Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/100107

Exploring the Solvability for Coupled System of Nonlinear Fractional Langevin Equations

Hamdy M. Ahmed and Ahmed S. Ghanem*

Higher Institute of Engineering, El-Shorouk Academy, P. O. 3 El-Shorouk City, Cairo, Egypt

Received: 2 Jun. 2022, Revised: 18 Oct. 2022, Accepted: 20 Nov. 2022
Published online: 1 Jan. 2024

Abstract: In this article, we investigate the existence and uniqueness of the solution to a nonlinear fractional coupled system of Langevin equations with boundary conditions. The required results are established based on fixed point theorem and fractional calculus. Finally, an example is given to illustrate the obtained results.

Keywords: Coupled system of fractional Langevin equations, boundary value problems, fixed point theorem.

1 Introduction

Numerous physicists, mathematicians, and engineers have been interested in fractional calculus, leading to substantial advances in the theory and applications of fractional systems. Coupled systems of fractional order differential equations are utilized to address physical problems that integer order differential equations cannot (see [1,2,3,4,5,6]). Many phenomena can be described by Langevin equation [7]. At present, many researchers have done excellent work on fractional order Langevin equation (see [8,9,10,11,12,13,14,15]).
In this paper, we discuss the existence and uniqueness of the solution to nonlinear fractional coupled system of Langevin equations in the following form:

$$\begin{cases} {}^c\mathbb{D}_{0+}^{\ell_1}({}^c\mathbb{D}_{0+}^{j_1}+\bar{h}_1)\Im_1(\varsigma)=\wp_1(\varsigma,\Im_1(\varsigma),\Im_2(\varsigma)),\ \varsigma\in J:=[0,\mathbb{T}], 1<j_1\leq 2,\ 1<\ell_1\leq 2 \\ {}^c\mathbb{D}_{0+}^{\ell_2}({}^c\mathbb{D}_{0+}^{j_2}+\bar{h}_2)\Im_2(\varsigma)=\wp_2(\varsigma,\Im_1(\varsigma),\Im_2(\varsigma)),\ \varsigma\in J:=[0,\mathbb{T}], 1<j_2\leq 2,\ 1<\ell_2\leq 2 \end{cases} \tag{1}$$

subject to the boundary conditions:

$$\begin{cases} \Im_1(0)=0,\ \Im_1(\mathbb{T})=\delta_1\Im_1(\eta_1),\ \Im_1'(\mathbb{T})=\varepsilon_1\Im_1'(\xi_1) \\ \Im_2(0)=0,\ \Im_2(\mathbb{T})=\delta_2\Im_2(\eta_2),\ \Im_2'(\mathbb{T})=\varepsilon_2\Im_2'(\xi_2) \end{cases} \tag{2}$$

where ${}^c\mathbb{D}_{0+}^{\ell_1}, {}^c\mathbb{D}_{0+}^{\ell_2}, {}^c\mathbb{D}_{0+}^{j_1}, {}^c\mathbb{D}_{0+}^{j_2}$ denote the Caputo fractional derivative of order $\ell_1,\ \ell_2,\ j_1$ and j_2 respectively, $\wp_1,\ \wp_2$: $[0,\mathbb{T}]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions, $\bar{h}_1,\ \bar{h}_2\in\mathbb{R}$ are the dissipative parameters and $\delta_i,\ \varepsilon_i\in\mathbb{R}$ and $0<\eta_i,\ \xi_i<1$ for $i=1,2$.

2 Preliminaries

In this part, we present the primary information's, concepts, and lemmas needed to organize the major outcomes of our study.

* Corresponding author e-mail: hamdy_17eg@yahoo.com

Definition 1.(see [16, 17]). *The fractional integral of order $\ell > 0$ with the lower limit zero for a function $\mathbb{F}$ is given by*

$$I^{\ell}\mathbb{F}(\varsigma) = \frac{1}{\Gamma(\ell)}\int_0^{\varsigma}\frac{\mathbb{F}(\vartheta)}{(\varsigma-\vartheta)^{1-\ell}}d\vartheta, \;\; \varsigma > 0 \tag{3}$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 2.(see [17]). *The Caputo derivative of order ℓ with the lower limit zero for a function $\mathbb{F}$ is given by*

$${}^{c}\mathbb{D}^{\ell}\mathbb{F}(\varsigma) = \frac{1}{\Gamma(n-\ell)}\int_0^{\varsigma}\frac{\mathbb{F}^{(n)}(\vartheta)}{(\varsigma-\vartheta)^{\ell+1-n}}d\vartheta = I^{n-\ell}\mathbb{F}^{(n)}(\varsigma), \;\; \varsigma > 0, \; 0 \le n-1 < \ell < n. \tag{4}$$

