}\delta\, {}_aD^{\alpha}_{x_\mu}\psi_\rho \\ + \frac{\partial \mathcal{L}}{\partial\, {}_aD^{\alpha}_{x_\mu}\, {}_aD^{\alpha}_{x_\sigma}\, {}_aD^{\alpha}_{x_\varepsilon}\psi_\rho}\delta\, {}_aD^{\alpha}_{x_\mu}\, {}_aD^{\alpha}_{x_\sigma}\, {}_aD^{\alpha}_{x_\varepsilon}\psi_\rho \\ + \frac{\partial \mathcal{L}}{\partial\, {}_{x_\mu}D^{\beta}_{b}\, {}_{x_\sigma}D^{\beta}_{b}\, {}_{x_\varepsilon}D^{\beta}_{b}\psi_\rho}\delta\, {}_{x_\mu}D^{\beta}_{b}\, {}_{x_\sigma}D^{\beta}_{b}\, {}_{x_\varepsilon}D^{\beta}_{b}\psi_\rho \end{bmatrix} d^4x = 0 \quad (7)$$

where$(\mu = 0, i)$,$(\sigma = 0, r)$, $(\varepsilon = 0, f)$, *and* $(\rho = 0, l)$.Now we can take the previous equation's integration over space $d\tau$ and transform it to summation, as follows:

$$\sum_i \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \psi_l} - {}_aD^{\alpha}_{x_i} \frac{\partial \mathcal{L}}{\partial(\, {}_aD^{\alpha}_{x_i}\psi_l)} + \\ - {}_aD^{2\alpha}_{t}\, {}_aD^{\alpha}_{x_f} \frac{\partial \mathcal{L}}{\partial\left(\, {}_aD^{2\alpha}_{t}\, {}_aD^{\alpha}_{x_f}\psi_l\right)} - {}_aD^{2\alpha}_{t}\, {}_aD^{\alpha}_{x_r} \frac{\partial \mathcal{L}}{\partial(\, {}_aD^{2\alpha}_{t}\, {}_aD^{\alpha}_{x_r}\psi_l)} \\ - {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{x_f} \frac{\partial \mathcal{L}}{\partial\left(\, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{x_f}\psi_l\right)} - {}_aD^{\alpha}_{x_i}\, {}_aD^{2\alpha}_{t} \frac{\partial \mathcal{L}}{\partial(\, {}_aD^{\alpha}_{x_i}\, {}_aD^{2\alpha}_{t}\psi_l)} \\ - {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_f} \frac{\partial \mathcal{L}}{\partial\left(\, {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_f}\psi_l\right)} - {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{t} \frac{\partial \mathcal{L}}{\partial(\, {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{t}\psi_l)} \\ - {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{x_f} \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_r \partial_0 \psi_l)} \end{bmatrix} \delta(\psi_l)_i \delta\tau_i$$

$$+\sum_i \left[\begin{array}{c} \frac{\partial \mathcal{L}}{\partial \psi_0} - {}_aD^{\alpha}_{x_i} \frac{\partial \mathcal{L}}{\partial \left({}_aD^{\alpha}_{x_i} \psi_0 \right)} + \\ - {}_aD^{2\alpha}_t \; {}_aD^{\alpha}_{x_f} \frac{\partial \mathcal{L}}{\partial \left({}_aD^{2\alpha}_t \; {}_aD^{\alpha}_{x_f} \psi_0 \right)} - {}_aD^{2\alpha}_t \; {}_aD^{\alpha}_{x_r} \frac{\partial \mathcal{L}}{\partial \left({}_aD^{2\alpha}_t \; {}_aD^{\alpha}_{x_r} \psi_0 \right)} \\ - {}_aD^{\alpha}_t \; {}_aD^{\alpha}_{x_r} \; {}_aD^{\alpha}_{x_f} \frac{\partial \mathcal{L}}{\partial \left({}_aD^{\alpha}_t \; {}_aD^{\alpha}_{x_r} \; {}_aD^{\alpha}_{x_f} \psi_0 \right)} - {}_aD^{\alpha}_{x_i} \; {}_aD^{2\alpha}_t \frac{\partial \mathcal{L}}{\partial \left({}_aD^{\alpha}_{x_i} \; {}_aD^{2\alpha}_t \psi_0 \right)} \\ - {}_aD^{\alpha}_{x_i} \; {}_aD^{\alpha}_t \; {}_aD^{\alpha}_{x_f} \frac{\partial \mathcal{L}}{\partial \left({}_aD^{\alpha}_{x_i} \; {}_aD^{\alpha}_t \; {}_aD^{\alpha}_{x_f} \psi_0 \right)} - {}_aD^{\alpha}_{x_i} \; {}_aD^{\alpha}_{x_r} \; {}_aD^{\alpha}_t \frac{\partial \mathcal{L}}{\partial \left({}_aD^{\alpha}_{x_i} \; {}_aD^{\alpha}_{x_r} \; {}_aD^{\alpha}_t \psi_0 \right)} \\ - {}_aD^{\alpha}_{x_i} \; {}_aD^{\alpha}_{x_r} \; {}_aD^{\alpha}_{x_f} \frac{\partial \mathcal{L}}{\partial (\partial_i \partial_r \partial_0 \psi_0)} \end{array} \right] \delta(\psi_0)_i \delta\tau_i$$

$$+ \sum_i \left[\frac{\partial \mathcal{L}}{\partial \left({}_aD^{\alpha}_t \psi_0 \right)} \right] \delta\left({}_aD^{\alpha}_t \psi_0 \right)_i \delta\tau_i + \sum_i \left[\frac{\partial \mathcal{L}}{\partial \left({}_aD^{3\alpha}_t \psi_0 \right)} \right] \delta\left({}_aD^{3\alpha}_t \psi_0 \right)_i$$

$$+ \sum_i \left[\frac{\partial \mathcal{L}}{\partial \left({}_aD^{\alpha}_t \psi_l \right)} \right] \delta\left({}_aD^{\alpha}_t \psi_l \right)_i \delta\tau_i + \sum_i \left[\frac{\partial \mathcal{L}}{\partial \left({}_aD^{3\alpha}_t \psi_l \right)} \right] \delta\left({}_aD^{3\alpha}_t \psi_l \right)_i = 0 \,. \tag{8}$$

In terms of Lagrangian density, we can express Eq. (8) as follows:

$$\sum [\delta \mathcal{L}]_i \; \delta\tau_i = 0 \tag{9}$$

Here the left-hand side in Eqs. (12) and (13) represents the variation of L (i.e. δL) which is now produced by independent variations in $\delta(\psi_0)_i, \delta\left({}_aD^{\alpha}_t \psi_0 \right)_i, \delta\left({}_aD^{3\alpha}_t \psi_0 \right)_i, and\ \delta(\psi_l)_i, \delta\left({}_aD^{\alpha}_t \psi_l \right)_i, \delta\left({}_aD^{3\alpha}_t \psi_l \right)_i.$

Suppose now that all $\delta(\psi_0)_i, \delta\left({}_aD^{\alpha}_t \psi_0 \right)_i, \delta\left({}_aD^{2\alpha}_t \psi_0 \right)_i, \delta\left({}_aD^{3\alpha}_t \psi_0 \right)_i and\ \delta(\psi_l)_i,\ \delta\left({}_aD^{\alpha}_t \psi_l \right)_i, \delta\left({}_aD^{3\alpha}_t \psi_l \right)_i$ are zeros except for a particular δψj . It is natural to define the functional derivative of the Lagrangian (∂L) with respect to $\delta(\psi_0)_i, \delta\left({}_aD^{3\alpha}_t \psi_0 \right)_i, and\ \ \delta(\psi_l)_i, \delta\left({}_aD^{\alpha}_t \psi_l \right)_i, \delta\left({}_aD^{3\alpha}_t \psi_l \right)_i$ for a point in the j -th cell to the ratio of δL to δψj [27].

$$\frac{\partial L}{\partial \; {}_aD^{\alpha}_t \psi_0} = \lim_{\delta\tau_j \to 0} \frac{\delta L}{\delta\left({}_aD^{\alpha}_t \psi_0 \right)_j \delta_{\tau j}} \quad , \tag{10}$$

$$\frac{\partial L}{\partial \; {}_aD^{3\alpha}_t \psi_0} = \lim_{\delta\tau_j \to 0} \frac{\delta L}{\delta\left({}_aD^{3\alpha}_t \psi_0 \right)_j \delta_{\tau j}} \; , \tag{11}$$

$$\frac{\partial L}{\partial \; {}_aD^{\alpha}_t \psi_l} = \lim_{\delta\tau_j \to 0} \frac{\delta L}{\delta\left({}_aD^{\alpha}_t \psi_l \right)_j \delta_{\tau j}} \quad , \tag{12}$$

$$\frac{\partial L}{\partial \; {}_aD^{3\alpha}_t \psi_l} = \lim_{\delta\tau_j \to 0} \frac{\delta L}{\delta\left({}_aD^{3\alpha}_t \psi_l \right)_j \delta_{\tau j}} \quad . \tag{13}$$

