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COVID-19 Mathematical Study with Environmental Reservoir and Three General Functions for Transmissions

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Abstract: In this paper, the ongoing new coronavirus (COVID-19) epidemic is being investigated using a mathematical model. The model depicts the dynamics of infection with several transmission pathways and general infection functions, plus it highlights the significance of the environment as a reservoir for the disease's propagation and dissemination. We have studied the qualitative behavior of the proposed model representing a system of fractional differential equations. Under a set of conditions on the general functions and the parameters, we have proven the global asymptotic stability of all steady states by using the Lyapunov method and LaSalle's invariance principle. We also carried some numerical results to confirm the analytical results we obtained.

Keywords: COVID-19 pandemic, Fractional-order differential equation, Environmental reservoir, General incidence, Global stability, Numerical simulations

1 Introduction

Coronavirus disease 2019 (COVID-19) is an infectious disease caused by severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). In the beginning, it was isolated December 2019 in Wuhan, China from some people who have pneumonia connected to the cluster of acute respiratory illness cases, and since then, it has spread over the world, culminating in the pandemic of 2020. When an infected individual cough, sneezes or exhales, the virus that causes COVID-19 is mostly transferred by droplets. These droplets are too heavy to float in the air and fall to the ground or other surfaces. COVID-19 affects different groups of people of different ages, but it is more prevalent in the less immune groups and those with chronic diseases. Also, most of the infected people have mild to moderate symptoms and recover without going to the hospital.

Because of thousands of confirmed infections and thousands of fatalities throughout the world, the COVID-19 pandemic is now regarded as the greatest global threat. In the weekly epidemiological update-8 December 2020 received by World Health Organization from national authorities, COVID-19 cases have remained stable at over 4 million new cases, but new fatalities have risen to around 73 000. Since the beginning of the pandemic, there have been about 65.8 million recorded illnesses and 1.5 million fatalities worldwide [1]. Many countries have followed China's lead and imposed curfews, closed borders, and halted all normal daily operations, such as school and workplace closures. The use of mathematical models to study social distancing techniques has proved their efficiency in limiting the spread of COVID-19 infection. Infectious disease transmission dynamics mathematical models are increasingly widely used. Models like this are useful for quantifying potential infectious disease prevention and mitigation techniques. For infectious diseases, there are a variety of models available, ranging from the very simple SIR model to more complicated ideas. Many researchers in the scientific community have conducted multidisciplinary investigations using various mathematical models to understand the virus spread pattern (see [2], [3], [4], [5]). However, a comprehensive approach of mathematical instrumentalization models in the characterization of the COVID-19 growth curve and its containment strategies are remaining drastically understudied in current literature.

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We supplement existing studies on the topic by extending SIR models and relying on conclusions gained from extant studies using SIR model extensions [6], [7], [8], [9]. These models were only concerned with direct human-to-human transmission [10], [11]. In addition, the role of the environment in COVID-19 transmission has been largely ignored in contemporary clinical and theoretical investigations and is seldom studied in modeling and simulation. As a result, our understanding of COVID-19 transmission mechanism and epidemiological features is still restricted. COVID-19 may be transferred between humans through direct touch, and both symptomatic and asymptomatic persons can infect others [12], [13], [14], [15]. Furthermore, the environment to human hosts indirect transmission is a highly likely method for propagate of coronavirus. Coughing and sneezing of infected people released respiratory droplets containing the coronavirus, and the majority of these droplets land on neighbouring surfaces and items. By contacting infected surfaces or items and then touching their faces, other people might get the virus. Meanwhile, coronaviruses generated by sick persons might float in the air as aerosols and be inhaled by those who pass by. Such environment-to-human transmission channels, and the effectiveness of such a form of transmission, are primarily dependent on the coronavirus's capacity to live and remain in the environment. The viability and duration of SARS-CoV in the environment were verified in [12] and [16]. New coronavirus (SARS-CoV-2) can stay alive and infectious in aerosols for hours and on surfaces for days, according to experimental research published in March 2020, indicating a high likelihood and large danger of environmental transmission. C. Yang and J. Wang [17] studied the effect of the environmental reservoirs by incorporating it into a model represented by a system of ordinary differential equations, however using fractional derivative to model a real process has piqued the interest of a number of authors from different fields (see e.g. [18], [19]) as fractional derivative is an ideal tool for describing real-world phenomena with memory, such as most biological systems.

The manuscript is structured as follows: In Section 2, we introduce fractional order differential equations preliminaries. In Section 3, a new fractional-order COVID-19 mathematical model is proposed and takes into account the influence of environmental reservoirs with three general functions of the transition, which are susceptible-exposed, susceptible-infected and susceptible-environmental transmissions, furthermore a qualitative analysis of the model is investigated in Subsections 3.1 and 3.2, also in Subsection 3.3, we calculate the basic reproduction number $\mathscr{R}_0$ for the model. In Section 4, we study the local and global stability of both disease-free and endemic steady states. The effect of parameters on the system is illustrated in Section 5 of numerical simulations, where we use real data. Section 6 brings us to a close with conclusions and discussion.

2 Preliminaries

In this section, we introduce the basic definitions and lemma of fraction calculus which is an important tool in modeling processes of biological systems, and has the ability to provide an exact description not only of the current state of the disease but also of all its historical states.

Definition 2.1. Define a function $f : [0,\infty) \longrightarrow R$ then fractional integral of it of order $\alpha \in (0,1]$ given as follows:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx,$$

where $\Gamma(.)$ is the gamma function [20], and the Caputo fractional derivative of order α is given by:

$$D^{\alpha} f(t) = I^{n-\alpha} D^n f(t),$$

where $n-1 < \alpha \leq n$ and f(t) is a continuous function [21]. In particular, when $0 < \alpha \leq 1$, one has

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(x)}{(t-x)^{\alpha}} dx.$$

For more properties of the fraction order derivatives (see e.g. [22] and [23]).

Lemma 2.1. Consider a fraction order system

$$D^{\alpha}(x) = f(x), \qquad x(0) = x_0,$$

with $0 < \alpha \leq 1$ and $x \in \{R^n\}$, evaluate the equilibrium points of the system by let $D^{\alpha}(x) = 0$ then this points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix of the system evaluated at the equilibrium points satisfy the following conditions: [24]

$$|\arg(\lambda_i)| > \alpha \frac{\pi}{2}. \tag{1}$$

3 Proposed COVID-19 fractional-order model

We will institute new COVID-19 model as a system of fractional order differential equations that includes five elements S, E, I, R and V represent the concentrations of the susceptible, exposed, infected and recovered individuals, respectively and V is the concentration of the coronavirus in the environment as follows:

$$D^{\alpha}S(t) = \lambda - \mu S - L_1(S,V) - L_2(S,E) - L_3(S,I) \tag{2}$$
$$D^{\alpha}E(t) = L_1(S,V) + L_2(S,E) + L_3(S,I) - (\delta+\mu)E \tag{3}$$
$$D^{\alpha}I(t) = \delta E - (\omega+\gamma+\mu)I \tag{4}$$
$$D^{\alpha}R(t) = \gamma I - \mu R \tag{5}$$
$$D^{\alpha}V(t) = \rho_1 E + \rho_2 I - \sigma V. \tag{6}$$

The parameter λ is the population influx, μ is the pace at which human hosts die naturally, δ^{-1} is the time between infection and emergence of symptoms (incubation period), ω represents the death rate as a result of disease, γ symbolizes the recovery from infection rate, ρ_1 and ρ_2 denote the contribution of exposed and infected individuals with coronavirus to the environmental reservoir rates, respectively, and σ denote the pace at which the virus is removed from the environment. The *SEIRV* model scheme are shown in Figure 1. The functions L_k, $k = 1,2,3$ are continuously differentiable and satisfy the following conditions:

(C1) $L_k(S,W) > 0$ and $L_k(S,0) = L_k(0,W) = 0$ for all $S > 0, W > 0$ where, $k = 1,2,3$.

(C2) $\frac{\partial L_k(S,W)}{\partial S} > 0$, $\frac{\partial L_k(S,W)}{\partial W} > 0$, $\frac{\partial L_k(S,W)}{\partial W}|_{W=0} > 0$, and $\frac{d}{dS}\left(\frac{\partial L_k(S,W)}{\partial W}|_{W=0}\right) > 0$ for all $S > 0, W > 0$ where, $k = 1,2,3$.

(C3) $\frac{\partial}{\partial V}\left(\frac{L_1(S,V)}{V}\right) \leq 0$, $\frac{\partial}{\partial E}\left(\frac{L_2(S,E)}{E}\right) \leq 0$ and $\frac{\partial}{\partial I}\left(\frac{L_3(S,I)}{I}\right) \leq 0$ for all $S,E,I,V > 0$.

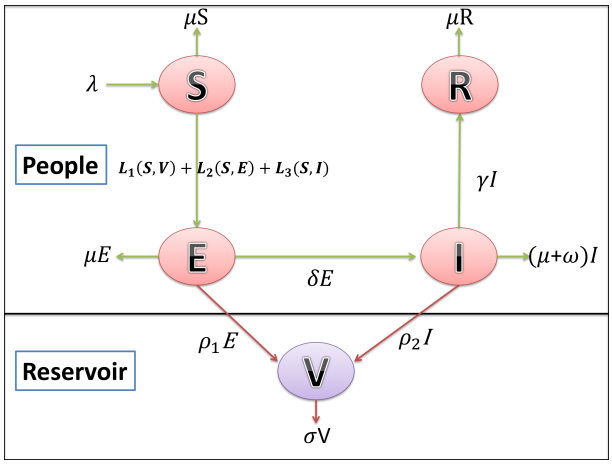

Fig. 1: *SEIRV* model scheme.