Lemma 1.*Let $\wp_1, \wp_2 \in C([0,\mathbb{T}],\mathbb{R})$ then the solution of linear system (1) is equivalent to the following system:*

$$\begin{aligned}\Im_i(\varsigma) = & \left(\frac{\rho_i\lambda_i - e_i\mu_i}{k_i\rho_i - e_id_i}\right)\frac{\varsigma^{j_i}}{\Gamma(j_i+1)} + \left(\frac{k_i\mu_i - d_i\lambda_i}{k_i\rho_i - e_id_i}\right)\frac{\varsigma^{j_i+1}}{\Gamma(j_i+2)} \\ & + \int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{\ell_i+j_i-1}}{\Gamma(\ell_i+j_i)}\wp_i(\vartheta)d\vartheta - \bar{h}_i\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{j_i-1}}{\Gamma(j_i)}\Im_i(\vartheta)d\vartheta, \;\; i=1,2\end{aligned} \tag{5}$$

where

$$k_i = \frac{\mathbb{T}^{j_i} - \delta_i\eta_i^{j_i}}{\Gamma(j_i+1)}, \;\; e_i = \frac{\mathbb{T}^{j_i+1} - \delta_i\eta_i^{j_i+1}}{\Gamma(j_i+2)}, \;\; d_i = \frac{\mathbb{T}^{j_i-1} - \varepsilon_i\xi_i^{j_i-1}}{\Gamma(j_i)}, \;\; \rho_i = \frac{\mathbb{T}^{j_i} - \varepsilon_i\xi_i^{j_i}}{\Gamma(j_i+1)},$$

$$\begin{aligned}\lambda_i = & \frac{1}{\Gamma(\ell_i+j_i)}\left[\delta_i\int_0^{\eta_i}(\eta_i-\vartheta)^{\ell_i+j_i-1}\wp_i(\vartheta)d\vartheta - \int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{\ell_i+j_i-1}\wp_i(\vartheta)d\vartheta\right] \\ & - \frac{\bar{h}_i}{\Gamma(j_i)}\left[\delta_i\int_0^{\eta_i}(\eta_i-\vartheta)^{j_i-1}\Im_i(\vartheta)d\vartheta - \int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{j_i-1}\Im_i(\vartheta)d\vartheta\right],\end{aligned}$$

$$\begin{aligned}\mu_i = & \frac{1}{\Gamma(\ell_i+j_i-1)}\left[\varepsilon_i\int_0^{\xi_i}(\xi_i-\vartheta)^{\ell_i+j_i-2}\wp_i(\vartheta)d\vartheta - \int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{\ell_i+j_i-2}\wp_i(\vartheta)d\vartheta\right] \\ & - \frac{\bar{h}_i}{\Gamma(j_i-1)}\left[\varepsilon_i\int_0^{\xi_i}(\xi_i-\vartheta)^{j_i-2}\Im_i(\vartheta)d\vartheta - \int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{j_i-2}\Im_i(\vartheta)d\vartheta\right].\end{aligned}$$

Proof. We know that

$$({}^{c}\mathbb{D}_{0+}^{j_1} + \bar{h}_1)\Im_1(\varsigma) = a_1 + b_1\varsigma + \int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{\ell_1-1}}{\Gamma(\ell_1)}\wp_1(\vartheta)d\vartheta$$

then

$${}^{c}\mathbb{D}_{0+}^{j_1}\Im_1(\varsigma) = a_1 + b_1\varsigma + \int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{\ell_1-1}}{\Gamma(\ell_1)}\wp_1(\vartheta)d\vartheta - \bar{h}_1\Im_1(\varsigma).$$

Hence,

$$\Im_1(\varsigma) = \frac{a_1\varsigma^{j_1}}{\Gamma(j_1+1)} + \frac{b_1\varsigma^{(j_1+1)}}{\Gamma(j_1+2)} + \int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{\ell_1+j_1-1}}{\Gamma(\ell_1+j_1)}\wp_1(\vartheta)d\vartheta - \bar{h}_1\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{j_1-1}}{\Gamma(j_1)}\Im_1(\vartheta)d\vartheta + \kappa_1. \tag{6}$$

Also,

$$\Im_2(\varsigma)=\frac{a_2\varsigma^{j_2}}{\Gamma(j_2+1)}+\frac{b_2\varsigma^{(j_2+1)}}{\Gamma(j_2+2)}+\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{\ell_2+j_2-1}}{\Gamma(\ell_2+j_2)}\wp_2(\vartheta)d\vartheta-\bar{h}_2\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{j_2-1}}{\Gamma(j_2)}\Im_2(\vartheta)d\vartheta+\kappa_2. \tag{7}$$

where a_i,b_i,κ_i $i=1,2$ are real constant.