The functional fractional derivative of L with respect to ${}_aD^{\alpha}_t \psi_0,\ {}_aD^{3\alpha}_t \psi_0\ ,\ {}_aD^{\alpha}_t \psi_l,\ {}_aD^{3\alpha}_t \psi_l$ is defined by setting all $the\ \delta\psi_i$ equal to zero except for a particular $\delta\psi_j$.

$$\frac{\partial L}{\partial \ {}_aD_t^{\alpha}\psi_0} = \lim_{\delta\tau_j\to 0}\frac{\delta L}{\delta({}_aD_t^{\alpha}\psi_0)_j\,\delta_{\tau j}} = \frac{\partial \mathcal{L}}{\partial \ {}_aD_t^{\alpha}\psi_0} \tag{14}$$

$$\frac{\partial L}{\partial \ {}_aD_t^{3\alpha}\psi_0} = \lim_{\delta\tau_j\to 0}\frac{\delta L}{\delta({}_aD_t^{3\alpha}\psi_0)_j\,\delta_{\tau j}} = \frac{\partial \mathcal{L}}{\partial \ {}_aD_t^{3\alpha}\psi_0} \tag{15}$$

$$\frac{\partial L}{\partial \ {}_aD_t^{\alpha}\psi_l} = \lim_{\delta\tau_j\to 0}\frac{\delta L}{\delta({}_aD_t^{\alpha}\psi_l)_j\,\delta_{\tau j}} = \frac{\partial \mathcal{L}}{\partial \ {}_aD_t^{\alpha}\psi_l} \tag{16}$$

$$\frac{\partial L}{\partial \ {}_aD_t^{3\alpha}\psi_l} = \lim_{\delta\tau_j\to 0}\frac{\delta L}{\delta({}_aD_t^{3\alpha}\psi_l)_j\,\delta_{\tau j}} = \frac{\partial \mathcal{L}}{\partial \ {}_aD_t^{3\alpha}\psi_l} \tag{17}$$

and

$$\frac{\partial L}{\partial \psi} = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \psi_0} + \frac{\partial \mathcal{L}}{\partial \psi_l} - {}_aD_{x_i}^{\alpha}\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\psi_0)} - {}_aD_{x_i}^{\alpha}\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\psi_l)} + \\ - {}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial \mathcal{L}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_0\right)} - {}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial \mathcal{L}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_l\right)} \\ - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\psi_0)} - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\psi_l)} \\ - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial \mathcal{L}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_0\right)} - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial \mathcal{L}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_l\right)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\psi_0)} - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\psi_l)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial \mathcal{L}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_0\right)} - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial \mathcal{L}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_l\right)} - \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\psi_0)} - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\psi_l)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial \mathcal{L}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_0\right)} - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial \mathcal{L}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_l\right)} \end{bmatrix} \tag{18}$$

We can rewrite the Eq. (5) Euler-Lagrange equation in terms of the Lagrangian L in fractional form using Eqs. ((14),(15),(16),(17), and(18).

$$\frac{\partial L}{\partial \psi} - {}_aD_t^{\alpha}\left(\frac{\partial L}{\partial \ {}_aD_t^{\alpha}\psi}\right) - {}_aD_t^{3\alpha}\left(\frac{\partial L}{\partial \ {}_aD_t^{3\alpha}\psi}\right) = 0 \tag{19}$$

For α →, Eq. (81) reduces to the standard Euler-Lagrange equation for classical fields, which is as follows:

$$\frac{\partial\ L}{\partial\ \psi}-\frac{\partial}{\partial t}\left(\frac{\partial\ L}{\partial\ \dot{\psi}}\right)-\frac{\partial^3}{\partial t^3}\left(\frac{\partial\ L}{\partial\ \dddot{\psi}}\right)=0.$$

We can write the variation of Lagrangian in terms of functional derivatives and variations of ${}_aD_t^{\alpha}\psi$, ${}_aD_t^{3\alpha}\psi_\rho$, and with the help of Eqs. (12) and (16) as:

$$\delta L=\int\left(\frac{\partial\ L}{\partial\ \psi_\rho}\delta\psi_\rho+\frac{\partial\ L}{\partial\ {}_aD_t^{\alpha}\psi_\rho}\delta\ {}_aD_t^{\alpha}\psi_\rho+\frac{\partial\ L}{\partial\ {}_aD_t^{3\alpha}\psi_\rho}\delta\psi_\rho\right)\mathrm{d}^3\mathrm{r} \tag{20}$$

In many cases, we take ${}_aD_{x_\mu}^{\alpha}\ {}_aD_{x_\sigma}^{\alpha}\ {}_aD_{x_\varepsilon}^{\alpha}\psi_\rho(x,t)={}_{x_\mu}D_b^{\beta}\ {}_{x_\sigma}D_b^{\beta}\ {}_{x_\varepsilon}D_b^{\beta}\psi_\rho(x,t)=0$ because we define (in the Lagrangian density and the Hamiltonian density) the time derivative in the right side as ${}_aD_t^{2\alpha}\psi,\ {}_tD_a^{2\alpha}\psi$, so that $\pi_{\alpha_2}=\frac{\partial\mathcal{L}}{\partial\left({}_aD_t^{2\alpha}\psi\right)}=0$ and $\pi_{\beta_2}=\frac{\partial\mathcal{L}}{\partial\left({}_tD_a^{2\alpha}\psi\right)}=0$. Therefore take π_α =0, and π_β=0;

5 Fractional Form of the Euler-Lagrange equation in terms of momentum density of the third order for Lee -Wick field

The right side of the fractional form of momentum is written as follows [25]:

$$P_j^{a}=\frac{\delta L}{\delta\ {}_aD_t^{\alpha}\psi_\rho} \tag{21}$$

$$P_j^{a}=\frac{\delta L}{\delta\ {}_aD_t^{3\alpha}\psi_\rho} \tag{22}$$

Using Eqs. ((19),(21),and(22)), we obtain:

$$\left(P_j^{a}\right)_1=\lim_{\delta\tau_j\to 0}\frac{\delta\mathcal{L}}{\delta\ {}_aD_t^{\alpha}\psi_\rho}\delta\tau_j+\lim_{\delta\tau_j\to 0}\ {}_aD_t^{2\alpha}\left[\frac{\delta\mathcal{L}}{\delta\ {}_aD_t^{3\alpha}\psi_\rho}\right]\delta\tau_j$$

$$=\frac{\partial\ L}{\partial\ {}_aD_t^{\alpha}\psi_\rho}\delta\tau_j+\ {}_aD_t^{2\alpha}\left[\frac{\partial\ \mathcal{L}}{\partial\ {}_aD_t^{3\alpha}\psi_\rho}\right]\delta\tau_j \tag{23}$$

$$\left(P_j^{a}\right)_3=\lim_{\delta\tau_j\to 0}\frac{\delta\mathcal{L}}{\delta\ {}_aD_t^{\alpha}\psi_\rho}\delta\tau_j=\frac{\partial\ L}{\partial\ {}_aD_t^{\alpha}\psi_\rho}\delta\tau_j \tag{24}$$

We may get the right side form of momentum density from Eqs. ((23),(24)) as follows:

$$(\pi)_1=\frac{\delta\mathcal{L}}{\delta\ {}_aD_t^{\alpha}\psi_\rho}+{}_aD_t^{2\alpha}\left[\frac{\delta\mathcal{L}}{\delta\ {}_aD_t^{3\alpha}\psi_\rho}\right]=\frac{\partial\ L}{\partial\ {}_aD_t^{\alpha}\psi_\rho}+\ {}_aD_t^{2\alpha}\left[\frac{\partial\ \mathcal{L}}{\partial\ {}_aD_t^{3\alpha}\psi_\rho}\right] \tag{25}$$

$$(\pi)_3=\frac{\delta\mathcal{L}}{\delta\ {}_aD_t^{3\alpha}\psi_\rho}=\frac{\partial\ L}{\partial\ {}_aD_t^{3\alpha}\psi_\rho} \tag{26}$$

Taking the left fractional derivative of Eqs. ((25),(26)), we get:

$${}_aD_t^{\alpha}(\pi_1)=\ {}_aD_t^{\alpha}\left[\frac{\partial\ L}{\partial\ {}_aD_t^{\alpha}\psi_\rho}\right]+\ {}_aD_t^{3\alpha}\left[\frac{\partial\ \mathcal{L}}{\partial\ {}_aD_t^{3\alpha}\psi_\rho}\right] \tag{27}$$

$$ {}_aD_t^{\alpha}(\pi_3) = {}_aD_t^{\alpha}\left[\frac{\partial L}{\partial\ {}_aD_t^{3\alpha}(\psi_\rho)}\right] \tag{28}$$

Now, substituting Eqs. ((27)and(28)) into Eq. (19), we get:

$$\frac{\partial L}{\partial \psi} = {}_aD_t^{\alpha}\pi_1 \tag{29}$$

The above equation represents the fractional form of the Euler- Lagrange equation in terms of momentum density and the functional derivative of the Lagrangian.