3.1 Nonnegativity and boundedness

Proposition 3.1.1. Suppose that for system (2)-(6) the conditions **(C1)**- **(C3)** are satisfied. Then the compact set

$$\Psi = \{(S,E,I,R,V) \in R^5_{\geq 0}, 0 \leq S(t), E(t) \leq \Gamma_1, 0 \leq I(t), R(t) \leq \Gamma_2, 0 \leq V(t) \leq \Gamma_3\}. \tag{7}$$

is positively invariant.

Proof. We have

$$\begin{aligned}
D^{\alpha}S(t)|_{S=0} &= \lambda > 0,\\
D^{\alpha}E(t)|_{E=0} &= L_1(S,V)+L_3(S,I) \geq 0, \quad \text{for all } S,I,V \geq 0,\\
D^{\alpha}I(t)|_{I=0} &= \delta E \geq 0, \qquad \text{for } E \geq 0,\\
D^{\alpha}R(t)|_{R=0} &= \gamma I \geq 0, \qquad \text{for } I \geq 0,\\
D^{\alpha}V(t)|_{V=0} &= \rho_1 E+\rho_2 I \geq 0, \qquad \text{for } E,I \geq 0.
\end{aligned}$$

This proves the nonnegativity of the solutions corresponding to the proposed model (2)-(6).

To prove the boundedness of the state variables, we let

$$\Upsilon = S+E+\frac{\rho_2\delta}{\rho_2\delta+\rho_1\mu}I+\frac{\rho_2\delta}{\rho_2\delta+\rho_1\mu}R+\frac{\mu\delta}{\rho_2\delta+\rho_1\mu}V. \tag{8}$$

Let $\rho_2\delta+\rho_1\mu = B$, then

$$\begin{aligned}
D^{\alpha}\Upsilon =& D^{\alpha}S+D^{\alpha}E+\frac{\rho_2\delta}{B}D^{\alpha}I+\frac{\rho_2\delta}{B}D^{\alpha}R+\frac{\mu\delta}{B}D^{\alpha}V\\
=&\lambda-\mu S-(\delta+\mu)E+\frac{\rho_2\delta}{B}\Big[\delta E-(\omega+\gamma+\mu)I\Big]+\frac{\rho_2\delta}{B}\Big[\gamma I-\mu R\Big]+\frac{\mu\delta}{B}\Big[\rho_1 E+\rho_2 I-\sigma V\Big]\\
=&\lambda-\mu S-\delta E-\mu E+\frac{\rho_2\delta^2}{B}E-\frac{\rho_2\delta[\omega+\mu]I}{B}-\frac{\rho_2\mu\delta}{B}R+\frac{\rho_1\mu\delta}{B}E+\frac{\rho_2\mu\delta}{B}I-\frac{\mu\delta\sigma}{B}V\\
=&\lambda-\mu S-\mu E-\frac{\rho_2\delta\omega}{B}I-\frac{\rho_2\mu\delta}{B}R-\frac{\mu\delta\sigma}{B}V\\
=&\lambda-\tau\left[S+E+\frac{\rho_2\delta}{B}I+\frac{\rho_2\delta}{B}R+\frac{\mu\delta}{B}V\right]\\
=&\lambda-\tau\Upsilon,
\end{aligned}$$

where, $\tau = \min\{\omega,\mu,\sigma\}$. Then

$$\Upsilon(t) \leq e^{-\tau t}\left(\Upsilon(0)-\frac{\lambda}{\tau}\right)+\frac{\lambda}{\tau}.$$

This yields, $0 \leq \Upsilon(t) \leq \Gamma_1$ for all $t \geq 0$ if $\Upsilon(0) \leq \Gamma_1$, where $\Gamma_1 = \frac{\lambda}{\tau}$. It follows that $0 \leq S(t),E(t) \leq \Gamma_1, 0 \leq I(t),R(t) \leq \Gamma_2$ and $0 \leq V(t) \leq \Gamma_3$ for all $t \geq 0$ if $S(0)+E(0)+\frac{\rho_2\delta}{B}I(0)+\frac{\rho_2\delta}{B}R(0)+\frac{\mu\delta}{B}V(0) \leq \Gamma_1$, where $\Gamma_2 = \frac{\lambda B}{\rho_2\delta\tau}$, $\Gamma_3 = \frac{\lambda B}{\mu\delta\tau}$ and $B = \rho_2\delta+\rho_1\mu$. This proves the boundedness of S,E,I,R and V.□

3.2 Steady states

This section researches the steady states of model (2)-(6) and extract the criteria for its existence. It is the positive solutions of the next equations.

$$0 = \lambda-\mu S-L_1(S,V)-L_2(S,E)-L_3(S,I), \tag{9}$$
$$0 = L_1(S,V)+L_2(S,E)+L_3(S,I)-(\delta+\mu)E, \tag{10}$$
$$0 = \delta E-(\omega+\gamma+\mu)I, \tag{11}$$
$$0 = \gamma I-\mu R, \tag{12}$$
$$0 = \rho_1 E+\rho_2 I-\sigma V. \tag{13}$$

Model (2)-(6) has a disease-free steady state $Q_0 = (S_0,0,0,0,0)$, which is always exist and $S_0 = \dfrac{\lambda}{\mu}$. The other positive steady state is evaluated as follows:

From equation (9)-(13), we obtain

$$\lambda-\mu S = L_1(S,V)+L_2(S,E)+L_3(S,I) = (\delta+\mu)E = \frac{(\delta+\mu)A}{\delta}I = \frac{\mu(\delta+\mu)A}{\gamma\delta}R = \frac{\sigma(\delta+\mu)A}{\rho_2\delta+\rho_1 A}V, \tag{14}$$

where, $A = \omega + \gamma + \mu$.

From the last equation, we get

$$V = h_1(S), \quad E = h_2(S), \quad I = h_3(S), \quad R = h_4(S), \tag{15}$$

where,

$$\begin{aligned} h_1(S) &= \frac{(\lambda - \mu S)(\rho_2\delta + \rho_1 A)}{\sigma(\delta+\mu)A}, \quad h_2(S) = \frac{(\lambda - \mu S)}{\delta + \mu}, \\ h_3(S) &= \frac{\delta(\lambda - \mu S)}{(\delta+\mu)A}, \quad h_4(S) = \frac{\gamma\delta(\lambda - \mu S)}{\mu(\delta+\mu)A}. \end{aligned} \tag{16}$$

It is clear that, $h_j(S) > 0$ for all $S \in [0, S_0)$ and $h_j(S_0) = 0, \quad j = 1,2,3,4$. Define

$$F_1(S) = L_1\big(S, h_1(S)\big) + L_2\big(S, h_2(S)\big) + L_3\big(S, h_3(S)\big) - \frac{\sigma(\delta+\mu)A}{\rho_2\delta + \rho_1 A} h_1(S). \tag{17}$$

From condition **C1**, we have

$$F_1(0) = -\frac{\sigma(\delta+\mu)A}{\rho_2\delta + \rho_1 A} h_1(0) = -\lambda < 0, \qquad F_1(S_0) = 0.$$

Moreover,

$$F_1'(S) = \frac{\partial L_1}{\partial S} + h_1'(S)\frac{\partial L_1}{\partial V} + \frac{\partial L_2}{\partial S} + h_2'(S)\frac{\partial L_2}{\partial E} + \frac{\partial L_3}{\partial S} + h_3'(S)\frac{\partial L_3}{\partial I} - \frac{\sigma(\delta+\mu)A}{\rho_2\delta + \rho_1 A} h_1'(S),$$

$$\begin{aligned} F_1'(S_0) = &\frac{\partial L_1(S_0,0)}{\partial S} + h_1'(S_0)\frac{\partial L_1(S_0,0)}{\partial V} + \frac{\partial L_2(S_0,0)}{\partial S} + h_2'(S_0)\frac{\partial L_2(S_0,0)}{\partial E} + \frac{\partial L_3(S_0,0)}{\partial S} \\ &+ h_3'(S_0)\frac{\partial L_3(S_0,0)}{\partial I} - \frac{\sigma(\delta+\mu)A}{\rho_2\delta + \rho_1 A} h_1'(S_0). \end{aligned}$$

Conditions **C1** and **C2** imply that $\frac{\partial L_k(S_0,0)}{\partial S} = 0, \quad k = 1,2,3$, then

$$F_1'(S_0) = h_1'(S_0)\frac{\partial L_1(S_0,0)}{\partial V} + h_2'(S_0)\frac{\partial L_2(S_0,0)}{\partial E} + h_3'(S_0)\frac{\partial L_3(S_0,0)}{\partial I} - \frac{\sigma(\delta+\mu)A}{\rho_2\delta + \rho_1 A} h_1'(S_0).$$

From equation (16), we obtain

$$F_1'(S_0) = -\mu\left[\frac{(\rho_2\delta + \rho_1 A)}{\sigma(\delta+\mu)A}\frac{\partial L_1(S_0,0)}{\partial V} + \frac{1}{\delta+\mu}\frac{\partial L_2(S_0,0)}{\partial E} + \frac{\delta}{(\delta+\mu)A}\frac{\partial L_3(S_0,0)}{\partial I} - 1\right].$$

If we have

$$\frac{1}{(\delta+\mu)}\left[\frac{(\rho_2\delta + \rho_1 A)}{\sigma A}\frac{\partial L_1(S_0,0)}{\partial V} + \frac{\partial L_2(S_0,0)}{\partial E} + \frac{\delta}{A}\frac{\partial L_3(S_0,0)}{\partial I}\right] > 1. \tag{18}$$

then, $F_1'(S_0) < 0$ and there exists $S^* \in (0, S_0)$ such that $F_1(S^*) = 0$. From equations (15) and (16), we get

$$V^* = \frac{(\lambda - \mu S^*)(\rho_2\delta + \rho_1 A)}{\sigma(\delta+\mu)A}, \quad E^* = \frac{(\lambda - \mu S^*)}{\delta+\mu}, \quad I^* = \frac{\delta(\lambda - \mu S^*)}{(\delta+\mu)A}, \quad R^* = \frac{\gamma\delta(\lambda - \mu S^*)}{\mu(\delta+\mu)A}. \tag{19}$$

It follows that system (2)-(6) has an endemic steady state $Q_1 = (S^*, E^*, I^*, R^*, V^*)$ if condition (18) is satisfied.