From (2), we have

$$\kappa_i=0,\ a_i=\frac{\rho_i\lambda_i-e_i\mu_i}{k_i\rho_i-e_id_i},\quad b_i=\frac{k_i\mu_i-d_i\lambda_i}{k_i\rho_i-e_id_i},\ i=1,2.$$

Inserting the values of $a_i,\ b_i,\kappa_i,\ i=1,2$ in (6) and (7), we get the solution (5).

3 Existence and Uniqueness

Let us introduce the space $\mathbb{Z}=\{\Im(\varsigma)|\Im(\varsigma)\in C([0,\mathbb{T}],\mathbb{R})\}$ through the norm $\|\Im\|=\sup\{\Im(\varsigma),\varsigma\in[0,\mathbb{T}]\}$. Obviously, $(\mathbb{Z},\|\cdot\|)$ is a Banach space. Then the product space $(\mathbb{Z}\times\mathbb{Z},\|(\Im_1,\Im_2)\|)$ is also a Banach space equipped with $\|(\Im_1,\Im_2)\|=\|\Im_1\|+\|\Im_2\|$.

From Lemma 1, we define $\mho:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ by

$$\mho(\Im_1,\Im_2)(\varsigma)=\begin{pmatrix}\mho_1(\Im_1,\Im_2)(\varsigma)\\ \mho_2(\Im_1,\Im_2)(\varsigma)\end{pmatrix},$$

where

$$\begin{aligned}\mho_i(\Im_1,\Im_2)(\varsigma)=&\left(\frac{\rho_i\lambda_{ii}-e_i\mu_{ii}}{k_i\rho_i-e_id_i}\right)\frac{\varsigma^{j_i}}{\Gamma(j_i+1)}+\left(\frac{k_i\mu_{ii}-d_i\lambda_{ii}}{k_i\rho_i-e_id_i}\right)\frac{\varsigma^{j_i+1}}{\Gamma(j_i+2)}\\&+\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{\ell_i+j_i-1}}{\Gamma(\ell_i+j_i)}\wp_i(s,\Im_1(\vartheta),\Im_2(\vartheta))d\vartheta-\bar{h}_i\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{j_i-1}}{\Gamma(j_i)}\Im_i(\vartheta)d\vartheta,\ \ i=1,2.\end{aligned}$$

where

$$\begin{aligned}\lambda_{ii}=&\frac{1}{\Gamma(\ell_i+j_i)}\left[\delta_i\int_0^{\eta_i}(\eta_i-\vartheta)^{\ell_i+j_i-1}\wp_i(s,\Im_1(\vartheta),\Im_2(\vartheta))d\vartheta\right.\\&\left.-\int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{\ell_i+j_i-1}\wp_i(s,\Im_1(\vartheta),\Im_2(\vartheta))d\vartheta\right]\\&-\frac{\bar{h}_i}{\Gamma(j_i)}\left[\delta_i\int_0^{\eta_i}(\eta_i-\vartheta)^{j_i-1}\Im_i(\vartheta)d\vartheta-\int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{j_i-1}\Im_i(\vartheta)d\vartheta\right],\end{aligned}$$

$$\begin{aligned}\mu_{ii}=&\frac{1}{\Gamma(\ell_i+j_i-1)}\left[\varepsilon_i\int_0^{\xi_i}(\xi_i-\vartheta)^{\ell_i+j_i-2}\wp_i(s,\Im_1(\vartheta),\Im_2(\vartheta))d\vartheta\right.\\&\left.-\int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{\ell_i+j_i-2}\wp_i(s,\Im_1(\vartheta),\Im_2(\vartheta))d\vartheta\right]\\&-\frac{\bar{h}_i}{\Gamma(j_i-1)}\left[\varepsilon_i\int_0^{\xi_i}(\xi_i-\vartheta)^{j_i-2}\Im_i(\vartheta)d\vartheta-\int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{j_i-2}\Im_i(\vartheta)d\vartheta\right].\end{aligned}$$