6 Equations of motion for Lee–Wick field in terms of Hamiltonian density in fractional form

We start with the general definition of the Hamiltonian density $\mathcal{H}$ in fractional form as [26]:

$$\mathcal{H} = \pi_{\alpha_1}\ {}_aD_t^{\alpha}\psi_\rho + \pi_{\alpha_3}\ {}_aD_t^{3\alpha}\psi_\rho - \mathcal{L} \tag{30}$$

Hamiltonian H can also be written in terms of Hamiltonian density $\mathcal{H}$ in the following way:

$$H = \sum_i \mathcal{H}_i\ \delta\tau_i \tag{31}$$

Substituting Eq. (27)into Eq. (28), one gets:

$$H = \sum_i (\pi_{\alpha_1})_i \left({}_aD_t^{\alpha}\psi_\rho\right)_i + (\pi_{\alpha_3})_i\left({}_aD_t^{3\alpha}\psi_\rho\right)_i \delta\tau_i - \sum_i \mathcal{L}_i\ \delta\tau_i \tag{32}$$

In continuous form, Eq. (26) can be written as follows:

$$H = \int\left[\pi_{\alpha_1}\ {}_aD_t^{\alpha}\psi_\rho + \pi_{\alpha_3}\ {}_aD_t^{3\alpha}\psi_\rho\right] d^3r - \int \mathcal{L}\, d^3r \tag{33}$$

Taking the variation of H, we get:

$$\delta H = \int \delta\left[\pi_{\alpha_1}\ {}_aD_t^{\alpha}\psi_\rho + \pi_{\alpha_3}\ {}_aD_t^{3\alpha}\psi_\rho\right]\ \mathrm{d}^3\mathrm{r} - \delta L \tag{34}$$

The above equation can be written as:

$$\delta L = \int\left[\left({}_aD_t^{\alpha}\pi_{\alpha_1} + {}_aD_t^{3\alpha}\pi_{\alpha_3}\right)\delta\psi_\rho + \pi_1\delta\ {}_aD_t^{\alpha}\psi_\rho + \pi_3\delta\ {}_aD_t^{3\alpha}\psi_\rho\right]\ \mathrm{d}^3\mathrm{r} \tag{35}$$

Substituting Eq. (51)into Eq. (50), one gets:

$$\int\left[\left({}_aD_t^{\alpha}\pi_{\alpha_1} + {}_aD_t^{3\alpha}\pi_{\alpha_3}\right)\delta\psi_\rho + {}_aD_t^{\alpha}\psi_\rho\delta\pi_{\alpha_1} + {}_aD_t^{3\alpha}\psi_\rho\delta\pi_{\alpha_3}\right]\ \mathrm{d}^3\mathrm{r} \tag{36}$$

By analogy with the variation in L (i.e: Eq. (20)), we can write the variation of Hamiltonian produced by variations of independent variables in terms of functional fractional derivative as follows in cases 1 and 2.

Case 1: **All variables are independent** $(\boldsymbol{\psi_\rho}, \boldsymbol{\pi_{\alpha_1}}, \boldsymbol{\pi_{\alpha_3}})$.
In this case, all variables $(\boldsymbol{\psi_\rho}, \boldsymbol{\pi_{\alpha_1}}, \boldsymbol{\pi_{\alpha_3}})$ can be shown to be independent, as follows:

$$\delta H = \int\left[\frac{\partial H}{\partial \psi_\rho}\delta\psi_\rho + \frac{\partial H}{\partial \pi_{\alpha_1}}\delta\pi_{\alpha_1} + \frac{\partial H}{\partial \pi_{\alpha_3}}\delta\pi_{\alpha_3}\right]\ \mathrm{d}^3\mathrm{r} \tag{37}$$

Comparing Eq. (37)with Eq. (36),we get Hamilton's equations of motion in a form of functional fractional derivatives (see appendix A).

Case 2: **π_1 depends on (ψ_ρ), and π_3 depends on (${}_aD_t^\alpha\, {}_aD_t^\alpha \psi_\rho$).**
So, if we only consider the variation for the independent variables ψ_ρ , ${}_aD_t^\alpha \psi_\rho$ and ${}_aD_t^{2\alpha}\psi_\rho$, we get:

$$\delta H = \int \left[\frac{\partial H}{\partial \psi_\rho} \delta\psi_\rho + \frac{\partial H}{\partial \left({}_aD_t^{2\alpha}\psi_\rho \right)} \delta\left({}_aD_t^{2\alpha}\psi_\rho \right) \right] d^3r \tag{38}$$

Details of the equations of motion from Eq. (39),are given in (Appendix B).

7 The Hamilton's equation in terms$\left(\boldsymbol{\phi}\ \textbf{and}\ \boldsymbol{A_i}\ \boldsymbol{and}\ \boldsymbol{A_j}\right)$

In this part, and utilizing the generic formulae in (case 1 and case 2), we construct further equations of motion from the variables in the other fields ($\boldsymbol{\phi}$ and $\boldsymbol{A_i}$and $\boldsymbol{A_j}$), where the following two situations are considered: **(see Appendix C)**.

8 Examples

In this section, we study two examples as applications on the formalism presented above.
The most general form of Lagrangian density for a four-vector field is given by the so-called Lee- Wick Lagrangian density [27]:

$$\mathcal{L}_{LW} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4m^2} F_{\mu\nu}\, \partial_\alpha \partial^\alpha F^{\mu\nu} - \frac{\partial_\mu A^\mu}{2\xi} - J_\mu A^\mu \tag{39}$$

where J_μ is an external source, m is a parameter that has mass dimension, and ξ is a gauge fixing parameter , it is important to mention that there are other covariant gauge conditions for this theory[26, 27].As discussed in many works [28-32],the Lagrangian (39) exhibits gauge invariance and two distinct poles, in momenta space, for the corresponding propagator; a massless one and a massive one
To rewrite the Lee- Wick Lagrangian density in Riemann – Liouville fractional form we use these relations:

$$\begin{bmatrix} F_{\mu\nu} = {}_aD_{x_\mu}^\alpha A_\nu - {}_aD_{x_\nu}^\alpha A_\mu \\ F^{\mu\nu} = {}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x^\nu}^\alpha A^\mu \end{bmatrix} \tag{40}$$

$$\begin{bmatrix} \partial_\alpha = {}_aD_{x_\mu}^\alpha = \left({}_aD_t^\alpha ,\ {}_aD_{x_i}^\alpha \right) \\ \partial^\alpha = {}_aD_{x^\mu}^\alpha = \left({}_aD_t^\alpha , -{}_aD_{x^i}^\alpha \right) \end{bmatrix} \tag{41}$$

$$F_{\mu\nu} F^{\mu\nu} = 2\left[{}_aD_{x_\mu}^\alpha A_\nu\ {}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x_\mu}^\alpha A_\nu\ {}_aD_{x^\nu}^\alpha A^\mu \right] \tag{42}$$

$$F_{\mu\nu}\, \partial_\alpha \partial^\alpha F^{\mu\nu} = 2\left[{}_aD_{x_\mu}^\alpha A_\nu\ {}_aD_{x_\alpha}^\alpha\ {}_aD_{x^\alpha}^\alpha\ {}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x_\nu}^\alpha A_\mu\ {}_aD_{x_\alpha}^\alpha\ {}_aD_{x^\alpha}^\alpha\ {}_aD_{x^\nu}^\alpha A^\mu \right] \tag{43}$$

$$\begin{bmatrix} A^\alpha = \left(\phi, \vec{A}\right) \\ A_\alpha = \left(\phi, -\vec{A}\right) \end{bmatrix} \tag{44}$$