3.3 The basic reproduction number $\mathscr{R}_0$

Now, we compute the basic reproduction number $\mathscr{R}_0$ for system (2)-(6) by the using the next generation matrix method.

Let $X=(E,I,V)^T$, then system (2)-(6) can be written as:

$$D^{\alpha}X=\aleph(X)-\hbar(X),$$

where,

$$\aleph(X)=\begin{pmatrix}L_1(S,V)+L_2(S,E)+L_3(S,I)\\0\\0\end{pmatrix},\qquad \hbar(X)=\begin{pmatrix}(\delta+\mu)E\\-\delta E+(\omega+\gamma+\mu)I\\-\rho_1E-\rho_2I+\sigma V\end{pmatrix}.$$

Jacobian matrices of $\aleph$ and $\hbar$ at the disease-free steady state Q_0 are

$$\mathscr{F}=\begin{pmatrix}\frac{\partial L_2(S_0,0)}{\partial E} & \frac{\partial L_3(S_0,0)}{\partial I} & \frac{\partial L_1(S_0,0)}{\partial V}\\0&0&0\\0&0&0\end{pmatrix},\qquad \mathscr{V}=\begin{pmatrix}(\delta+\mu)&0&0\\-\delta&(\omega+\gamma+\mu)&0\\-\rho_1&-\rho_2&\sigma\end{pmatrix}.$$

Then, the next generation matrix is

$$\mathscr{F}\mathscr{V}^{-1}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\0&0&0\\0&0&0\end{pmatrix},$$

where,

$$\begin{aligned}a_{11}&=\frac{1}{\delta+\mu}\left[\frac{(\rho_2\delta+\rho_1A)}{\sigma A}\frac{\partial L_1(S_0,0)}{\partial V}+\frac{\partial L_2(S_0,0)}{\partial E}+\frac{\delta}{A}\frac{\partial L_3(S_0,0)}{\partial I}\right]\\a_{12}&=\frac{1}{A}\left[\frac{\partial L_3(S_0,0)}{\partial I}+\frac{\rho_2}{\sigma}\frac{\partial L_1(S_0,0)}{\partial V}\right]\\a_{13}&=\frac{1}{\sigma}\frac{\partial L_1(S_0,0)}{\partial V}.\end{aligned}$$

The basic reproduction number of system (2)-(6) is the spectral radius of $(\mathscr{F}\mathscr{V}^{-1})$, which is given as follows:

$$\begin{aligned}\mathscr{R}_0&=\frac{1}{\delta+\mu}\left[\frac{(\rho_2\delta+\rho_1A)}{\sigma A}\frac{\partial L_1(S_0,0)}{\partial V}+\frac{\partial L_2(S_0,0)}{\partial E}+\frac{\delta}{A}\frac{\partial L_3(S_0,0)}{\partial I}\right]\\&=\mathscr{R}_{01}+\mathscr{R}_{02}+\mathscr{R}_{03}.\end{aligned}\tag{20}$$

Based on the above review, we have the next outcome:

Lemma 3.3.1. Suppose that for system (2)-(6), the conditions **C1-C3** are satisfied. Then there exists a positive threshold parameter $\mathscr{R}_0$ such that
(i) if $\mathscr{R}_0\leq 1$; then there exists only one steady state Q_0,
(ii) if $\mathscr{R}_0>1$; then there exist two steady states Q_0 and Q_1.

4 Stability of steady states

4.1 Local stability

This subsection discusses the local stability of the proposed model (2)-(6). Because the fourth equation of system (2)-(6) is independent of the other system equations, we can simplify analysis by reducing system (2)-(6) to the following sub-system.

$$D^{\alpha}S(t)=\lambda-\mu S-L_1(S,V)-L_2(S,E)-L_3(S,I)\tag{21}$$
$$D^{\alpha}E(t)=L_1(S,V)+L_2(S,E)+L_3(S,I)-(\delta+\mu)E\tag{22}$$
$$D^{\alpha}I(t)=\delta E-(\omega+\gamma+\mu)I\tag{23}$$
$$D^{\alpha}V(t)=\rho_1E+\rho_2I-\sigma V.\tag{24}$$

The Jacobian matrix of system (21)-(24) at a point $Q=(S,E,I,V)$ takes the following form:

$$J(Q)=\begin{pmatrix} -\mu-\mathscr{Z} & -\frac{\partial L_2(S,E)}{\partial E} & -\frac{\partial L_3(S,I)}{\partial I} & -\frac{\partial L_1(S,V)}{\partial V} \\ \mathscr{Z} & \frac{\partial L_2(S,E)}{\partial E}-(\delta+\mu) & \frac{\partial L_3(S,I)}{\partial I} & \frac{\partial L_1(S,V)}{\partial V} \\ 0 & \delta & -A & 0 \\ 0 & \rho_1 & \rho_2 & -\sigma \end{pmatrix}, \tag{25}$$

where,

$$\mathscr{Z}=\frac{\partial L_1(S,V)}{\partial S}+\frac{\partial L_2(S,E)}{\partial S}+\frac{\partial L_3(S,I)}{\partial S}. \tag{26}$$

Theorem 4.1.1. If $\mathscr{R}_0<1$, then the disease-free steady state $Q_0=(S_0,0,0,0)$ of system (21)-(24) is locally asymptotically stable if the condition $A>\sigma$ holds.

Proof. The Jacobian matrix (25) at $Q_0=(S_0,0,0,0)$ is

$$J(Q_0)=\begin{pmatrix} -\mu & -\frac{\partial L_2(S_0,0)}{\partial E} & -\frac{\partial L_3(S_0,0)}{\partial I} & -\frac{\partial L_1(S_0,0)}{\partial V} \\ 0 & \frac{\partial L_2(S_0,0)}{\partial E}-(\delta+\mu) & \frac{\partial L_3(S_0,0)}{\partial I} & \frac{\partial L_1(S_0,0)}{\partial V} \\ 0 & \delta & -A & 0 \\ 0 & \rho_1 & \rho_2 & -\sigma \end{pmatrix}.$$

The disease-free steady state Q_0 is locally asymptotically stable if all eigenvalues $\xi_i, i=1,2,3,4$ of $J(Q_0)$ satisfy the condition given in (1). These eigenvalues are the roots of the characteristic equation corresponding to $J(Q_0)$, which is given from $det(J(Q_0)-\xi I_4)=0$, where I_4 is a square identity matrix of order 4, as follows:

$$|J(Q_0)-\xi I|=(\mu+\xi)\begin{vmatrix} -c_{11}-\xi & c_{12} & c_{13} \\ \delta & -A-\xi & 0 \\ \rho_1 & \rho_2 & -\sigma-\xi \end{vmatrix}=0,$$

where,

$$\begin{aligned} c_{11}&=(\delta+\mu)-\frac{\partial L_2(S_0,0)}{\partial E}>0, \quad if \quad \mathscr{R}_0<1, \\ c_{12}&=\frac{\partial L_3(S_0,0)}{\partial I}>0, \\ c_{13}&=\frac{\partial L_1(S_0,0)}{\partial V}>0. \end{aligned} \tag{27}$$

Hence,

$$det(J(Q_0)-\xi I):=(\xi+\mu)(\xi^3+F\xi^2+G\xi+H)=0. \tag{28}$$

Clearly, one of the roots of $J(Q_0)$ is $-\mu$, which is a negative. The remaining roots of $J(Q_0)$ can be obtained from the following equation:

$$\Phi(\xi):=\xi^3+F\xi^2+G\xi+H=0, \tag{29}$$

where,

$$\begin{aligned} F&=\sigma+A+c_{11}, \\ G&=A(c_{11}+\sigma)+\sigma c_{11}-\delta c_{12}-\rho_1 c_{13} \\ &=\sigma A+\sigma c_{11}+\frac{\rho_2\delta}{\sigma}c_{13}+\frac{\rho_1}{\sigma}c_{13}(A-\sigma)+A(\delta+\mu)(1-\mathscr{R}_0), \\ H&=\sigma A c_{11}-\delta\sigma c_{12}-c_{13}(\delta\rho_2+\rho_1 A) \\ &=\sigma A(\delta+\mu)(1-\mathscr{R}_0), \end{aligned}$$

$$\begin{aligned} FG-H&=(\sigma+A+c_{11})\left(\sigma A+\sigma c_{11}+\frac{\rho_2\delta}{\sigma}c_{13}+\frac{\rho_1}{\sigma}c_{13}(A-\sigma)+A(\delta+\mu)(1-\mathscr{R}_0)\right)-\sigma A(\delta+\mu)(1-\mathscr{R}_0) \\ &=(\sigma+A+c_{11})\left(\sigma A+\sigma c_{11}+\frac{\rho_2\delta}{\sigma}c_{13}+\frac{\rho_1}{\sigma}c_{13}(A-\sigma)\right)+A(\delta+\mu)(A+c_{11})(1-\mathscr{R}_0). \end{aligned}$$