For convenience, we set

$$\sigma_i = \frac{\delta_i \eta_i^{\ell_i+j_i} + \mathbb{T}^{\ell_i+j_i}}{\Gamma(\ell_i+j_i+1)}, \quad \tau_i = \frac{\varepsilon_i \xi_i^{\ell_i+j_i-1} + \mathbb{T}^{\ell_i+j_i-1}}{\Gamma(\ell_i+j_i)},$$
$$M_i = C_i\sigma_i + \frac{\bar{h}_i(\delta_i\eta_i^{j_i} + \mathbb{T}^{j_i})}{\Gamma(j_i+1)}, \quad \upsilon_i = C_i\tau_i + \frac{\bar{h}_i(\varepsilon_i\xi_i^{j_i-1} + \mathbb{T}^{j_i-1})}{\Gamma(j_i)},$$
$$\Pi_i = \frac{(|\rho_i|M_i + |e_i|\upsilon_i)(j_i+1)\mathbb{T}^{j_i} + (|k_i|\upsilon_i + |d_i|M_i)T^{j_i+1}}{|k_i\rho_i - e_id_i|\Gamma(j_i+2)} + \frac{\mathbb{T}^{\ell_i+j_i}C_i}{\Gamma(\ell_i+j_i+1)} + \frac{\bar{h}_i\mathbb{T}^{j_i}}{\Gamma(j_i+1)},$$
$$\Lambda_i = \frac{(|\rho_i|\sigma_i + |e_i|\tau_i)(j_i+1)\mathbb{T}^{j_i} + (|k_i|\tau_i + |d_i|\sigma_i)T^{j_i+1}}{|k_i\rho_i - e_id_i|\Gamma(j_i+2)} + \frac{\mathbb{T}^{\ell_i+j_i}}{\Gamma(\ell_i+j_i+1)}.$$

Let's discuss the main theorem by introducing the necessary assumption: :
(H) $\wp_i : [0,\mathbb{T}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\exists\, C_i > 0$ such that, $\forall\, \varsigma \in [0,\mathbb{T}]$ and $u_i,\ v_i,\ i = 1,2$ we have

$$|\wp_i(\varsigma,u_1,u_2)) - \wp_i(\varsigma,v_1,v_2)| \le C_i(|u_1 - v_1| + |u_2 - v_2|).$$

Theorem 1. *If (H) is valid, then, the system (1)-(2) has a unique solution provided that $\nu_1 + \nu_2 < 1$ where $\nu_1 = \Pi_1 + C_2\Lambda_2,\ \ \nu_2 = \Pi_2 + C_1\Lambda_1$.*

Proof. Define $\sup_{\varsigma\in[0,\mathbb{T}]} |\wp_i(\varsigma,0,0)| = \aleph_i < \infty$ and $r > 0$ such that

$$r > \frac{\Lambda_1\aleph_1 + \Lambda_2\aleph_2}{1 - \Xi},$$

where $\Xi = \Pi_1 + \Pi_2$.

Let $B_r = \{(\Im_1(\varsigma),\Im_2(\varsigma)) \in Z\times Z : \sup_{\varsigma\in[0,\mathbb{T}]} |(\Im_1(\varsigma),\Im_2(\varsigma))| = \|(\Im_1,\Im_2)\| \le r\}$.
We show that $\mho B_r \subset B_r$.
By using (H), for $(\Im_1,\Im_2) \in B_r,\ \varsigma \in [0,\mathbb{T}]$, we have

$$\begin{aligned}\sup_{\varsigma\in[0,\mathbb{T}]} |\wp_i(\varsigma,\Im_1(\varsigma),\Im_2(\varsigma))| &\le \sup_{\varsigma\in[0,\mathbb{T}]} |\wp_i(\varsigma,\Im_1(\varsigma),\Im_2(\varsigma)) - \wp_i(\varsigma,0,0)| + \sup_{\varsigma\in[0,\mathbb{T}]} |\wp_i(\varsigma,0,0)| \\ &\le C_i(\sup_{\varsigma\in[0,\mathbb{T}]} |\Im_1(\varsigma)| + \sup_{\varsigma\in[0,\mathbb{T}]} |\Im_2(\varsigma)|) + \aleph_i \\ &\le C_i(\|\Im_1\| + \|\Im_2\|) + \aleph_i \\ &\le C_ir + \aleph_i,\ i = 1,2\end{aligned}$$

which lead to

$$\begin{aligned}\|\lambda_{ii}\| = \sup_{\varsigma\in J}\Bigg|&\frac{1}{\Gamma(\ell_i+j_i)}\left[\delta_i\int_0^{\eta_i}(\eta_i-\vartheta)^{\ell_i+j_i-1}\wp_i(s,\Im_1(\vartheta),\Im_2(\vartheta))d\vartheta\right. \\ &\left.-\int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{\ell_i+j_i-1}\wp_i(s,\Im_1(\vartheta),\Im_2(\vartheta))d\vartheta\right] \\ &-\frac{\bar{h}_i}{\Gamma(j_i)}\left[\delta_i\int_0^{\eta_i}(\eta_i-\vartheta)^{j_i-1}\Im_i(\vartheta)d\vartheta - \int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{j_i-1}\Im_i(\vartheta)d\vartheta\right]\Bigg| \\ &\le \frac{1}{\Gamma(\ell_i+j_i+1)}\left[\delta_i\eta_i^{\ell_i+j_i} + \mathbb{T}^{\ell_i+j_i}\right][C_ir + \aleph_i] + \frac{r\bar{h}_i}{\Gamma(j_i+1)}\left[\delta_i\eta_i^{j_i} + T^{j_i}\right] \\ &\le M_ir + \sigma_i\aleph_i,\end{aligned}$$