Where $\mu = 0, i = 1,2,3\ and\ \nu = 0, j = 1,2,3\ and\ \alpha = 0, k = 1,2,3$.
Expand μ, ν , α in termes of(0, i), 0, j and (0, k) respectively, and use definition of left Riemann – Liouville fractional derivative, the fractional electromagnetic lagrangian density formulation takes the form :

$$\mathcal{L} =$$

$$-\frac{2}{4}\begin{bmatrix} -\left({}_aD_t^\alpha A_j\right)^2 + {}_aD_t^\alpha A_j\ {}_aD_{x_j}^\alpha \phi \\ -\left({}_aD_{x_i}^\alpha \phi\right)^2 + {}_aD_{x_i}^\alpha \phi\ {}_aD_t^\alpha A_i \\ +\left({}_aD_{x_i}^\alpha A_j\right)^2 - {}_aD_{x_i}^\alpha A_j\ {}_aD_{x_i}^\alpha A_i \end{bmatrix} - \frac{2}{4m^2}\begin{bmatrix} -{}_aD_t^\alpha A_j\ {}_aD_t^{3\alpha} A_j - {}_aD_t^\alpha A_j\ {}_aD_{x_k}^{2\alpha}\ {}_aD_t^\alpha A_j \\ + {}_aD_t^\alpha A_j\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^\alpha \phi + {}_aD_t^\alpha A_j\ {}_aD_{x_k}^{2\alpha}\ {}_aD_{x_i}^\alpha \phi - \\ {}_aD_{x_i}^\alpha \phi\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^\alpha \phi - {}_aD_{x_i}^\alpha \phi\ {}_aD_{x_k}^{2\alpha}\ {}_aD_{x_i}^\alpha \phi + \\ {}_aD_{x_i}^\alpha \phi\ {}_aD_t^{3\alpha} A_i + {}_aD_{x_i}^\alpha \phi\ {}_aD_{x_k}^{2\alpha}\ {}_aD_t^\alpha A_i + \\ {}_aD_{x_i}^\alpha A_j\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^\alpha A^j - {}_aD_{x^i}^\alpha A^j\ {}_aD_{x_k}^{2\alpha}\ {}_aD_{x_i}^\alpha A_j - \\ {}_aD_{x_i}^\alpha A_j\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^\alpha A_i + {}_aD_{x_i}^\alpha A^j\ {}_aD_{x_k}^{2\alpha}\ {}_aD_{x_i}^\alpha A_i \end{bmatrix}$$

$$-\frac{{}_aD_t^\alpha \phi}{2\xi} + \frac{{}_aD_{x_i}^\alpha A_i}{2\xi} + J_0\phi - J_iA_i \qquad (45)$$

9 Fractional forms of Euler-Lagrange equations of Lee-Wick density

Let us begin with the definition of fractional Lee- Wick Lagrangian density and then utilize the generalization formula of Euler – Lagrange equation (5) to derive the equations of motion from Lee- Wick Lagrangian density. Take the first field variable ϕ, then:

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \phi} - {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial\ {}_aD_t^\alpha \phi} - {}_aD_{x_i}^\alpha \frac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^\alpha \phi} + \\ {}_aD_t^{3\alpha} \frac{\partial \mathcal{L}}{\partial\ {}_aD_t^{3\alpha} \phi} - {}_aD_t^{2\alpha}\ {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_f}^\alpha \phi} \\ -{}_aD_t^{2\alpha}\ {}_aD_{x_r}^\alpha \frac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_r}^\alpha \phi} - {}_aD_t^\alpha\ {}_aD_{x_r}^\alpha\ {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\ {}_aD_t^\alpha\ {}_aD_{x_r}^\alpha\ {}_aD_{x_f}^\alpha \phi} \\ -{}_aD_t^{2\alpha}\ {}_aD_{x_i}^\alpha \frac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^\alpha \phi} - {}_aD_{x_i}^\alpha\ {}_aD_t^\alpha\ {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^\alpha\ {}_aD_t^\alpha\ {}_aD_{x_f}^\alpha \phi} \\ -{}_aD_{x_i}^\alpha\ {}_aD_{x_r}^\alpha\ {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^\alpha\ {}_aD_{x_r}^\alpha\ {}_aD_t^\alpha \phi} - {}_aD_{x_i}^\alpha\ {}_aD_{x_r}^\alpha\ {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^\alpha\ {}_aD_{x_r}^\alpha\ {}_aD_{x_f}^\alpha \phi} \end{bmatrix} = 0 \qquad (46)$$

Calculating these derivatives with respect to ϕ, we get:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial\ {}_aD_t^\alpha \phi} = -\frac{1}{2\xi} \\ \frac{\partial \mathcal{L}}{\partial\ {}_aD_t^{3\alpha} \phi} = 0 \end{cases} \qquad (47)$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^\alpha \phi} = -\left(-{}_aD_{x_i}^\alpha \phi + {}_aD_t^\alpha A_i\right) \\ -\frac{2}{4m^2}\begin{pmatrix} -{}_aD_t^{2\alpha}\ {}_aD_{A_j}^\alpha \phi - {}_aD_{x_k}^{2\alpha}\ {}_aD_{x^i}^\alpha \phi \\ + {}_aD_t^{3\alpha} A_i + {}_aD_{x_k}^{2\alpha}\ {}_aD_t^\alpha A_i \end{pmatrix} \end{cases} \qquad (48)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{xf}^{\alpha}\phi} = -\dfrac{2}{4m^2}\left({}_aD_t^{\alpha}A_f - {}_aD_{x_f}^{\alpha}\phi\right) \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_r}^{\alpha}\phi} = -\dfrac{2}{4m^2}\left({}_aD_t^{\alpha}A_r - {}_aD_{x_r}^{\alpha}\phi\right) \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^{\alpha}\phi} = -\dfrac{2}{4m^2}\left({}_aD_t^{\alpha}A_i - {}_aD_{x_i}^{\alpha}\phi\right) \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha}\phi} = -\dfrac{2}{4m^2}\left({}_aD_t^{\alpha}A_f - {}_aD_{x_i}^{\alpha}\phi\right) \end{cases} \tag{49}$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha}\phi} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha}\ {}_aD_t^{\alpha}\ {}_aD_{x_f}^{\alpha}\phi} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_t^{\alpha}\phi} = 0 \end{cases} \tag{50}$$

Substituting equations ((47), (48), (49), and (50)) into equation (46), we get:

$$\begin{bmatrix} J_0 = {}_aD_t^{\alpha}\left(\dfrac{1}{2\xi}\right) - {}_aD_{x_i}^{\alpha}\left(-\ {}_aD_{x_i}^{\alpha}\phi + {}_aD_t^{\alpha}A_i\right) \\ +\dfrac{2}{4m^2}\ {}_aD_{x_i}^{\alpha}\left(-\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^{\alpha}\phi - {}_aD_{x_k}^{2\alpha}\ {}_aD_{x_i}^{\alpha}\phi + {}_aD_t^{3\alpha}A_i + {}_aD_{x_k}^{2\alpha}\ {}_aD_t^{\alpha}A_i\right) \\ +\dfrac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_f}^{\alpha}\left({}_aD_t^{\alpha}A_f - {}_aD_{x_f}^{\alpha}\phi\right) + \dfrac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_r}^{\alpha}\left({}_aD_t^{\alpha}A_r - {}_aD_{x_r}^{\alpha}\phi\right) \\ +\dfrac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^{\alpha}\left({}_aD_t^{\alpha}A_i - {}_aD_{x_i}^{\alpha}\phi\right) + \dfrac{2}{4m^2}\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha}\left({}_aD_t^{\alpha}A_i - {}_aD_{x_f}^{\alpha}\phi\right) \end{bmatrix} \tag{51}$$

This represents the first non- homogeneous equation in fractional form.
Now use the general formula (5)to obtain other equations of motion from the other fields' variables A_i and A_j .

$$0 = \begin{bmatrix} \dfrac{\partial \mathcal{L}}{\partial A_i} - {}_aD_t^{\alpha}\dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{\alpha}A_i} - {}_aD_{x_i}^{\alpha}\dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha}A_i} \\ -\ {}_aD_t^{3\alpha}\dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{3\alpha}A_i} - {}_aD_t^{2\alpha}\ {}_aD_{x_f}^{\alpha}\dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_f}^{\alpha}A_i} \\ -\ {}_aD_t^{2\alpha}\ {}_aD_{x_r}^{\alpha}\dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_r}^{\alpha}A_i} - {}_aD_t^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha}\dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha}A_i} \\ -\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^{\alpha}\dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^{\alpha}A_i} - {}_aD_{x_i}^{\alpha}\ {}_aD_t^{\alpha}\ {}_aD_{x_f}^{\alpha}\dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha}\ {}_aD_t^{\alpha}\ {}_aD_{x_f}^{\alpha}A_i} \\ -\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_t^{\alpha}\dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_t^{\alpha}A_i} - {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha}\dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha}A_i} \end{bmatrix} \tag{52}$$

