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Extended Maxwell–Chern–Simons Lagrangian Density in Riemann–Liouville Fractional Derivatives

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Abstract: The Hamiltonian formulation for higher derivatives is reformulated using fractional derivatives. More precisely, the extended Maxwell–Chern–Simons Lagrangian density is reformulated using the Riemann–Liouville fractional derivative. The equations of motion resulting from the extended Maxwell–Chern–Simons Lagrangian density are obtained. Furthermore, the Hamiltonian of the system is constructed. When fractional derivatives are replaced by integer order derivatives, the classical results are obtained.

Keywords: Fractional derivatives; Extended Maxwell–Chern–Simons Lagrangian density; Lagrangian formulation; Hamiltonian formulation; Euler–Lagrange equations

1 Introduction

Fractional derivatives, which have many applications in physics, science, engineering, control, economics, mechanics, and other fields, can be viewed as a generalization of classical calculus[1,2,3,4]. Actually, recently, physicists have applied a fractional derivatives approach in order to deal with problems that cannot be solved by classical methods. Therefore, fractional calculus appears as one of the most influential and commonly applicable methods for describing and explaining a set of complicated physical systems. [5,6,7]. Generally, higher derivative theories are an essential field in theoretical physics. Higher derivative theories were first proposed to remove infinities associated to point particles [8]. They can enhance ultraviolet features in quantum field theories [9]. Furthermore, they have been obtained from string theory [10] and non commutative theory [11] and have been utilized in electrodynamics[12], dark energy physics[13], and Lee-Wick models [14].

The extended Maxwell–Chern–Simons model is an attractive instance of higher derivative field theory [15] [16]. Generally, the Maxwell–Chern–Simons theory is a (2+1) dimensional field model, which defines charged fermions interactions with each other and with topologically massive propagating photons [17].

The fractional derivatives method was used by Jarab'ah and Nawafleh[18] to investigate nonconservative systems with second order Lagrangian. They obtained the fractional Hamilton's equations for nonconservative systems. The generated equations were similar to fractional Euler– Lagrange equations.

Alawaida et al.[19] developed the Hamiltonian formulation of continuous field systems with third order using the fractional derivatives method. They generated the fractional Euler and fractional Hamilton equations for these systems from the fractional variational principle. In addition, Al-Oqali [20] reformulated Podolsky's Lagrangian density in fractional form using the Riemann–Liouville fractional derivative. He gained the equations of motion using the fractional Euler–Lagrange equation. The Hamiltonian and the energy stress tensor are also generated in fractional form from the Lagrangian density.

The remainder of this paper is organized as follows: Section 2 presents some basic definitions of fractional derivatives. In Section 3, the Hamiltonian formulation of higher derivative field theories is reformulated using the Riemann–Liouville fractional derivative. In Section 4, the fractional form of the Euler–Lagrange equation of the extended Maxwell–Chern–Simons Lagrangian density is obtained. In section 5, the Hamiltonian density of the extended Maxwell–Chern–Simons model in fractional form is constructed. Finally, Section 6 is devoted to the paper's conclusion.

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2 Definitions of fractional derivatives

In this section, we briefly present some basic definitions utilized in this work. The left and right Riemann–Liouville fractional derivative is defined as [21]:

The left Riemann–Liouville fractional derivative

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x (x-\tau)^{n-\alpha+1} f(\tau)\, d\tau \tag{1} $$

The right Riemann-Liouville fractional derivative

$$ {}_xD_b^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{-d}{\mathrm{d}x}\right)^n \int_a^x (\tau-x)^{n-\alpha+1} f(\tau)\, d\tau. \tag{2} $$

where α represents the order of the derivative such that $n-1 \leq \alpha < n$ and Γ represents the gamma function. If α is an integer, these derivatives are defined in the usual sense, i.e.:

$$ {}_aD_x^{\alpha} f(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x) \tag{3} $$

$$ {}_aD_t^{\alpha} f(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(t) \tag{4} $$

α=1,2,...

3 The Hamiltonian formulation of higher derivative field theories with Riemann–Liouville fractional derivative

In this section, we will refomulate the generalized Euler–Lagrange equation, the generalized energy–momentum tensor, and the Hamiltonian density of higher derivative theories using fractional derivatives [22].