and

$$\begin{aligned}\|\mu_{ii}\| = \sup_{\varsigma\in J}\Bigg|&\frac{1}{\Gamma(\ell_i+j_i-1)}\left[\varepsilon_i\int_0^{\xi_i}(\xi_i-\vartheta)^{\ell_i+j_i-2}\wp_i(s,\Im_1(\vartheta),\Im_2(\vartheta))d\vartheta\right.\\&\left.-\int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{\ell_i+j_i-2}\wp_i(s,\Im_1(\vartheta),\Im_2(\vartheta))d\vartheta\right]\\&-\frac{\bar{h}_i}{\Gamma(j_i-1)}\left[\varepsilon_i\int_0^{\xi_i}(\xi_i-\vartheta)^{j_i-2}\Im_i(\vartheta)d\vartheta-\int_0^{\mathbb{T}}(\mathbb{T}-\vartheta)^{j_i-2}\Im_i(\vartheta)d\vartheta\right]\Bigg|\\&\le\frac{[\varepsilon_i\xi_i^{\ell_i+j_i-1}+\mathbb{T}^{\ell_i+j_i-1}][C_ir+\aleph_i]}{\Gamma(\ell_i+j_i)}+\frac{r\bar{h}_i[\varepsilon_i\xi_i^{j_i-1}+\mathbb{T}^{j_i-1}]}{\Gamma(j_i)}\\&\le\upsilon_ir+\tau_i\aleph_i.\end{aligned}$$

Hence

$$\begin{aligned}\|\mho_i(\Im_1,\Im_2)\| &= \sup_{\varsigma\in[0,\mathbb{T}]}|\mho_i(\Im_1,\Im_2)(\varsigma)|\\&=\sup_{\varsigma\in[0,\mathbb{T}]}\Bigg|\left(\frac{\rho_i\lambda_{ii}-e_i\mu_{ii}}{k_i\rho_i-e_id_i}\right)\frac{\varsigma^{j_i}}{\Gamma(j_i+1)}+\left(\frac{k_i\mu_{ii}-d_i\lambda_{ii}}{k_i\rho_i-e_id_i}\right)\frac{\varsigma^{j_i+1}}{\Gamma(j_i+2)}\\&+\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{\ell_i+j_i-1}}{\Gamma(\ell_i+j_i)}\wp_i(s,\Im_1(\vartheta),\Im_2(\vartheta))d\vartheta-\bar{h}_i\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{j_i-1}}{\Gamma(j_i)}\Im_i(\vartheta)d\vartheta\Bigg|\\&\le\left(\frac{|\rho_i|\|\lambda_{ii}\|+|e_i|\|\mu_{ii}\|}{|k_i\rho_i-e_id_i|}\right)\frac{\mathbb{T}^{j_i}}{\Gamma(j_i+1)}+\left(\frac{|k_i|\|\mu_{ii}\|+\|d_i\|\|\lambda_{ii}\|}{|k_i\rho_i-e_id_i|}\right)\frac{\mathbb{T}^{j_i+1}}{\Gamma(j_i+2)}\\&+\frac{\mathbb{T}^{\ell_i+j_i}}{\Gamma(\ell_i+j_i+1)}(C_ir+\aleph_i)+\frac{(\mathbb{T}^{j_i}r\bar{h}_i}{\Gamma(j_i+1)}\\&\le\frac{[|\rho_i|(M_ir+\sigma_i\aleph_i)+|e_i|(\upsilon_ir+\tau_i\aleph_i)]T^{j_i}}{|k_i\rho_i-e_id_i|\Gamma(j_i+1)}\\&+\frac{[|k_i|(\upsilon_ir+\tau_i\aleph_i)+|d_i|(M_ir+\sigma_i\aleph_i)]\mathbb{T}^{j_i+1}}{|k_i\rho_i-e_id_i|\Gamma(j_i+2)}\\&+\frac{\mathbb{T}^{\ell_i+j_i}(C_ir+\aleph_i)}{\Gamma(\ell_i+j_i+1)}+\frac{\bar{h}_i\mathbb{T}^{j_i}r}{\Gamma(j_i+1)}\\&\le\Pi_ir+\Lambda_i\aleph_i.\end{aligned}$$

Consequently,

$$\|\mho(\Im_1,\Im_2)\|\le(\Pi_1+\Pi_2)r+\Lambda_1\aleph_1+\Lambda_2\aleph_2\le r.$$