Calculating these derivatives yields:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial A_i} = -J_i \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{\alpha} A_i} = -\dfrac{2}{4}\left({}_aD_{x_i}^{\alpha} \phi \right) \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha} A_i} = -\dfrac{2}{4}\left({}_aD_{x_j}^{\alpha} A_j \right) \end{cases} \quad (53)$$

$$\left\{ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{3\alpha} A_i} = 0 \right. \quad (54)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha} A_i} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha}\ {}_aD_t^{\alpha}\ {}_aD_{x_f}^{\alpha} A_i} = 0 \end{cases} \quad (55)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_f}^{\alpha} A_i} = -\dfrac{2}{4m^2}\left({}_aD_{x_i}^{\alpha} A_f \right) \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_r}^{\alpha} A_i} = -\dfrac{2}{4m^2}\left({}_aD_{x_i}^{\alpha} A_r \right) \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^{\alpha} A_i} = -\dfrac{2}{4m^2}\left({}_aD_{x_i}^{\alpha} A_i \right) \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_t^{\alpha} A_i} = -\dfrac{2}{4m^2}\left({}_aD_{x_k}^{\alpha} \phi \right) \\ \dfrac{\partial \mathcal{L}}{\partial\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha} A_i} = -\dfrac{2}{4m^2}\left({}_aD_{x_k}^{\alpha} A_i \right) \end{cases} \quad (56)$$

Substituting equations ((53), (54), (55),and (56)) in equation (52); we get:

$$\left[\begin{array}{c} -J_i = \dfrac{2}{4}\ {}_aD_t^{\alpha}\left({}_aD_{x_i}^{\alpha} \phi \right) + \dfrac{2}{4}\ {}_aD_{x_i}^{\alpha}\left({}_aD_{x_j}^{\alpha} A_j \right) + \dfrac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_f}^{\alpha}\left({}_aD_{x_i}^{\alpha} A_j \right) \\ + \dfrac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_f}^{\alpha}\left({}_aD_{x_i}^{\alpha} A_r \right) + \dfrac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_r}^{\alpha}\left({}_aD_{x_i}^{\alpha} A_r \right) \\ + \dfrac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^{\alpha}\left({}_aD_{x_i}^{\alpha} A_i \right) + \dfrac{2}{4m^2}\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_t^{\alpha}\left({}_aD_{x_k}^{\alpha} \phi \right) \\ + \dfrac{2}{4m^2}\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha}\left({}_aD_{x_k}^{\alpha} A_i \right) \end{array} \right] \quad (57)$$

Take the field variable A_j, then:

$$0=\left[\begin{array}{c}\frac{\partial\mathcal{L}}{\partial A_j}-{}_aD_t^\alpha\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^\alpha A_j}-{}_aD_{x_i}^\alpha\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha A_j}\\ -{}_aD_t^{3\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{3\alpha}A_j}-{}_aD_t^{2\alpha}\,{}_aD_{x_f}^\alpha\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^\alpha A_j}\\ -{}_aD_t^{2\alpha}\,{}_aD_{x_r}^\alpha\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^\alpha A_j}-{}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha A_j}\\ -{}_aD_t^{2\alpha}\,{}_aD_{x_i}^\alpha\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^\alpha A_j}-{}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha A_j}\\ -{}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha A_j}-{}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha A_j}\end{array}\right] \tag{58}$$

Calculating these derivatives yields:

$$\left\{\frac{\partial\mathcal{L}}{\partial A_j}=0\right. \tag{59}$$

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^\alpha A_j}=-\frac{2}{4}\left({}_aD_{x_j}^\alpha\phi+2\,{}_aD_t^\alpha A_j\right)\\ -\frac{2}{4m^2}\begin{pmatrix}{}_aD_t^{3\alpha}A_j-{}_aD_{x_k}^{2\alpha}\,{}_aD_t^\alpha A_j\\ {}_aD_t^{2\alpha}\,{}_aD_{x_j}^\alpha\phi+{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_j}^\alpha\phi\end{pmatrix}\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha A_j}=-\frac{2}{4}\left(2\,{}_aD_{x_i}^\alpha A_j-{}_aD_{x_j}^\alpha A_i\right)\\ -\frac{2}{4m^2}\begin{pmatrix}{}_aD_t^{2\alpha}\,{}_aD_{x_i}^\alpha A_j-{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_i}^\alpha A_j\\ -{}_aD_t^{2\alpha}\,{}_aD_{x_j}^\alpha\phi+{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_j}^\alpha A_i\end{pmatrix}\end{cases} \tag{60}$$

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{3\alpha}A_j}=0\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha A_j}=0\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha A_j}=0\end{cases} \tag{61}$$

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^\alpha A_j}=-\frac{2}{4m^2}\left({}_aD_{x_f}^\alpha A_j\right)\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^\alpha A_j}=-\frac{2}{4m^2}\left({}_aD_{x_r}^\alpha A_j\right)\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^\alpha A_j}=-\frac{2}{4m^2}\left({}_aD_{x_i}^\alpha A_j\right)\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha A_j}=\frac{2}{4m^2}\left({}_aD_{x_k}^\alpha A_i\right)\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha A_j}=\frac{2}{4m^2}\left({}_aD_{x_f}^\alpha A_j\right)\end{cases} \tag{62}$$

Substituting equations ((59), (60), (61) and (62)) in equation (58) we get:

$$\left[\begin{array}{c} 0=\frac{2}{4}\,{}_aD_t^{\alpha}\left({}_aD_{x_j}^{\alpha}\phi+2\,{}_aD_t^{\alpha}A_j\right) \\ +\frac{2}{4m^2}\,{}_aD_t^{\alpha}\left({}_aD_t^{3\alpha}A_j-{}_aD_{x_k}^{2\alpha}\,{}_aD_t^{\alpha}A_j+{}_aD_t^{2\alpha}\,{}_aD_{x_j}^{\alpha}\phi+{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_j}^{\alpha}\phi\right) \\ +\frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left(2\,{}_aD_{x_i}^{\alpha}A_j-{}_aD_{x_j}^{\alpha}A_i\right)+ \\ \frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\left({}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_j-{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_j-{}_aD_t^{2\alpha}\,{}_aD_{x_j}^{\alpha}\phi+{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_j}^{\alpha}A_i\right) \\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\left({}_aD_{x_f}^{\alpha}A_j\right)+\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\left({}_aD_{x_r}^{\alpha}A_j\right) \\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_j\right)-\frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\left({}_aD_{x_k}^{\alpha}A_i\right)- \\ \frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\left({}_aD_{x_f}^{\alpha}A_j\right) \end{array}\right] \tag{60}$$

This represents the first homogeneous equation in fractional form. If α goes to 1, Eq. (58) go to the standard equations. The conjugate momenta are defined as[23]:

$$\begin{cases} \pi_1^1=-\frac{1}{2\xi} \\ \pi_1^2=-\frac{2}{4}\left({}_aD_{x_i}^{\alpha}\phi\right) \\ \pi_1^3=-\frac{2}{4}\left({}_aD_{x_j}^{\alpha}\phi+2\,{}_aD_t^{\alpha}A_j\right) \\ -\frac{2}{4m^2}\begin{pmatrix} {}_aD_t^{3\alpha}A_j-{}_aD_{x_k}^{2\alpha}\,{}_aD_t^{\alpha}A_j \\ {}_aD_t^{2\alpha}\,{}_aD_{x_j}^{\alpha}\phi+{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_j}^{\alpha}\phi \end{pmatrix} \\ \pi_3^1=0 \\ \pi_3^2=0 \\ \pi_3^3=0 \end{cases} \tag{61}$$

Then, using Eq. (27), the Hamiltonian density can be written as:

$$\begin{aligned} \mathcal{H}=\frac{1}{4}&\begin{bmatrix} -\left({}_aD_t^{\alpha}A_j\right)^2-\left({}_aD_{x_i}^{\alpha}\phi\right)^2 \\ +\left({}_aD_{x_i}^{\alpha}A_j\right)^2-{}_aD_{x_i}^{\alpha}A_j\,{}_aD_{x_i}^{\alpha}A_i \end{bmatrix}+ \\ &\frac{2}{4m^2}\begin{bmatrix} {}_aD_{x_i}^{\alpha}\phi\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\phi-{}_aD_{x_i}^{\alpha}\phi\,{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_i}^{\alpha}\phi+ \\ {}_aD_{x_i}^{\alpha}\phi\,{}_aD_t^{3\alpha}A_i+{}_aD_{x_i}^{\alpha}\phi\,{}_aD_{x_k}^{2\alpha}\,{}_aD_t^{\alpha}A_i+ \\ {}_aD_{x_i}^{\alpha}A_j\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}A^j-{}_aD_{x^i}^{\alpha}A^j\,{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_j- \\ {}_aD_{x_i}^{\alpha}A_j\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_i+{}_aD_{x_i}^{\alpha}A^j\,{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_i \end{bmatrix} \\ &+\frac{{}_aD_t^{\alpha}\phi}{2\xi}-\frac{{}_aD_{x_i}^{\alpha}A_i}{2\xi}-J_0\phi+J_iA_i \end{aligned} \tag{62}$$