The discriminant $D(\Phi)$ of $\Phi(\xi)$ given in (29) is:

$$D(\Phi)=\begin{pmatrix}1 & F & G & H & 0\\ 0 & 1 & F & G & H\\ 3 & 2F & G & 0 & 0\\ 0 & 3 & 2F & G & 0\\ 0 & 0 & 3 & 2F & G\end{pmatrix}=18FGH+(FG)^2-4H(F)^3-4(G)^3-27(H)^2.$$

It is clear that, $F>0$. Also, we have $G>0,\quad H>0$ and $FG-H>0$ if $\mathscr{R}_0<1$ and $A>\sigma$. Following Ahmed et al. [25], for the fractional Routh-Hurwitz conditions, the all the eigenvalues associated with $J(Q_0)$ have negative real parts and therefore, Q_0 is locally asymptotically stable if $D(\Phi)>0 \ \ for \ \ 0<\alpha\leq 1$. This ends the proof.□

Now, we are analysing the stability of the endemic steady state Q_1 of the model (21)-(24). The Jacobian matrix (25), calculated at the endemic steady state Q_1, is shown as below.

$$J(Q_1)=\begin{pmatrix}-\mu-\mathscr{B} & -\frac{\partial L_2(S^*,E^*)}{\partial E} & -\frac{\partial L_3(S^*,I^*)}{\partial I} & -\frac{\partial L_1(S^*,V^*)}{\partial V}\\ \mathscr{B} & \frac{\partial L_2(S^*,E^*)}{\partial E}-(\delta+\mu) & \frac{\partial L_3(S^*,I^*)}{\partial I} & \frac{\partial L_1(S^*,V^*)}{\partial V}\\ 0 & \delta & -A & 0\\ 0 & \rho_1 & \rho_2 & -\sigma\end{pmatrix},$$

where,

$$\mathscr{B}=\frac{\partial L_1(S^*,V^*)}{\partial S}+\frac{\partial L_2(S^*,E^*)}{\partial S}+\frac{\partial L_3(S^*,I^*)}{\partial S}.$$

Adding row 2 to row 1, we have

$$J(Q_1)\sim\begin{pmatrix}-\mu & -(\delta+\mu) & 0 & 0\\ \mathscr{B} & \frac{\partial L_2(S^*,E^*)}{\partial E}-(\delta+\mu) & \frac{\partial L_3(S^*,I^*)}{\partial I} & \frac{\partial L_1(S^*,V^*)}{\partial V}\\ 0 & \delta & -A & 0\\ 0 & \rho_1 & \rho_2 & -\sigma\end{pmatrix}.$$

Multiply row 1 by $\dfrac{\mathscr{B}}{\mu}$ and add it to row 2 implies

$$J(Q_1)\sim\begin{pmatrix}-\mu & -(\delta+\mu) & 0 & 0\\ 0 & -d_{22} & d_{23} & d_{24}\\ 0 & \delta & -A & 0\\ 0 & \rho_1 & \rho_2 & -\sigma\end{pmatrix},$$

where,

$$\begin{aligned}d_{22}&=\Big(\frac{\delta+\mu}{\mu}\Big)\frac{\partial}{\partial S}\Big(L_1(S^*,V^*)+L_2(S^*,E^*)+L_3(S^*,I^*)\Big)-\frac{\partial L_2(S^*,E^*)}{\partial E}+(\delta+\mu),\\ d_{23}&=\frac{\partial L_3(S^*,I^*)}{\partial I},\qquad d_{24}=\frac{\partial L_1(S^*,V^*)}{\partial V}.\end{aligned}\tag{30}$$

Then the characteristic equation of $J(Q_1)$ is

$$(\xi+\mu)(\xi^3+L\xi^2+M\xi+N)=0.\tag{31}$$

One of the roots is obviously negative, which is $-\mu$. The remaining roots can be extracted from the next equation.

$$\Psi(\xi):=\xi^3+L\xi^2+M\xi+N=0,\tag{32}$$

where,

$$\begin{aligned}L&=\sigma+A+d_{22},\\ M&=\sigma A+(\sigma+A)d_{22}-\delta d_{23}-\rho_1 d_{24},\\ N&=\sigma A d_{22}-\sigma\delta d_{23}-(\delta\rho_2+\rho_1 A)d_{24}.\end{aligned}\tag{33}$$

The discriminant $D(\Psi)$ of $\Psi(\xi)$ reads:

$$D(\Psi)=\begin{pmatrix}1 & L & M & N & 0\\ 0 & 1 & L & M & N\\ 3 & 2L & M & 0 & 0\\ 0 & 3 & 2L & M & 0\\ 0 & 0 & 3 & 2L & M\end{pmatrix}=18LMN+(LM)^2-4N(L)^3-4(M)^3-27(N)^2.$$

Following Ahmed et al. [25], we have the following result.

Theorem 4.1.2. The endemic steady state Q_1 is locally asymptotically stable if one of the following requirements is met:
(i) $D(\Psi)>0, L>0, N>0, LM>N$;
(ii) $D(\Psi)<0, L\geq 0, M\geq 0, N>0, \quad for \quad \alpha<2/3$;
(iii) $D(\Psi)<0, L>0, M>0, LM=N for \quad \alpha\in(0,1)$.
Also, Q_1 is unstable if $\quad D(\Psi)<0, L<0, M<0, \alpha>2/3$.

4.2 Global Stability

In this subsection, we develop Lyapunov functionals to demonstrate the global asymptotic stability of disease-free and endemic steady states, define

$$G_1(S)=\lim_{V\to 0^+}\frac{L_1(S,V)}{V},\quad G_2(S)=\lim_{E\to 0^+}\frac{L_2(S,E)}{E},\quad G_3(S)=\lim_{I\to 0^+}\frac{L_3(S,I)}{I}. \tag{34}$$

From condition **C2**, we obtain

$$G_1(S)=\frac{\partial L_1(S,0)}{\partial V}>0,\quad G_2(S)=\frac{\partial L_2(S,0)}{\partial E}>0,\quad G_3(S)=\frac{\partial L_3(S,0)}{\partial I}>0,\quad \text{for any}\quad S>0. \tag{35}$$

Moreover,

$$\acute{G}_k(S)>0\quad \text{for all}\ \ k=1,2,3. \tag{36}$$

Therefore, the basic reproduction number can be rewritten as

$$\mathscr{R}_0=\frac{(\rho_2\delta+\rho_1 A)G_1(S_0)}{\sigma(\delta+\mu)A}+\frac{\delta G_2(S_0)}{(\delta+\mu)}+\frac{\delta G_3(S_0)}{(\delta+\mu)A}. \tag{37}$$

The following condition is required to survey the next theorem [26]:

Condition **(C4)**
(i) The supremum of $\frac{G_2(S)}{G_1(S)}$ is achieved at $S=S_0$ for all $S\in(0,S_0]$,
(ii) The supremum of $\frac{G_3(S)}{G_1(S)}$ is achieved at $S=S_0$ for all $S\in(0,S_0]$.

Theorem 4.2.1. If $\mathscr{R}_0<1$ and constraints **C1**-**C4** for system (2)-(6) are met, then Q_0 is globally asymptotic stable.

Proof. Constructing a Lyapunov functional as follows:

$$P_0=S-S_0-\int_{S_0}^{S}\frac{G_1(S_0)}{G_1(\zeta)}d\zeta+E+\frac{[\rho_2 G_1(S_0)+\sigma G_3(S_0)]}{A\sigma}I+\frac{G_1(S_0)}{\sigma}V. \tag{38}$$

We note that $P_0(S,E,I,V)>0\quad$ for all $\quad S,E,I,V>0$ and $P_0(S_0,0,0,0)=0$. We calculate $D^\alpha P_0$ along the system (2)-(6) solutions as:

$$\begin{aligned}D^\alpha P_0=&\left(1-\frac{G_1(S_0)}{G_1(S)}\right)\left[\lambda-\mu S(t)-L_1(S,V)-L_2(S,E)-L_3(S,I)\right]+L_1(S,V)+L_2(S,E)+L_3(S,I)\\&-(\delta+\mu)E+\frac{\left[\rho_2 G_1(S_0)+\sigma G_3(S_0)\right]}{A\sigma}\left[\delta E-AI\right]+\frac{G_1(S_0)}{\sigma}\left[\rho_1 E+\rho_2 I-\sigma V\right].\end{aligned} \tag{39}$$

From condition **C3** and Equation (34), we get

$$\frac{L_1(S,V)}{V} \leq \lim_{V\to 0^+} \frac{L_1(S,V)}{V} = G_1(S),$$
$$\frac{L_2(S,E)}{E} \leq \lim_{E\to 0^+} \frac{L_2(S,E)}{E} = G_2(S),$$
$$\frac{L_3(S,I)}{I} \leq \lim_{I\to 0^+} \frac{L_3(S,I)}{I} = G_3(S).$$