For $(\Im_1,\Im_2)$, $(\bar{\Im}_1,\bar{\Im}_2)\in\mathbb{Z}\times\mathbb{Z}$ and for any $\varsigma\in[0,\mathbb{T}]$, we get

$$\begin{aligned}
&\|\mho_1(\bar{\Im}_1,\bar{\Im}_2)-\mho_1(\Im_1,\Im_2)\| \\
&\leq \left(\frac{|\rho_1|[M_1\|\bar{\Im}_1-\Im_1\|+C_1\sigma_1\|\bar{\Im}_2-\Im_2\|]+|e_1|[\upsilon_1\|\bar{\Im}_1-\Im_1\|+C_1\tau_1\|\bar{\Im}_2-\Im_2\|]}{|k_1\rho_1-e_1d_1|}\right)\frac{\mathbb{T}^{j_1}}{\Gamma(j_1+1)} \\
&+\left(\frac{|k_1|[\upsilon_1\|\bar{\Im}_1-\Im_1\|+C_1\tau_1\|\bar{\Im}_2-\Im_2\|]+|d_1|[M_1\|\bar{\Im}_1-\Im_1\|+C_1\sigma_1\|\bar{\Im}_2-\Im_2\|]}{|k_1\rho_1-e_1d_1|}\right)\frac{\mathbb{T}^{j_1+1}}{\Gamma(j_1+2)} \\
&+\frac{\mathbb{T}^{\ell_1+j_1}C_1[\|\bar{\Im}_1-\Im_1\|+\|\bar{\Im}_2-\Im_2\|]}{\Gamma(\ell_1+j_1+1)}+\bar{h}_1\frac{\mathbb{T}^{j_1}\|\bar{\Im}_1-\Im_1\|}{\Gamma(j_1+1)} \\
&\leq\left\{\frac{(|\rho_1|M_1+|e_1|\upsilon_1)(j_1+1)\mathbb{T}^{j_1}+(|k_1|\upsilon_1+|d_1|M_1)\mathbb{T}^{j_1+1}}{|k_1\rho_1-e_1d_1|\Gamma(j_1+2)}\right. \\
&\left.+\frac{C_1T^{\ell_1+j_1}}{\Gamma(\ell_1+j_1+1)}+\frac{\bar{h}_1\mathbb{T}^{j_1}}{\Gamma(j_1+1)}\right\}\|\bar{\Im}_1-\Im_1\| \\
&+C_1\left\{\frac{(|\rho_1|\sigma_1+|e_1|\tau_1)(j_1+1)\mathbb{T}^{j_1}+(|k_1|\tau_1+|d_1|\sigma_1)T^{j_1+1}}{|k_1\rho_1-e_1d_1|\Gamma(j_1+2)}+\frac{\mathbb{T}^{\ell_1+j_1}}{\Gamma(\ell_1+j_1+1)}\right\}\|\bar{\Im}_2-\Im_2\|, \\
&\leq \Pi_1\|\bar{\Im}_1-\Im_1\|+C_1\Lambda_1\|\bar{\Im}_2-\Im_2\|.
\end{aligned}\tag{8}$$

Also,

$$\begin{aligned}
&\|\mho_2(\bar{\Im}_1,\bar{\Im}_2)-\mho_2(\Im_1,\Im_2)\| \\
&\leq \left(\frac{|\rho_2|[M_2\|\bar{\Im}_2-\Im_2\|+C_2\sigma_2\|\bar{\Im}_1-\Im_1\|]+|e_2|[\upsilon_2\|\bar{\Im}_2-\Im_2\|+C_2\tau_2\|\bar{\Im}_1-\Im_1\|]}{|k_2\rho_2-e_2d_2|}\right)\frac{\mathbb{T}^{j_2}}{\Gamma(j_2+1)} \\
&+\left(\frac{|k_2|[\upsilon_2\|\bar{\Im}_2-\Im_2\|+C_2\tau_2\|\bar{\Im}_1-\Im_1\|]+|d_2|[M_2\|\bar{\Im}_2-\Im_2\|+C_2\sigma_2\|\bar{\Im}_1-\Im_1\|]}{|k_2\rho_2-e_2d_2|}\right)\frac{\mathbb{T}^{j_2+1}}{\Gamma(j_2+2)} \\
&+\frac{\mathbb{T}^{\ell_2+j_2}C_2[\|\bar{\Im}_1-\Im_1\|+\|\bar{\Im}_2-\Im_2\|]}{\Gamma(\ell_2+j_2+1)}+\bar{h}_2\frac{\mathbb{T}^{j_2}\|\bar{\Im}_2-\Im_2\|}{\Gamma(j_2+1)} \\
&\leq\left\{\frac{(|\rho_2|M_2+|e_2|\upsilon_2)(j_2+1)\mathbb{T}^{j_2}+(|k_2|\upsilon_2+|d_2|M_2)\mathbb{T}^{j_2+1}}{|k_2\rho_2-e_2d_2|\Gamma(j_2+2)}\right. \\
&\left.+\frac{C_2\mathbb{T}^{\ell_2+j_2}}{\Gamma(\ell_2+j_2+1)}+\frac{\bar{h}_2\mathbb{T}^{j_2}}{\Gamma(j_2+1)}\right\}\|\bar{\Im}_2-\Im_2\| \\
&+C_2\left\{\frac{(|\rho_2|\sigma_2+|e_2|\tau_2)(j_2+1)\mathbb{T}^{j_2}+(|k_2|\tau_2+|d_2|\sigma_2)\mathbb{T}^{j_2+1}}{|k_2\rho_2-e_2d_2|\Gamma(j_2+2)}+\frac{\mathbb{T}^{\ell_2+j_2}}{\Gamma(\ell_2+j_2+1)}\right\}\|\bar{\Im}_1-\Im_1\| \\
&\leq \Pi_2\|\bar{\Im}_2-\Im_2\|+C_2\Lambda_2\|\bar{\Im}_1-\Im_1\|.
\end{aligned}\tag{9}$$