Using Hamiltonian equation $(\boldsymbol{D_1})$ in case (1) , by taking the derivative with respect to ϕ, we get:

$$\left[\begin{array}{c} J_0 = -\left(-\,{}_aD^{\alpha}_{x_i}\phi + {}_aD^{\alpha}_t A_i\right) \\ +\frac{2}{4m^2}\,{}_aD^{\alpha}_{x_i}\left(-\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_i}\phi - {}_aD^{2\alpha}_{x_k}\,{}_aD^{\alpha}_{x_i}\phi + {}_aD^{3\alpha}_t A_i + {}_aD^{2\alpha}_{x_k}\,{}_aD^{\alpha}_t A_i\right) \\ +\frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_f}\left({}_aD^{\alpha}_t A_f - {}_aD^{\alpha}_{x_f}\phi\right) + \frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_r}\left({}_aD^{\alpha}_t A_r - {}_aD^{\alpha}_{x_r}\phi\right) \\ +\frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_i}\left({}_aD^{\alpha}_t A_i - {}_aD^{\alpha}_{x_i}\phi\right) + \frac{2}{4m^2}\,{}_aD^{\alpha}_{x_i}\,{}_aD^{\alpha}_{x_r}\,{}_aD^{\alpha}_{x_f}\left({}_aD^{\alpha}_t A_i - {}_aD^{\alpha}_{x_f}\phi\right) \end{array}\right] \quad (63)$$

Eq. (63) is exactly the same as the equation that has been derived by (Eq. (49)) in fractional form.
Using Hamiltonian equation ($\boldsymbol{D_2}$), in case1 by taking the derivative with respect to A_i, we get:

$$\left[\begin{array}{c} -J_i = \frac{2}{4}\,{}_aD^{\alpha}_t\left({}_aD^{\alpha}_{x_i}\phi\right) + \frac{2}{4}\,{}_aD^{\alpha}_{x_i}\left({}_aD^{\alpha}_{x_j}A_j\right) + \frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_f}\left({}_aD^{\alpha}_{x_i}A_j\right) \\ +\frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_f}\left({}_aD^{\alpha}_{x_i}A_r\right) + \frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_r}\left({}_aD^{\alpha}_{x_i}A_r\right) \\ +\frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_i}\left({}_aD^{\alpha}_{x_i}A_i\right) + \frac{2}{4m^2}\,{}_aD^{\alpha}_{x_i}\,{}_aD^{\alpha}_{x_r}\,{}_aD^{\alpha}_t\left({}_aD^{\alpha}_{x_k}\phi\right) \\ +\frac{2}{4m^2}\,{}_aD^{\alpha}_{x_i}\,{}_aD^{\alpha}_{x_r}\,{}_aD^{\alpha}_{x_f}\left({}_aD^{\alpha}_{x_k}A_i\right) \end{array}\right] \quad (64)$$

And using Eq.($\boldsymbol{D_3}$),with respect to A_j, we get:

$$\left[\begin{array}{c} 0 = \frac{2}{4}\,{}_aD^{\alpha}_t\left({}_aD^{\alpha}_{x_j}\phi + 2\,{}_aD^{\alpha}_t A_j\right) \\ +\frac{2}{4m^2}\,{}_aD^{\alpha}_t\left({}_aD^{3\alpha}_t A_j - {}_aD^{2\alpha}_{x_k}\,{}_aD^{\alpha}_t A_j + {}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_j}\phi + {}_aD^{2\alpha}_{x_k}\,{}_aD^{\alpha}_{x_j}\phi\right) \\ +\frac{2}{4}\,{}_aD^{\alpha}_{x_i}\left(2\,{}_aD^{\alpha}_{x_i}A_j - {}_aD^{\alpha}_{x_j}A_i\right) + \\ \frac{2}{4m^2}\,{}_aD^{\alpha}_{x_i}\left({}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_i}A_j - {}_aD^{2\alpha}_{x_k}\,{}_aD^{\alpha}_{x_i}A_j - {}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_j}\phi + {}_aD^{2\alpha}_{x_k}\,{}_aD^{\alpha}_{x_j}A_i\right) \\ +\frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_f}\left({}_aD^{\alpha}_{x_f}A_j\right) + \frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_r}\left({}_aD^{\alpha}_{x_r}A_j\right) \\ +\frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_i}\left({}_aD^{\alpha}_{x_i}A_j\right) - \frac{2}{4m^2}\,{}_aD^{\alpha}_{x_i}\,{}_aD^{\alpha}_{x_r}\,{}_aD^{\alpha}_t\left({}_aD^{\alpha}_{x_k}A_i\right) - \\ \frac{2}{4m^2}\,{}_aD^{\alpha}_{x_i}\,{}_aD^{\alpha}_{x_r}\,{}_aD^{\alpha}_{x_f}\left({}_aD^{\alpha}_{x_f}A_j\right) \end{array}\right] \quad (65)$$

And using Eq. ($\boldsymbol{D_4}$) in case (2),with respect to ϕ , then we obtain :

$$\left[\begin{array}{c} J_0 = -\left(-\,{}_aD^{\alpha}_{x_i}\phi + {}_aD^{\alpha}_t A_i\right) \\ +\frac{2}{4m^2}\,{}_aD^{\alpha}_{x_i}\left(-\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_i}\phi - {}_aD^{2\alpha}_{x_k}\,{}_aD^{\alpha}_{x_i}\phi + {}_aD^{3\alpha}_t A_i + {}_aD^{2\alpha}_{x_k}\,{}_aD^{\alpha}_t A_i\right) \\ +\frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_f}\left({}_aD^{\alpha}_t A_f - {}_aD^{\alpha}_{x_f}\phi\right) + \frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_r}\left({}_aD^{\alpha}_t A_r - {}_aD^{\alpha}_{x_r}\phi\right) \\ +\frac{2}{4m^2}\,{}_aD^{2\alpha}_t\,{}_aD^{\alpha}_{x_i}\left({}_aD^{\alpha}_t A_i - {}_aD^{\alpha}_{x_i}\phi\right) + \frac{2}{4m^2}\,{}_aD^{\alpha}_{x_i}\,{}_aD^{\alpha}_{x_r}\,{}_aD^{\alpha}_{x_f}\left({}_aD^{\alpha}_t A_i - {}_aD^{\alpha}_{x_f}\phi\right) \end{array}\right] \quad (66)$$

Applying the Hamiltonian equation Eq. ($\boldsymbol{D_5}$) in case 2, then we get:

$$\begin{bmatrix} -J_i = \frac{2}{4}\, {}_aD_t^{\alpha}\left({}_aD_{x_i}^{\alpha}\phi\right) + \frac{2}{4}\, {}_aD_{x_i}^{\alpha}\left({}_aD_{x_j}^{\alpha}A_j\right) + \frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_j\right) \\ + \frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_r\right) + \frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_r}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_r\right) \\ + \frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_i}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_i\right) + \frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha}\left({}_aD_{x_k}^{\alpha}\phi\right) \\ + \frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_k}^{\alpha}A_i\right) \end{bmatrix} \tag{67}$$

And using Eq. ($\boldsymbol{D_6}$) in case(2), with respect to A_j,we get:

$$\begin{bmatrix} 0 = \frac{2}{4}\, {}_aD_t^{\alpha}\left({}_aD_{x_j}^{\alpha}\phi + 2\, {}_aD_t^{\alpha}A_j\right) \\ + \frac{2}{4m^2}\, {}_aD_t^{\alpha}\left({}_aD_t^{3\alpha}A_j - {}_aD_{x_k}^{2\alpha}\, {}_aD_t^{\alpha}A_j + {}_aD_t^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi + {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi\right) \\ + \frac{2}{4}\, {}_aD_{x_i}^{\alpha}\left(2\, {}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i\right) + \\ \frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\left({}_aD_t^{2\alpha}\, {}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_i}^{\alpha}A_j - {}_aD_t^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi + {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_j}^{\alpha}A_i\right) \\ + \frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_f}^{\alpha}A_j\right) + \frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_r}^{\alpha}\left({}_aD_{x_r}^{\alpha}A_j\right) \\ + \frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_i}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_j\right) - \frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha}\left({}_aD_{x_k}^{\alpha}A_i\right) - \\ \frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_f}^{\alpha}A_j\right) \end{bmatrix} \tag{68}$$

This represents the second homogeneous equation in fractional form.