Then Equation (39) can be rewritten as:

$$D^\alpha P_0 \leq \Big(1-\frac{G_1(S_0)}{G_1(S)}\Big)(\lambda-\mu S) - (\delta+\mu)E + \frac{G_1(S_0)}{G_1(S)}\Big[G_1(S)V + G_2(S)E + G_3(S)I\Big] + \frac{\rho_2\delta}{A\sigma}G_1(S_0)E$$
$$+\frac{\delta}{A}G_3(S_0)E - \frac{\rho_2}{\sigma}G_1(S_0)I - G_3(S_0)I + \frac{\rho_1}{\sigma}G_1(S_0)E + \frac{\rho_2}{\sigma}G_1(S_0)I - G_1(S_0)V.$$

From condition **C4** and Equation (36), we find

$$\frac{G_1(S_0)G_2(S)}{G_1(S)} \leq G_1(S_0)\frac{G_2(S_0)}{G_1(S_0)} = G_2(S_0),$$
$$\frac{G_1(S_0)G_3(S)}{G_1(S)} \leq G_1(S_0)\frac{G_3(S_0)}{G_1(S_0)} = G_3(S_0), \quad \text{for } 0 < S \leq S_0.$$

Applying disease-free steady state condition $\lambda = \mu S_0$, we obtain

$$D^\alpha P_0 \leq \mu\Big(1-\frac{G_1(S_0)}{G_1(S)}\Big)(S_0-S) + \Big[G_2(S_0) + \frac{\rho_2\delta}{A\sigma}G_1(S_0) + \frac{\delta}{A}G_3(S_0) + \frac{\rho_1}{\sigma}G_1(S_0) - (\delta+\mu)\Big]E$$
$$= \mu\Big(1-\frac{G_1(S_0)}{G_1(S)}\Big)(S_0-S) + (\delta+\mu)(\mathscr{R}_0-1)E.$$

Conditions **C1** and **C2** imply that $\Big(1-\frac{G_1(S_0)}{G_1(S)}\Big)\Big(1-\frac{S}{S_0}\Big) \leq 0$. Clearly, if $\mathscr{R}_0 < 1$ then $D^\alpha P_0 \leq 0$ for all $S,E,I,V > 0$. Moreover, $D^\alpha P_0 = 0$ if and only if $S(t) = S_0$ and $E(t) = 0$. Let $\mathscr{F}_0 = \{(S,E,I,R,V) : D^\alpha P_0 = 0\}$ and $\hat{\mathscr{F}}_0$ be the largest invariant subset of $\mathscr{F}_0$. Therefore, the solutions of model (2)-(6) tend to $\hat{\mathscr{F}}_0$. For each element in $\hat{\mathscr{F}}_0$ we set $S(t) = S_0$ and $E(t) = 0$. Thus Equation (4) yields: $D^\alpha I = 0 = \delta E(t) - AI(t)$, hence $I(t) = 0$. From Equation (5), we have $D^\alpha R(t) = 0 = \gamma I(t) - \mu R(t)$, then $R(t) = 0$. Also from Equation (6), we conclude that $V(t) = 0$. It follows that $\hat{\mathscr{F}}_0$ contains a single point which is $(S_0,0,0,0,0)$. LaSalle's invariance principle (LIP) implies that Q_0 is globally asymptotic stable when $\mathscr{R}_0 < 1$.□

Remark 4.2.1. From conditions **C1**- **C3**, we obtain

$$\Big(L_1(S,V) - L_1(S,V^*)\Big)\left(\frac{L_1(S,V)}{V} - \frac{L_1(S,V^*)}{V^*}\right) \leq 0, \qquad S,V,V^* > 0,$$

and this leads to

$$\left(1-\frac{L_1(S,V^*)}{L_1(S,V)}\right)\left(\frac{L_1(S,V)}{L_1(S,V^*)} - \frac{V}{V^*}\right) \leq 0, \qquad S,V,V^* > 0. \tag{40}$$

Define the next functions [5]:

$$\mathscr{H}_E(S,E) = \frac{L_2(S,E)}{L_1(S,V^*)}, \qquad \mathscr{H}_I(S,I) = \frac{L_3(S,I)}{L_1(S,V^*)}. \tag{41}$$

We state the following condition:
Condition(C5)

$$\textbf{(i)}\Big(\mathscr{H}_E(S,E)-\mathscr{H}_E(S^*,E^*)\Big)\left(\frac{\mathscr{H}_E(S,E)}{E}-\frac{\mathscr{H}_E(S^*,E^*)}{E^*}\right)\leq 0,$$
$$\textbf{(ii)}\Big(\mathscr{H}_I(S,I)-\mathscr{H}_I(S^*,I^*)\Big)\left(\frac{\mathscr{H}_I(S,I)}{I}-\frac{\mathscr{H}_I(S^*,I^*)}{I^*}\right)\leq 0,$$

for all $E,E^*,I,I^*>0$ and $S\in(0,S_0)$. Hence, we get the following remark:

Remark 4.2.2.

$$\begin{aligned}\left(1-\frac{\mathscr{H}_E(S^*,E^*)}{\mathscr{H}_E(S,E)}\right)\left(\frac{\mathscr{H}_E(S,E)}{\mathscr{H}_E(S^*,E^*)}-\frac{E}{E^*}\right)\leq 0,\quad E,E^*>0,S\in(0,S_0],\\ \left(1-\frac{\mathscr{H}_I(S^*,I^*)}{\mathscr{H}_I(S,I)}\right)\left(\frac{\mathscr{H}_I(S,I)}{\mathscr{H}_I(S^*,I^*)}-\frac{I}{I^*}\right)\leq 0,\quad I,I^*>0,S\in(0,S_0].\end{aligned}\tag{42}$$

Theorem 4.2.2. For model (2)-(6) if the endemic steady state Q_1 exists, then it is globally asymptotic stable if the conditions **C1**-**C3** and **C5** are hold.

Proof. Constructing a Lyapunov function $P_1(S,E,I,V)$ as:

$$\begin{aligned}P_1=&\left(S-S^*-\int_{S^*}^{S}\frac{L_1(S^*,V^*)}{L_1(\zeta,V^*)}d\zeta\right)+\left(E-E^*-E^*\ln\left(\frac{E}{E^*}\right)\right)+\left(\frac{L_3(S^*,I^*)}{\delta E^*}+\frac{\rho_2L_1(S^*,V^*)}{(\rho_1A+\rho_2\delta)E^*}\right)\\&\times\left(I-I^*-I^*\ln\left(\frac{I}{I^*}\right)\right)+\frac{AL_1(S^*,V^*)}{(\rho_1A+\rho_2\delta)E^*}\left(V-V^*-V^*\ln\left(\frac{V}{V^*}\right)\right).\end{aligned}$$

It is clear that, $P_1(S,E,I,V)>0$ for all $S,E,I,V>0$ and $P_1(S^*,E^*,I^*,V^*)=0$. Moreover,

$$\begin{aligned}D^\alpha P_1=&\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\Big[\lambda-\mu S-L_1(S,V)-L_2(S,E)-L_3(S,I)\Big]\\&+\left(1-\frac{E^*}{E}\right)\Big[L_1(S,V)+L_2(S,E)+L_3(S,I)-(\delta+\mu)E\Big]\\&+\left(\frac{L_3(S^*,I^*)}{\delta E^*}+\frac{\rho_2L_1(S^*,V^*)}{(\rho_1A+\rho_2\delta)E^*}\right)\left(1-\frac{I^*}{I}\right)\Big[\delta E-AI\Big]\\&+\frac{AL_1(S^*,V^*)}{(\rho_1A+\rho_2\delta)E^*}\left(1-\frac{V^*}{V}\right)\Big[\rho_1E+\rho_2I-\sigma V\Big]\\=&\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\Big[\lambda-\mu S\Big]+\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\Big[L_1(S,V)+L_2(S,E)+L_3(S,I)\Big]\\&-(\delta+\mu)E-\frac{E^*}{E}\Big[L_1(S,V)+L_2(S,E)+L_3(S,I)\Big]+(\delta+\mu)E^*\\&+\left(\frac{L_3(S^*,I^*)}{\delta E^*}+\frac{\rho_2L_1(S^*,V^*)}{(\rho_1A+\rho_2\delta)E^*}\right)\left(1-\frac{I^*}{I}\right)\Big[\delta E-AI\Big]\\&+\frac{AL_1(S^*,V^*)}{(\rho_1A+\rho_2\delta)E^*}\left(1-\frac{V^*}{V}\right)\Big[\rho_1E+\rho_2I-\sigma V\Big].\end{aligned}$$

Applying the equilibrium conditions for the endemic steady state Q_1, we obtain that

$$\begin{aligned}\lambda=&\mu S^*+L_1(S^*,V^*)+L_2(S^*,E^*)+L_3(S^*,I^*),\\(\delta+\mu)E^*=&L_1(S^*,V^*)+L_2(S^*,E^*)+L_3(S^*,I^*),\\\delta E^*=&AI^*,\\\sigma V^*=&\rho_1E^*+\rho_2I^*.\end{aligned}$$