From (8) and (9), we obtain

$$\|\mho(\bar{\Im}_1,\bar{\Im}_2)-\mho(\Im_1,\Im_2)\|\leq(\nu_1+\nu_2)(\|\bar{\Im}_1-\Im_1\|+\|\bar{\Im}_2-\Im_2\|).$$

Since $\nu_1+\nu_2<1$, therefore, $\mho$ is a contraction operator. So, from Banach's fixed point theorem, the operator $\mho$ has a unique fixed point, which is the unique solution of system (1), (2). This completes the proof.

4 Example

Consider the upcoming coupled system of nonlinear fractional Langevin equation

$$\begin{cases}{}^cD^{1.5}_{0+}({}^cD^{1.5}_{0+}+2\times10^{-4})\Im_1(\varsigma)=\frac{(\varsigma+3)^2|\Im_1(\varsigma)|^2}{10(\varsigma+4)^2)(1+|\Im_1(\varsigma)|^2)}+\frac{1}{40}\sin^2(\Im_2(\varsigma)),\ \varsigma\in J:=[0,1],\\ {}^cD^{1.5}_{0+}({}^cD^{1.5}_{0+}+3\times10^{-3})\Im_2(\varsigma)=\frac{1}{16\pi}\sin(2\pi\Im_1(\varsigma))+\frac{(\varsigma+1)^2|\Im_2(\varsigma)|^2}{20(\varsigma+2)^2)(1+|\Im_2(\varsigma)|^2)},\end{cases}\tag{10}$$

subject to the boundary conditions

$$\begin{cases} \Im_1(0) = 0, \ \Im_1(1) = \frac{1}{2}\Im_1(0.8), \ \Im_1'(1) = \frac{1}{4}\Im_1'(0.4) \\ \Im_2(0) = 0, \ \Im_2(1) = \frac{1}{2}\Im_2(0.8), \ \Im_2'(1) = \frac{1}{4}\Im_2'(0.4) \end{cases} \tag{11}$$

Here, $\ell_1 = \ell_2 = j_1 = j_2 = 1.5$, $\bar{h}_1 = 2\times 10^{-4}$, $\bar{h}_2 = 3\times 10^{-3}$, $\delta_1 = \delta_2 = \frac{1}{2}$, $\varepsilon_1 = \varepsilon_2 = \frac{1}{4}$, $\eta_1 = \eta_2 = 0.8$, $\xi_1 = \xi_2 = 0.4$, $\wp_1(\varsigma, \Im_1(\varsigma), \Im_2(\varsigma)) = \frac{(\varsigma+3)^2|\Im_1(\varsigma)|^2}{10(\varsigma+4)^2)(1+|\Im_1(\varsigma)|^2)} + \frac{1}{40}\sin^2(\Im_2(\varsigma))$ and $\wp_2(\varsigma, \Im_1(\varsigma), \Im_2(\varsigma)) = \frac{1}{16\pi}\sin(2\pi\Im_1(\varsigma)) + \frac{(\varsigma+1)^2|\Im_2(\varsigma)|^2}{20(\varsigma+2)^2)(1+|\Im_2(\varsigma)|^2)}$. Take $C_1 = 0.01$, $C_2 = 0.02$. With given data, we find that

$$\begin{aligned} & k_1 = k_2 = 0.48312, \ e_1 = e_2 = 0.21478, \ d_1 = d_2 = 0.95, \ \rho_1 = \rho_2 = 0.705, \ \sigma_1 = \sigma_2 = 0.21, \\ & \tau_1 = \tau_2 = 0.52, \ \Lambda_1 = \Lambda_2 = 2.4240.10488, \\ & \quad M_1 = 2.3\times 10^{-3}, \ M_2 = 7.3\times 10^{-3}, \ \upsilon_1 = 5.46\times 10^{-3}, \ \upsilon_2 = 0.01432, \\ & \quad \Pi_1 = 0.027836, \ \Pi_2 = 0.0542, \\ & \quad \nu_1 = \Pi_1 + C_2\Lambda_2 = 0.0299336, \ \nu_2 = \Pi_2 + C_1\Lambda_1 = 0.055248 \end{aligned} \tag{12}$$