10 Conclusion

We developed the Hamiltonian formulation of continuous field systems. Our results are identical to those obtained using the Euler-Lagrange model. There are two kinds of conjugate momenta: field dependent and field independent. The equations of motion discoveries can be shown to agree with the Lagrangian formulation of continuous systems as a specific case, with derivatives of integer orders only. This method can be used to calculate the interaction energy between a stationary point charge and a conducting plate. This method can be developed in order to calculate interaction of two charged conducting parallel plates.

References

[1] A. S. Shaikh and K. S. Nisar, Transmission dynamics of fractional order typhoid fever model using Caputo–Fabrizio operator, *Chaos Solit. Fract.* **128**, 355-365 (2019).

[2] G. Rahman, A. Khan, T. Abdeljawad and K. S. Nisar, The Minkowski inequalities via generalized proportional fractional integral operators, *Adv. Differ. Equ.* (1), 1-14 (2019).

[3] G. Rahman, T. Abdeljawad, F. Jarad, A. Khan and K. S. Nisar, Certain inequalities via generalized proportional Hadamard fractional integral operators, *Adv. Differ. Equ.* (1), 1-10 (2019).

[4] D. Baleanu, M. Samraiz, Z. Perveen, S. Iqbal, K. S. Nisar and G. Rahman, Hermite-Hadamard-Fejer type inequalities via fractional integral of a function concerning another function, *AIMS Math.* **6**(5), 4280-4295 (2021).

[5] G. Rahman, K. S. Nisar, S. U. Khan, D. Baleanu and V. Vijayakumar, On the weighted fractional integral inequalities for Chebyshev functionals, *Adv. Differ. Equ.* (1), 1-19 (2021).

[6] S. Mubeen, R. S. Ali, I. Nayab, G. Rahman, K. S. Nisar and D. Baleanu, Some generalized fractional integral inequalities with nonsingular function as a kernel, *AIMS Math*. **6**(4), 3352-3377 (2021).

[7] T. Kang-Bo, L. Chang-Hong and D. Xiao-Jie, Electrodynamic analysis of dissipative electromagnetic materials based on fractional derivative, *Chinese Phy. Lett.* **24,** 847 (2007).

[8] J. C. Gutiérrez-Vega, Fractionalization of optical beams: I. Planar analysis, *Optic. Lett.* **32**, 1521-1523 (2007).

[9] J. C. Gutiérrez-Vega, Fractionalization of optical beams: II. Elegant Laguerre-Gaussian modes, *Optic. Express.* **15**, 6300-6313 (2007).

[10] A. Gaies and A. El-Akrmi, Fractional variational principle in macroscopic picture, *Phys. Scripta* **70**, 7 (2004).

[11] R. Subashini, K. Jothimani, K. S. Nisar and C. Ravichandran, New results on nonlocal functional integro-differential equations via Hilfer fractional derivative, *Alexandria Eng. J.* **59** (5), 2891-2899 (2020).
[12] M. M. Khater, B. Ghanbari, K. S. Nisar and D. Kumar, Novel exact solutions of the fractional Bogoyavlensky–Konopelchenko equation involving the Atangana-Baleanu-Riemann derivative, *Alexandria Engin. J.* **59**(5), 2957-2967 (2020).
[13] A. A. Diab, R. Hijjawi, J. Asad and J. M. Khalifeh, Hamiltonian formulation of classical fields with fractional derivatives: revisited, *Meccanica* **48**, 323-330 (2013).
[14] J. Z. Simon, Higher-derivative Lagrangians, nonlocality, problems, and solutions, *Phys. Rev. D* **41**, 3720 (1990).
[15] Y. Alawaideh, R. Hijjawi and J. Khalifeh, Jordan J. Phys. **13**, 67-72 (2020).
[16] H. A. El-Zalan, S. I. Muslih, E. M. Rabei and D. Baleanu, Hamilton formulation for continuous systems with second order derivatives, *Int. J. Theor. Phys.* **47**, 2195-2202 (2008).
[17] A. Kumar, H. V. S. Chauhan, C. Ravichandran, K. S. Nisar and D. Baleanu, Existence of solutions of non-autonomous fractional differential equations with integral impulse condition, *Adv. Differ. Equ.* (1), 1-14 (2020).
[18] K. S. Nisar and D. Baleanu, Fractional calculus and application of generalized Struve function. *SpringerPlus* **5**(1), 1-13 (2016).
[19] A. Tassaddiq, I. Khan and K. S. Nisar, Heat transfer analysis in sodium alginate based nanofluid using MoS2 nanoparticles: Atangana–Baleanu fractional model, *Chaos Solit. Fract.* **130**, 109445.
[20] K. A. Abro, I. Khan K. S. Nisar, Novel technique of Atangan Baleanu for heat dissipation in transmission line of electrical circuit, *Chaos Soliton. Fract.* **129**, 40-45 (2019).
[21] V. V. Kulish and J. L. Lage, Application of fractional calculus to fluid mechanics, *J. Fluids Eng.* **124**, 803-806 (2020).
[22] M. M. Khater, B. Ghanbari, K. S. Nisar and D. Kumar, Novel exact solutions of the fractional Bogoyavlensky–Konopelchenko equation involving the Atangana-Baleanu-Riemann derivative, *Alexandria Eng. J.* **59**, 2957-2967 (2020).
[23] O. Khan, N. Khan, D. Baleanu and K. S. Nisar, Computable solution of fractional kinetic equations using Mathieu-type series, *Adv. Differ. Equ.,* 1-13 (2019).
[24] L. I. Schiff, Quantum Mechanics, *American J. Phy.* **41**, 599-600 (1973).
[25] P. M. Morse and H. Feshbach, *Methods of theoretical physics: Philip M. Morse, Herman Feshbach, McGraw-Hill*, 1953.
[26] H. Goldstein, *Classical Mechanics*, Ch. 5, Reading: Addison-Wesley, 1980.
[27] F. Barone and A. Nogueira, A conducting surface in Lee–Wick electrodynamics, The European Phy. J. C. **75**, 1-11 (2015).
[28] C. A. Galvao and B. Pimentel, The canonical structure of Podolsky generalized electrodynamics, *Canadian J. Phy.* **66**, 460-466 (1988).
[29] B. Podolsky, A generalized electrodynamics part I—non-quantum, *Phys. Rev.* **62,** 68 (1942).
[30] B. Podolsky and P. Schwed, Review of a generalized electrodynamics, *Rev. Mod. Phy.* **20**, 40 (1948).
[31] T. Lee and G. Wick, Negative metric and the unitarity of the S-matrix, *Nuclear Phys. B.* **9**, 209-243 (1969).
[32] F. Barone, G. Flores-Hidalgo and A. Nogueira, External sources in Lee-Wick theories, *Phys. Rev. D.* **88**, 105031 (2013).

Appendix A

The Hamilton equations of motion, where all variables are independent (ψ_ρ,π_1 ,π_3)

In this (Appendix), we obtain equations (A 1, A 2) and A 3 in terms of functional fractional derivatives.
When comparing (36) and (36), we obtain the equations of motion in terms of the Hamiltonian as:

$$\frac{\partial H}{\partial \psi_\rho} = -\ {}_aD_t^{\alpha}\pi_{\alpha_1} -\ {}_aD_t^{3\alpha}\pi_{\alpha_3} \qquad (A_1)$$

$$\frac{\partial H}{\partial \pi_{\alpha_1}} =\ {}_aD_t^{\alpha}\psi_\rho \qquad (A_2)$$

$$\frac{\partial H}{\partial \pi_{\alpha_3}} =\ {}_aD_t^{3\alpha}\psi_\rho \qquad (A_3)$$

By analogy with Eq. (30) for functional fractional derivative of Lagranian in terms of derivative of Lagrangian density, we can simply define the functional derivative of *H* in terms of derivative of Hamiltonian density with respect to the general variable field φ as [24].

$$\frac{\partial H}{\partial \phi} = \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \phi_0} + \frac{\partial \mathcal{H}}{\partial \phi_l} - {}_aD_{x_i}^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\phi_0)} - {}_aD_{x_i}^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\phi_l)} + \\ - {}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi_0\right)} - {}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi_l\right)} \\ - {}_aD_{x_r}^{\alpha}\,{}_aD_t^{2\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\phi_0)} - {}_aD_{x_r}^{\alpha}\,{}_aD_t^{2\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\phi_l)} \\ - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi_0\right)} - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi_l\right)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{2\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\phi_0)} - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{2\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\phi_l)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi_0\right)} - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi_l\right)} - \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\phi_0)} - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\phi_l)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi_0\right)} - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi_l\right)} \end{bmatrix} \quad (A_4)$$

Using the definition given in Eq. (**A** $_4$) and simplifying, we rewrite the equations of motion (i.e. Eq. (**A** $_1$)) in terms of the Hamiltonian density as follows:

$$\begin{bmatrix} \frac{\partial H}{\partial \psi_\rho} = \frac{\partial \mathcal{H}}{\partial \psi_0} + \frac{\partial \mathcal{H}}{\partial \psi_l} - {}_aD_{x_i}^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\psi_0)} - {}_aD_{x_i}^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\psi_l)} + \\ - {}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_0\right)} - {}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_l\right)} \\ - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\psi_0)} - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\psi_l)} \\ - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_0\right)} - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_l\right)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\psi_0)} - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\psi_l)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_0\right)} - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_l\right)} - \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\psi_0)} - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\psi_l)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_0\right)} - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\psi_l\right)} \end{bmatrix}$$
$$= -\,{}_aD_t^{\alpha}\pi_{\alpha_1} + {}_aD_t^{2\alpha}\pi_{\alpha_2} - {}_aD_t^{3\alpha}\pi_{\alpha_3} \quad (A_5)$$

Appendix B

The Hamiltons equations of motion (all variables are dependent (ψ_ρ,π_1 ,π_3)).