Then, we get

$$
\begin{aligned}
D^{\alpha}P_1 =& \mu S^*\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\left(1-\frac{S}{S^*}\right)+2L_1(S^*,V^*)+2L_2(S^*,E^*)+2L_3(S^*,I^*)\\
&-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\left[L_1(S^*,V^*)+L_2(S^*,E^*)+L_3(S^*,I^*)\right]+\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\left[L_1(S,V)+L_2(S,E)+L_3(S,I)\right]\\
&-\frac{E}{E^*}\left[L_1(S^*,V^*)+L_2(S^*,E^*)+L_3(S^*,I^*)\right]-\frac{E^*}{E}\left[L_1(S,V)+L_2(S,E)+L_3(S,I)\right]\\
&+\delta E^*\left(\frac{L_3(S^*,I^*)}{\delta E^*}+\frac{\rho_2 L_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)E^*}\right)\left(1+\frac{E}{E^*}-\frac{I}{I^*}-\frac{I^*E}{IE^*}\right)\\
&+\frac{AL_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)E^*}\left[\rho_1 E+\rho_2 I-\frac{V}{V^*}\left(\rho_1 E^*+\frac{\rho_2\delta E^*}{A}\right)-\rho_1 E\frac{V^*}{V}-\rho_2 I\frac{V^*}{V}+\rho_1 E^*+\frac{\rho_2\delta E^*}{A}\right].
\end{aligned}
$$

Rearranging the last equation and add same terms, we obtain

$$
\begin{aligned}
D^{\alpha}P_1 =& \mu S^*\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\left(1-\frac{S}{S^*}\right)+L_1(S^*,V^*)\left[3-\frac{V}{V^*}-\frac{E}{E^*}-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{L_1(S,V)}{L_1(S,V^*)}-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}\right]\\
&+L_2(S^*,E^*)\left[2-\frac{E}{E^*}-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{E^*L_2(S,E)}{EL_2(S^*,E^*)}+\frac{L_1(S^*,V^*)L_2(S,E)}{L_1(S,V^*)L_2(S^*,E^*)}\right]\\
&+L_3(S^*,I^*)\left[2-\frac{E}{E^*}-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{E^*L_3(S,I)}{EL_3(S^*,I^*)}+\frac{L_1(S^*,V^*)L_3(S,I)}{L_1(S,V^*)L_3(S^*,I^*)}\right]\\
&+L_3(S^*,I^*)\left[1+\frac{E}{E^*}-\frac{I}{I^*}-\frac{EI^*}{E^*I}\right]+\frac{\rho_2\delta L_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)}\left[1+\frac{E}{E^*}-\frac{I}{I^*}-\frac{EI^*}{E^*I}\right]\\
&+\frac{\rho_1 AL_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)}\frac{E}{E^*}+\frac{\rho_2\delta L_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)}\frac{I}{I^*}-\frac{\rho_1 AL_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)}\frac{EV^*}{E^*V}-\frac{\rho_2\delta L_1(S^*,V^*)}{(\rho_1 A+\rho_2\delta)}\frac{IV^*}{I^*V}.
\end{aligned}
$$

The last Equation can be simplified as:

$$
\begin{aligned}
D^{\alpha}P_1 =& \mu S^*\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\left(1-\frac{S}{S^*}\right)+\frac{\rho_1 AL_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[3-\frac{V}{V^*}-\frac{E}{E^*}-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{L_1(S,V)}{L_1(S,V^*)}\right.\\
&\left.-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}\right]+\frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[4-\frac{V}{V^*}-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{L_1(S,V)}{L_1(S,V^*)}-\frac{EI^*}{E^*I}-\frac{IV^*}{I^*V}-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}\right]\\
&+L_2(S^*,E^*)\left[3-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{E^*L_2(S,E)}{EL_2(S^*,E^*)}-\frac{EL_2(S^*,V^*)L_1(S,V^*)}{E^*L_2(S,E)L_1(S^*,V^*)}\right]\\
&+L_3(S^*,I^*)\left[4-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EI^*}{E^*I}-\frac{IL_1(S,V^*)L_3(S^*,I^*)}{I^*L_1(S^*,V^*)L_3(S,I)}-\frac{E^*L_3(S,I)}{EL_3(S^*,I^*)}\right]\\
&+L_2(S^*,E^*)\left[-1-\frac{E}{E^*}+\frac{L_1(S^*,V^*)L_2(S,E)}{L_1(S,V^*)L_2(S^*,E^*)}+\frac{EL_2(S^*,V^*)L_1(S,V^*}{E^*L_2(S,E)L_1(S^*,V^*)}\right]\\
&+L_3(S^*,I^*)\left[-1+\frac{L_1(S^*,V^*)L_3(S,I)}{L_1(S,V^*)L_3(S^*,I^*)}-\frac{I}{I^*}+\frac{IL_1(S,V^*)L_3(S^*,I^*)}{I^*L_1(S^*,V^*)L_3(S,I)}\right].
\end{aligned}
$$

Adding and subtracting terms, it follows that

$$
\begin{aligned}
D^{\alpha}P_1 =& \mu S^*\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\left(1-\frac{S}{S^*}\right)+\frac{\rho_1 A L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[4-\frac{EV^*}{E^*V}-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}\right.\\
&\left.-\frac{VL_1(S,V^*)}{V^*L_1(S,V)}\right]+\frac{\rho_1 A L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left(1-\frac{L_1(S,V^*)}{L_1(S,V)}\right)\left(\frac{L_1(S,V)}{L_1(S,V^*)}-\frac{V}{V^*}\right)\\
&+\frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[5-\frac{EI^*}{E^*I}-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{IV^*}{I^*V}-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}-\frac{VL_1(S,V^*)}{V^*L_1(S,V)}\right]\\
&+\frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left(1-\frac{L_1(S,V^*)}{L_1(S,V)}\right)\left(\frac{L_1(S,V)}{L_1(S,V^*)}-\frac{V}{V^*}\right)\\
&+L_2(S^*,E^*)\left[3-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{E^*L_2(S,E)}{EL_2(S^*,E^*)}-\frac{EL_2(S^*,V^*)L_1(S,V^*)}{E^*L_2(S,E)L_1(S^*,V^*)}\right]\\
&+L_3(S^*,I^*)\left[4-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EI^*}{E^*I}-\frac{E^*L_3(S,I)}{EL_3(S^*,I^*)}-\frac{IL_1(S,V^*)L_3(S^*,I^*)}{I^*L_1(S^*,V^*)L_3(S,I)}\right]\\
&+L_2(S^*,E^*)\left(1-\frac{L_1(S,V^*)L_2(S^*,E^*)}{L_1(S^*,V^*)L_2(S,E)}\right)\left(\frac{L_1(S^*,V^*)L_2(S,E)}{L_1(S,V^*)L_2(S^*,E^*)}-\frac{E}{E^*}\right)\\
&+L_3(S^*,I^*)\left(1-\frac{L_1(S,V^*)L_3(S^*,I^*)}{L_1(S^*,V^*)L_3(S,I)}\right)\left(\frac{L_1(S^*,V^*)L_3(S,I)}{L_1(S,V^*)L_3(S^*,I^*)}-\frac{I}{I^*}\right).
\end{aligned}
$$

We can rewrite as:

$$
\begin{aligned}
D^{\alpha}P_1 =& \mu S^*\left(1-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}\right)\left(1-\frac{S}{S^*}\right)+\frac{\rho_1 A L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[4-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EV^*}{E^*V}-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}\right.\\
&\left.-\frac{VL_1(S,V^*)}{V^*L_1(S,V)}\right]+L_1(S^*,V^*)\left(1-\frac{L_1(S,V^*)}{L_1(S,V)}\right)\left(\frac{L_1(S,V)}{L_1(S,V^*)}-\frac{V}{V^*}\right)\\
&+\frac{\rho_2\delta L_1(S^*,V^*)}{\rho_1 A+\rho_2\delta}\left[5-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EI^*}{E^*I}-\frac{IV^*}{I^*V}-\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}-\frac{VL_1(S,V^*)}{V^*L_1(S,V)}\right]\\
&+L_2(S^*,E^*)\left[3-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{E^*L_2(S,E)}{EL_2(S^*,E^*)}-\frac{EL_2(S^*,V^*)L_1(S,V^*)}{E^*L_2(S,E)L_1(S^*,V^*)}\right]\\
&+L_3(S^*,I^*)\left[4-\frac{L_1(S^*,V^*)}{L_1(S,V^*)}-\frac{EI^*}{E^*I}-\frac{E^*L_3(S,I)}{EL_3(S^*,I^*)}-\frac{IL_1(S,V^*)L_3(S^*,I^*)}{I^*L_1(S^*,V^*)L_3(S,I)}\right]\\
&+L_2(S^*,E^*)\left(1-\frac{L_1(S,V^*)L_2(S^*,E^*)}{L_1(S^*,V^*)L_2(S,E)}\right)\left(\frac{L_1(S^*,V^*)L_2(S,E)}{L_1(S,V^*)L_2(S^*,E^*)}-\frac{E}{E^*}\right)\\
&+L_3(S^*,I^*)\left(1-\frac{L_1(S,V^*)L_3(S^*,I^*)}{L_1(S^*,V^*)L_3(S,I)}\right)\left(\frac{L_1(S^*,V^*)L_3(S,I)}{L_1(S,V^*)L_3(S^*,I^*)}-\frac{I}{I^*}\right).
\end{aligned}
$$

Using the geometrical and arithmetical means relationship, we obtain

$$
\begin{aligned}
4 &\leq \frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{EV^*}{E^*V}+\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}+\frac{VL_1(S,V^*)}{V^*L_1(S,V)},\\
5 &\leq \frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{EI^*}{E^*I}+\frac{IV^*}{I^*V}+\frac{E^*L_1(S,V)}{EL_1(S^*,V^*)}+\frac{VL_1(S,V^*)}{V^*L_1(S,V)},\\
3 &\leq \frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{E^*L_2(S,E)}{EL_2(S^*,E^*)}+\frac{EL_2(S^*,V^*)L_1(S,V^*)}{E^*L_2(S,E)L_1(S^*,V^*)},\\
4 &\leq \frac{L_1(S^*,V^*)}{L_1(S,V^*)}+\frac{EI^*}{E^*I}+\frac{E^*L_3(S,I)}{EL_3(S^*,I^*)}+\frac{IL_1(S,V^*)L_3(S^*,I^*)}{I^*L_1(S^*,V^*)L_3(S,I)}.
\end{aligned}
$$