Note that

$$\begin{aligned} & |\wp_1(\varsigma, u_1, u_2)) - \wp_1(\varsigma, v_1, v_2)| \le 0.01(|u_1 - v_1| + |u_2 - v_2|), \\ & |\wp_2(\varsigma, u_1, u_2)) - \wp_2(\varsigma, v_1, v_2)| \le 0.02(|u_1 - v_1| + |u_2 - v_2|). \end{aligned}$$

and $\nu_1 + \nu_2 = 0.0299336 + 0.055248 < 1$. Thus all conditions of Theorem 1 are satisfied.

Hence, the system (10), (11) has a unique solution on $[0, 1]$.

References

[1] K. Shah, J. Wang, H. Khalil and R. A. Khan, Existence and numerical solutions of a coupled system of integral BVP for fractional differential equations, *Adv. Differ. Equ.* **2018** (1), 1–21 (2018).

[2] A. Khan, Y. Li, K. Shah and T. S. Khan, On coupled-laplacian fractional differential equations with nonlinear boundary conditions, *Complexity* **2017**, (2017).

[3] D. Luo, A. Zada, S. Shaleena and M. Ahmad, Analysis of a coupled system of fractional differential equations with non-separated boundary conditions, *Adv. Differ. Equ.* **2020** (1), 1–24 (2020).

[4] I. M. Sokolov, J. Klafter and A. Blumen, Fractional kinetics, *Phys. Today* **55** (11), 48–54 (2002).

[5] I. Petras and R. L. Magin, Simulation of drug uptake in a two compartmental fractional model for a biological system, *Commun. Nonlin. Sci. Numer. Simul.* **16** (12), 4588–4595 (2011).

[6] A. Carvalho and C. M. A. Pinto, A delay fractional order model for the co-infection of malaria and HIV/AIDS, *Int. J. Dyn. Cont.* **5** (1), 168–186 (2017).

[7] W. Coffey and Y. P. Kalmykov, The Langevin equation: with applications to stochastic problems in physics, chemistry and electrical engineering, *World Scientific* **27**, (2012).

[8] H. Baghani and J. J. Nieto, Applications of the Mittag–Leffler function in solvability and stability of a class of fractional Langevin equations with two fractional orders, *The J. Anal.*, 1–15 (2023).

[9] M. Awadalla, M. Subramanian and K. Abuasbeh, Existence and Ulam–Hyers stability results for a system of coupled generalized Liouville–Caputo fractional Langevin equations with multipoint boundary conditions, *Symmetry* **15** (1), 198 (2023).

[10] C. Derbazi, H. Hammouche, A. Salim and M. Benchohra, Weak solutions for fractional Langevin equations involving two fractional orders in banach spaces, *Afrika Mat.* **34** (1), (2023).

[11] M. Aydin and N. I. Mahmudov, ψ-Caputo type time-delay Langevin equations with two general fractional orders, *Math. Meth. Appl. Sci.* **46** (8), 9187–9204 (2023).

[12] K. Zhao, Existence and UH-stability of integral boundary problem for a class of nonlinear higher-order Hadamard fractional Langevin equation via Mittag-Leffler functions, *Filomat* **37** (4), 1053-1063 (2023).

[13] M. M. Matar, J. Alzabut and J. M. Jonnalagadda, A coupled system of nonlinear Caputo–Hadamard Langevin equations associated with nonperiodic boundary conditions, *Math. Meth. Appl. Sci.*, (2020).

[14] D. Baleanu, J. Alzabut, J. M. Jonnalagadda, Y. Adjabi and M. M. Matar, A coupled system of generalized Sturm–Liouville problems and Langevin fractional differential equations in the framework of nonlocal and nonsingular derivatives, *Adv. Differ. Equ.* **2020** (1), 1–30 (2020).

[15] H. Baghani, J. Alzabut, J. Farokhi-Ostad and J. J. Nieto, Existence and uniqueness of solutions for a coupled system of sequential fractional differential equations with initial conditions, *J. Pseudo Differ. Oper. App.* **11** (4), 1731–1741 (2020).
[16] K. S. Miller and B. Ross, *An Introduction to The Fractional Calculus and Fractional Differential Equations*, John-Wiley and Sons, Inc. New York, 1993.
[17] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.