To state the equations of motion from Eq. (37) , let us define and $\pi_1 = g(\psi_\rho)$, $\pi_2 = f({}_aD_t^{\alpha}\psi_\rho)$ $\pi_3 = k({}_aD_t^{2\alpha}\psi_\rho)$ So that, we can write their variations as:

$$\delta\pi_{\alpha_1} = \frac{\partial g}{\partial \psi_\rho}\delta\psi_\rho \qquad (B_1)$$

$$\delta\pi_{\alpha_2} = \frac{\partial f}{\partial\left({}_aD_t^\alpha\psi_\rho\right)}\delta\left({}_aD_t^\alpha\psi_\rho\right) \qquad (B_2)$$

$$\delta\pi_{\alpha_3} = \frac{\partial k}{\partial\left({}_aD_t^{3\alpha}\psi_\rho\right)}\delta\left({}_aD_t^{2\alpha}\psi_\rho\right) \qquad (\boldsymbol{B}_3)$$

Now, by substituting Eqs. (($\boldsymbol{B}_1$), ($\boldsymbol{B}_1$), and ($\boldsymbol{B}_1$) in Eq. (37) and comparing with Eq.(63), we obtain the general equations of the Hamiltonian density for this case:

$$\begin{bmatrix} \frac{\partial H}{\partial \psi_\rho} = \frac{\partial\mathcal{H}}{\partial\psi_0} + \frac{\partial\mathcal{H}}{\partial\psi_l} - {}_aD_{x_i}^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^\alpha\psi_0\right)} - {}_aD_{x_i}^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^\alpha\psi_l\right)} + \\ - {}_aD_t^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_t^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha\psi_0\right)} - {}_aD_t^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_t^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha\psi_l\right)} \\ - {}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha\psi_0\right)} - {}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha\psi_l\right)} \\ - {}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha\psi_0\right)} - {}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_t^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha\psi_l\right)} \\ - {}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_t^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_t^\alpha\psi_0\right)} - {}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_t^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_t^\alpha\psi_l\right)} \\ - {}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha\psi_0\right)} - {}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^\alpha\,{}_aD_t^\alpha\,{}_aD_{x_f}^\alpha\psi_l\right)} - \\ - {}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha\psi_0\right)} - {}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_t^\alpha\psi_l\right)} \\ - {}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha\psi_0\right)} - {}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha \frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^\alpha\,{}_aD_{x_r}^\alpha\,{}_aD_{x_f}^\alpha\psi_l\right)} \end{bmatrix}$$

$$= -{}_aD_t^\alpha\pi_{\alpha_1} - {}_aD_t^{3\alpha}\pi_{\alpha_3} + {}_aD_t^\alpha\psi_\rho\frac{\partial g}{\partial\psi_\rho} \qquad (\boldsymbol{B}_4)$$

Appendix D

The Hamilton's equation in terms($\boldsymbol{\phi}$,$\boldsymbol{A_i}$, $\boldsymbol{A_j}$)

In this Appendix, by using the fields′ variables ($\boldsymbol{\phi}$ *and* $\boldsymbol{A_i}$ *and* $\boldsymbol{A_j}$), we can write the Hamilton's equation in terms of functional derivative as follows (***cases* 1 *and* 2**),We get :

Case 1: All variables are independent

$$\left[\begin{array}{c} \frac{\partial H}{\partial \phi} = \frac{\partial \mathcal{H}}{\partial \phi} - {}_aD_{x_i}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\phi\right)} \\ - {}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi\right)} - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\phi\right)} - \\ {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi\right)} - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\phi\right)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi\right)} - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\phi\right)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi\right)} \end{array}\right]$$
$$= - {}_aD_t^{\alpha}\pi_1^1 - {}_aD_t^{3\alpha}\pi_3^1 \qquad (\boldsymbol{D_1})$$

$$\left[\begin{array}{c} \frac{\partial H}{\partial A_j} = \frac{\partial \mathcal{H}}{\partial A_j} - {}_aD_{x_i}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}A_j\right)} + \\ - {}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j\right)} - {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}A_j\right)} - \\ {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j\right)} - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}A_j\right)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j\right)} - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}A_j\right)} \\ - {}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j\right)} \end{array}\right]$$
$$= - {}_aD_t^{\alpha}\pi_1^3 - {}_aD_t^{3\alpha}\pi_3^3 \qquad (\mathbf{D_3})$$

Case2 : $\boldsymbol{\pi_1^1, \pi_1^2, \pi_1^3}$, **depend on** $\boldsymbol{{}_aD_t^{\alpha}\phi, {}_aD_t^{\alpha}A_i, {}_aD_t^{\alpha}A_j}$) **and** $\boldsymbol{\pi_3^1, \pi_3^2, \pi_3^3}$,**depend (** $\boldsymbol{{}_aD_t^{2\alpha}\phi, {}_aD_t^{2\alpha}A_i, {}_aD_t^{2\alpha}A_j}$), we get the general equations of the Hamiltonian density for this case:

$$\begin{bmatrix} \frac{\partial H}{\partial \phi} = \frac{\partial \mathcal{H}}{\partial \phi} - {}_aD_{x_i}^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_{x_i}^{\alpha} \phi \right)} + \\ - {}_aD_t^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_t^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_{x_f}^{\alpha} \phi \right)} - {}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha} \phi \right)} - \\ {}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha} \phi \right)} - {}_aD_{x_i}^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_{x_i}^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_t^{\alpha} \phi \right)} \\ - {}_aD_{x_i}^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_{x_i}^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_{x_f}^{\alpha} \phi \right)} - {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha} A_0 \right)} \\ - {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha} \phi \right)} \end{bmatrix}$$

$$= - {}_aD_t^{\alpha} \pi_1^1 - {}_aD_t^{3\alpha} \pi_3^1 + {}_aD_t^{\alpha} \phi \left(\frac{\partial g}{\partial \phi} \right) \qquad (\mathbf{D_4})$$

$$\begin{bmatrix} \frac{\partial H}{\partial A_i} = \frac{\partial \mathcal{H}}{\partial A_i} - {}_aD_{x_i}^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_{x_i}^{\alpha} A_i \right)} \\ - {}_aD_t^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_t^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_{x_f}^{\alpha} A_i \right)} - {}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha} A_i \right)} - \\ {}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha} A_i \right)} - {}_aD_{x_i}^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_{x_i}^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_t^{\alpha} A_i \right)} \\ - {}_aD_{x_i}^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_{x_i}^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_{x_f}^{\alpha} A_i \right)} - {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha} A_i \right)} \\ - {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha} \frac{\partial \mathcal{H}}{\partial \left({}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha} A_i \right)} \end{bmatrix}$$

$$= - {}_aD_t^{\alpha} \pi_1^2 - {}_aD_t^{3\alpha} \pi_3^2 + {}_aD_t^{\alpha} A_i \left(\frac{\partial g}{\partial A_i} \right) \qquad (\mathbf{D_5})$$

$$\left[\begin{array}{c}\frac{\partial H}{\partial A_j}=\frac{\partial\mathcal{H}}{\partial A_j}-{}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}A_j\right)}\\ -{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j\right)}-{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial\mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}A_j\right)}-\\ {}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{H}}{\partial\left({}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j\right)}-{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_t^{\alpha}A_j\right)}\\ -{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j\right)}-{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}A_j\right)}\\ -{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j\right)}\end{array}\right]$$

$$=-{}_aD_t^{\alpha}\pi_1^2-{}_aD_t^{3\alpha}\pi_3^2+{}_aD_t^{\alpha}A_j\left(\frac{\partial g}{\partial A_j}\right) \quad (\mathbf{D_6})$$