Also, from condition **C5**, we have

$$0 \geq \left(1-\frac{L_1(S,V^*)L_2(S^*,E^*)}{L_1(S^*,V^*)L_2(S,E)}\right)\left(\frac{L_1(S^*,V^*)L_2(S,E)}{L_1(S,V^*)L_2(S^*,E^*)}-\frac{E}{E^*}\right)$$
$$0 \geq \left(1-\frac{L_1(S,V^*)L_3(S^*,I^*)}{L_1(S^*,V^*)L_3(S,I)}\right)\left(\frac{L_1(S^*,V^*)L_3(S,I)}{L_1(S,V^*)L_3(S^*,I^*)}-\frac{I}{I^*}\right).$$

We conclude that $D^\alpha P_1(t) \leq 0$ and $D^\alpha P_1(t) = 0$ at the point $Q_1 = (S^*,E^*,I^*,R^*,V^*)$. Let $\acute{\mathscr{F}}_1$ be the largest invariant subset of the set $\{(S,E,I,R,V) : D^\alpha P_1(t) = 0\}$. Thus, the solutions of the model tend to $\acute{\mathscr{F}}_1$. It is clear that $\acute{\mathscr{F}}_1$ contains unique point, which is Q_1. The globally asymptotically stable of Q_1 follows from LaSalle's invariance principle (LIP).□

5 Numerical Simulations

In this section, we introduce the following COVID-19 model example as a special case of system (2)-(6):

$$\begin{aligned}
D^\alpha S(t) &= \lambda - \mu S - \frac{S}{1+\varepsilon S}\left(\frac{\beta_1 V}{1+\kappa_1 V}+\frac{\beta_2 E}{1+\kappa_2 E}+\frac{\beta_3 I}{1+\kappa_3 I}\right),\\
D^\alpha E(t) &= \frac{S}{1+\varepsilon S}\left(\frac{\beta_1 V}{1+\kappa_1 V}+\frac{\beta_2 E}{1+\kappa_2 E}+\frac{\beta_3 I}{1+\kappa_3 I}\right)-(\delta+\mu)E,\\
D^\alpha I(t) &= \delta E-(\omega+\gamma+\mu)I,\\
D^\alpha R(t) &= \gamma I-\mu R,\\
D^\alpha V(t) &= \rho_1 E+\rho_2 I-\sigma V.
\end{aligned} \tag{43}$$

The three functions for the transmission rates of infection are given by:

$$L_1(S,V)=\frac{\beta_1 SV}{(1+\varepsilon S)(1+\kappa_1 V)},\quad L_2(S,E)=\frac{\beta_2 SE}{(1+\varepsilon S)(1+\kappa_2 E)},\quad L_3(S,I)=\frac{\beta_3 SI}{(1+\varepsilon S)(1+\kappa_3 I)}. \tag{44}$$

The parameters β_j indicate maximum transmission rates and κ_j allow transmission speeds to be adjusted and are all positive constants, where $j = 1,2,3$. The parameters $\lambda,\mu,\varepsilon,\delta,\omega,\gamma,\rho_1,\rho_2$ and σ are positive constants.

Checking the conditions **C1**-**C5**

(C1) Obviously,

$$\begin{aligned}
&L_1(S,V)>0,\ \ L_2(S,E)>0,\ \ L_3(S,I)>0 \quad for\ all\ \ S,E,I,V>0,\\
&L_1(S,0)=L_2(S,0)=L_3(S,0)=0 \quad for\ S>0,\\
&L_1(0,V)=L_2(0,E)=L_3(0,I)=0 \quad for\ all\ \ \ E,I,V>0.
\end{aligned}$$

(C2)

$$\begin{aligned}
&\frac{\partial L_1(S,V)}{\partial S}=\frac{\beta_1 V}{(1+\varepsilon S)^2(1+\kappa_1 V)}>0,\quad \frac{\partial L_2(S,E)}{\partial S}=\frac{\beta_2 E}{(1+\varepsilon S)^2(1+\kappa_2 E)}>0\\
&\frac{\partial L_3(S,I)}{\partial S}=\frac{\beta_3 I}{(1+\varepsilon S)^2(1+\kappa_3 I)}>0,\quad \frac{\partial L_1(S,V)}{\partial V}=\frac{\beta_1 S}{(1+\varepsilon S)(1+\kappa_1 V)^2}>0,\\
&\frac{\partial L_2(S,E)}{\partial E}=\frac{\beta_2 S}{(1+\varepsilon S)(1+\kappa_2 E)^2}>0,\quad \frac{\partial L_3(S,I)}{\partial I}=\frac{\beta_3 S}{(1+\varepsilon S)(1+\kappa_3 I)^2}>0,\\
&\frac{\partial L_1(S,0)}{\partial V}=\frac{\beta_1 S}{(1+\varepsilon S)}>0,\quad \frac{\partial L_2(S,0)}{\partial E}=\frac{\beta_2 S}{(1+\varepsilon S)}>0,\quad \frac{\partial L_3(S,0)}{\partial I}=\frac{\beta_3 S}{(1+\varepsilon S)}>0,\quad for\ all\ \ S,E,I,V>0,
\end{aligned}$$

furthermore,

$$\begin{aligned}
&\frac{d}{dS}\left(\frac{\partial L_1(S,0)}{\partial V}\right)=\frac{\beta_1}{(1+\varepsilon S)^2}>0,\quad \frac{d}{dS}\left(\frac{\partial L_2(S,0)}{\partial E}\right)=\frac{\beta_2}{(1+\varepsilon S)^2}>0,\\
&\frac{d}{dS}\left(\frac{\partial L_3(S,0)}{\partial I}\right)=\frac{\beta_3}{(1+\varepsilon S)^2}>0,\quad for\ all\ \ S,E,I,V>0.
\end{aligned}$$

(C3)

$$\frac{\partial}{\partial V}\Big(\frac{L_1(S,V)}{V}\Big)=\frac{\partial}{\partial V}\Big(\frac{\beta_1 S}{(1+\varepsilon S)(1+\kappa_1 V)}\Big)=\frac{-\kappa_1\beta_1 S}{(1+\varepsilon S)(1+\kappa_1 V)^2}\le 0,$$
$$\frac{\partial}{\partial E}\Big(\frac{L_2(S,E)}{E}\Big)=\frac{\partial}{\partial E}\Big(\frac{\beta_2 S}{(1+\varepsilon S)(1+\kappa_2 E)}\Big)=\frac{-\kappa_2\beta_2 S}{(1+\varepsilon S)(1+\kappa_2 E)^2}\le 0,$$
$$\frac{\partial}{\partial I}\Big(\frac{L_3(S,I)}{I}\Big)=\frac{\partial}{\partial I}\Big(\frac{\beta_3 S}{(1+\varepsilon S)(1+\kappa_3 I)}\Big)=\frac{-\kappa_3\beta_3 S}{(1+\varepsilon S)(1+\kappa_3 I)^2}\le 0,\quad \textit{for all } S,E,I,V>0.$$

(C4) We have

$$G_1(S)=\frac{\partial L_1(S,0)}{\partial V}=\frac{\beta_1 S}{(1+\varepsilon S)},\qquad G_2(S)=\frac{\partial L_2(S,0)}{\partial E}=\frac{\beta_2 S}{(1+\varepsilon S)},$$
$$G_3(S)=\frac{\partial L_3(S,0)}{\partial I}=\frac{\beta_3 S}{(1+\varepsilon S)}.$$

Thus, $\dfrac{G_2(S)}{G_1(S)}=\dfrac{\beta_2}{\beta_1}$ and $\dfrac{G_3(S)}{G_1(S)}=\dfrac{\beta_3}{\beta_1}$.

(C5)

$$\mathscr{H}_E(S,E)=\frac{L_2(S,E)}{L_1(S,V^*)}=\frac{\beta_2(1+\kappa_1V^*)E}{\beta_1(1+\kappa_2E)V^*},\quad \mathscr{H}_E(S^*,E^*)=\frac{L_2(S^*,E^*)}{L_1(S^*,V^*)}=\frac{\beta_2(1+\kappa_1V^*)E^*}{\beta_1(1+\kappa_2E^*)V^*},$$
$$\mathscr{H}_I(S,I)=\frac{L_3(S,I)}{L_1(S,V^*)}=\frac{\beta_3(1+\kappa_1V^*)I}{\beta_1(1+\kappa_3I)V^*},\quad \mathscr{H}_I(S^*,I^*)=\frac{L_3(S^*,I^*)}{L_1(S^*,V^*)}=\frac{\beta_3(1+\kappa_1V^*)I^*}{\beta_1(1+\kappa_3I^*)V^*},$$

$$\Big(\mathscr{H}_E(S,E)-\mathscr{H}_E(S^*,E^*)\Big)\left(\frac{\mathscr{H}_E(S,E)}{E}-\frac{\mathscr{H}_E(S^*,E^*)}{E^*}\right)=-\frac{\kappa_2\beta_2^2(1+\kappa_1V^*)^2(E-E^*)^2}{\beta_1^2(V^*)^2(1+\kappa_2E^*)^2(1+\kappa_2E)^2}\le 0,$$
$$\Big(\mathscr{H}_I(S,I)-\mathscr{H}_I(S^*,I^*)\Big)\left(\frac{\mathscr{H}_I(S,I)}{I}-\frac{\mathscr{H}_I(S^*,I^*)}{I^*}\right)=-\frac{\kappa_3\beta_3^2(1+\kappa_1V^*)^2(I-I^*)^2}{\beta_1^2(V^*)^2(1+\kappa_3I^*)^2(1+\kappa_3I)^2}\le 0,$$

for all $E,I>0$, $S\in(0,S_0)$.

As a result, the validity of the conditions **C1**-**C5** ensures that the results of global stability shown in Theorems 4.2.1 and 4.2.2 are true in this example. Therefore, the basic reproduction number of model (43) is:

$$\begin{aligned}\mathscr{R}_0=&\frac{S_0}{\sigma A(\delta+\mu)(1+\varepsilon S_0)}\Big((\rho_2\delta+\rho_1A)\beta_1+\sigma A\beta_2+\sigma\delta\beta_3\Big)\\ =&\mathscr{R}_{01}+\mathscr{R}_{02}+\mathscr{R}_{03}.\end{aligned}\tag{45}$$

Specifically,

$$\mathscr{R}_{01}=\frac{\beta_1(\rho_2\delta+\rho_1A)S_0}{\sigma A(\delta+\mu)(1+\varepsilon S_0)},\quad \mathscr{R}_{02}=\frac{\beta_2S_0}{(\delta+\mu)(1+\varepsilon S_0)},\quad \mathscr{R}_{03}=\frac{\beta_3\delta S_0}{A(\delta+\mu)(1+\varepsilon S_0)}.\tag{46}$$

Case (I): In this case, we run computational simulations for real-world data beginning from June 5, 2021 to September 11, 2021. We assume that the global influx and death rates in 2021 are 18.077 and 7.612 per 1000 people, respectively as the same as 2020 [27], [28]. On June 5, 2021, the total population of the world was N = 7794798739. So, $\lambda=\frac{18.077\times N}{1000\times 365}=3.8605\times10^5$ and $\mu=\frac{7.612}{1000\times365}=2.0855\times10^{-5}$. According to [1], the initial condition is set as $I(0)=13032161$, $R(0)=157029051$ and we assume $E(0)=3.3\times10^7$ then from $S(0)+E(0)+I(0)+R(0)=N(0)$, we have $S(0)=7.59174\times10^9$. Figure 2 depicts the fitted curve and the reported global cumulative number of COVID-19 from June 5 to September 11 2021. A comparison is also provided in Figure 3 between the integer-order one when $\alpha=1$, fractional order model with $\alpha=0.95$, and the actual active infected cases with COVID-19 in the world at the same period. The achieved results show that the response of the fractional-order model matches real data and show the benefit of using the derivative of the fractional-order instead of the integer-order in conjunction with the results of Table 1.

Table 1: Worldwide approximate parameters for the COVID-19 model (43).

Parameter	Value	Source	Parameter	Value	Source
β_1	$1\times10^{-}11$	[17],[27]	ρ_1	0.2	Assumed
β_2	$3\times10^{-}11$	[17],[27]	ρ_2	2	Assumed
β_3	$1\times10^{-}11$	[17],[27]	κ_1	1×10^{-9}	fitting by data
ω	0.0003	fitting by data	κ_2	2.2×10^{-10}	fitting by data
δ	0.0000435	fitting by data	κ_3	5×10^{-11}	fitting by data
γ	0.009	fitting by data	ε	0.00064	fitting by data
σ	1	[17]			

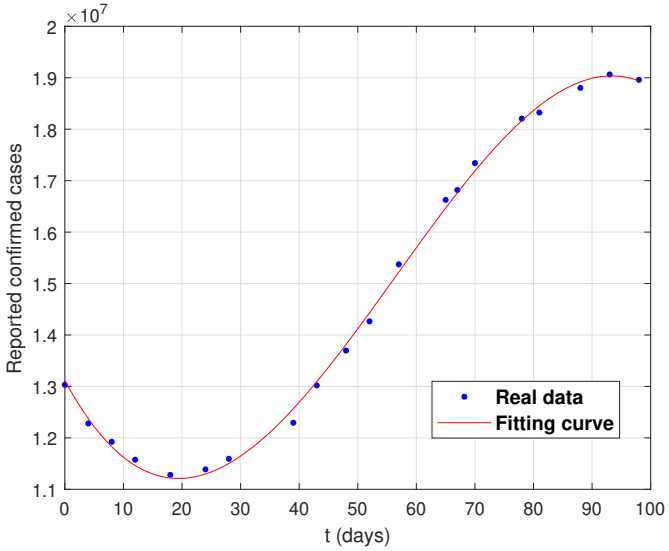

Fig. 2: The fitted curve and the reported COVID-19 cases in the world from June 5 to September 11, 2021.

Case (II): To study the effect of environmental reservoirs on the COVID-19 transmissions, we will assume new set of initial conditions as $I(0) = 24545$, $R(0) = 907$ and $S(0) = 7.6100\times10^9$ [27] and a set of parameters in Table 2. We can calculate the basic reproduction number $\mathscr{R}_0 = 5.3837$ using the parameter values from Table 2. We find, in particular, that $\mathscr{R}_{01} = 1.5558$, $\mathscr{R}_{02} = 2.5304$, $\mathscr{R}_{03} = 1.2974$.

The largest of these three components $\mathscr{R}_{02}$ comes from exposed-to-susceptible transmission, since exposed persons display no symptoms and can transmit the disease to others easily in close proximity, even without their awareness. In the meantime, if we suppose that the infected to susceptible transmission rate is equal to transmission rate from environmental factors to susceptible persons, we find that the rate of impact of transmission from the environment is greater than its effect from infected persons on basic reproduction number, likely as a result of the symptomatic infected persons being isolated. Furthermore, this indicating that the environmental reservoir plays a significant role in the overall risk for infection.

6 Conclusion and discussion

In this paper, a mathematical model has been introduced to examine the ongoing novel coronavirus pandemic. We have proposed a fractional-order *SEIRV* epidemic model that uses general infection rates and incorporates the environmental reservoir into the dynamics of disease transmission that alter with the epidemiological state and environmental factors. We started by applying the Caputo derivative to create a general *SEIRV* model that is appropriate for initial-value problems. We have shown the system's feasibility area and calculated its steady states. The basic reproduction number $\mathscr{R}_0$ is obtained using the next generation technique, and it is made up of three components that represent the three different mechanisms of infection, namely, exposed people, infected people, and environmental reservoirs, to susceptible people. Based on characteristic equations and suitable Lyapunov functions, we examined local and global stability

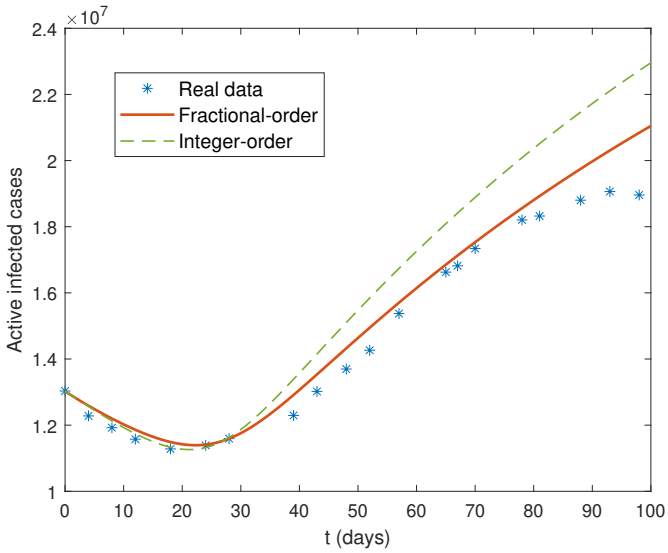

Fig. 3: Comparison between the results of the fractional-order derivative $\alpha = 0.9$ and the integer-order derivative $\alpha = 1$ and with real data.

Table 2: Approximate parameters for the COVID-19 model (43).

Parameter	Value	Parameter	Value
λ	3.7690×10^5	ρ_1	2.3
μ	2.0855×10^{-5}	ρ_2	0.2
β_1	$1.012 \times 10^- 11$	κ_1	1.01×10^{-4}
β_2	$2.11 \times 10^- 11$	κ_2	1.01×10^{-4}
β_3	$0.82 \times 10^- 11$	κ_3	1.01×10^{-4}
δ	$1/7$	ε	0.00064
ω	0.034	σ	2
γ	$1/15$		

analyzes of equilibrium points in detail. In the disease-free case, the underlying model is locally and globally asymptotically stable when $\mathscr{R}_0 < 1$, when $\mathscr{R}_0 > 1$, the positive endemic steady state is both locally and globally asymptotically stable. Based on real-world data, numerical simulations of an example with general infection functions have been presented. The results of our simulations show that our model can be applied to the COVID-19 outbreak in the world as it fits the supplied data really well. We can estimate the fundamental reproduction number using data fitting. Moreover, the environmental reservoir is important in shaping the outbreak risk. In the numerical portion, we also looked at the benefits of applying the fractional-order instead of the integer-order by comparing the results of our model with real data and integer-order and fractional-order, we obtained that the fractional-order has the better result that depends on the historical states of the disease and increases the stability region of the solution.

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