Let us start with a Lagrangian

$$ \mathscr{L}(\phi, {}_aD_{x_\mu}^k \phi, {}_aD_{x_\mu}^k {}_aD_{x_\nu}^k \phi, \ldots\ldots, {}_aD_{x_{\mu_1}}^k {}_aD_{x_{\mu_2}}^k \cdots {}_aD_{x_{\mu_N}}^k \phi) \tag{5} $$

Which for simplicity we consider to be a function of a scalar field and to depend on a finite number of N derivatives. In this case, the action function corresponding to the above Lagrangian is given by

$$ \begin{aligned} S = \int d^4x \mathscr{L}(\phi, {}_aD_{x_\mu}^k \phi, {}_aD_{x_\mu}^k {}_aD_{x_\nu}^k \phi, \ldots\ldots, \\ {}_aD_{x_{\mu_1}}^k {}_aD_{x_{\mu_2}}^k \cdots {}_aD_{x_{\mu_N}}^k \phi) \end{aligned} \tag{6} $$

The generalized Euler–Lagrange equation in fractional form is obtained by following the typical approach of extremizing the action.

$$ \begin{aligned} \frac{\partial \mathscr{L}}{\partial \phi} - {}_aD_{x_\mu}^k \left(\frac{\partial \mathscr{L}}{\partial({}_aD_{x_\mu}^k \phi)}\right) + {}_aD_{x_\mu}^k {}_aD_{x_\nu}^k \left(\frac{\partial \mathscr{L}}{\partial({}_aD_{x_\mu}^k {}_aD_{x_\nu}^k \phi)}\right) - \ldots \\ + (-1)^n {}_aD_{x_{\mu_1}}^k {}_aD_{x_{\mu_2}}^k \cdots {}_aD_{x_{\mu_N}}^k \left(\frac{\partial \mathscr{L}}{\partial({}_aD_{x_{\mu_1}}^k {}_aD_{x_{\mu_2}}^k \cdots {}_aD_{x_{\mu_N}}^k \phi)}\right) = 0 \end{aligned} \tag{7} $$

and the generalized energy–momentum tensor in the same way

$$
\begin{aligned}
T^{\mu}_{\nu} = &-\delta^{\mu}_{\nu}\mathscr{L} + \left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_\mu}\phi)}\right){}_aD^k_{x_\nu}\phi - \\
&[{}_aD^k_{x_{\mu_1}}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_\mu}{}_aD^k_{x_{\mu_1}}\phi)}\right){}_aD^k_{x_\nu}\phi - \\
&\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_\mu}{}_aD^k_{x_{\mu_1}}\phi)}\right){}_aD^k_{x_{\mu_1}}{}_aD^k_{x_\nu}\phi] \\
&+[{}_aD^k_{x_{\mu_1}}{}_aD^k_{x_{\mu_2}}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}{}_aD^k_{x_{\mu_2}}{}_aD^k_{x_\mu}\phi)}\right){}_aD^k_{x_\nu}\phi - \\
&{}_aD^k_{x_{\mu_1}}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}{}_aD^k_{x_{\mu_2}}{}_aD^k_{x_\mu}\phi)}\right){}_aD^k_{x_{\mu_2}}{}_aD^k_{x_\nu}\phi + \\
&\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}{}_aD^k_{x_{\mu_2}}{}_aD^k_{x_\mu}\phi)}\right){}_aD^k_{x_{\mu_1}}{}_aD^k_{x_{\mu_2}}{}_aD^k_{x_\nu}\phi] + \\
&...(-1)^{(N-1)}[{}_aD^k_{x_{\mu_1}}\cdots{}_aD^k_{x_{\mu_{N-1}}}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}\cdots{}_aD^k_{x_{\mu N-1}}{}_aD^k_{x_\mu}\phi)}\right) \\
&{}_aD^k_{x_\nu}\phi - {}_aD^k_{x_{\mu_1}}\cdots{}_aD^k_{x_{\mu_{N-2}}} \\
&\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}\cdots{}_aD^k_{x_{\mu N-1}}{}_aD^k_{x_\mu}\phi)}\right){}_aD^k_{x_{\mu N-1}}{}_aD^k_{x_\nu}\phi + \\
&...(-1)^{(N-1)}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}\cdots{}_aD^k_{x_{\mu N-1}}{}_aD^k_{x_\mu}\phi)}\right) \\
&{}_aD^k_{x_{\mu_1}}\cdots{}_aD^k_{x_{\mu N-1}}{}_aD^k_{x_\nu}\phi]
\end{aligned} \tag{8}
$$

The Hamiltonian density in fractional form, which corresponds to the component T_0^0, is now given by

$$\mathscr{H} = -\mathscr{L} + \frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_0}\phi)}\,{}_aD^k_{x_0}\phi - [{}_aD^k_{x_{\mu_1}}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_0}\,{}_aD^k_{x_{\mu_1}}\phi)}\right){}_aD^k_{x_0}\phi - \left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_0}\,{}_aD^k_{x_{\mu_1}}\phi)}\right){}_aD^k_{x_{\mu_1}}\,{}_aD^k_{x_0}\phi] + [{}_aD^k_{x_{\mu_1}}\,{}_aD^k_{x_{\mu_2}}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}\,{}_aD^k_{x_{\mu_2}}\,{}_aD^k_{x_0}\phi)}\right){}_aD^k_{x_0}\phi - {}_aD^k_{x_{\mu_1}}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}\,{}_aD^k_{x_{\mu_2}}\,{}_aD^k_{x_0}\phi)}\right){}_aD^k_{x_{\mu_2}}\,{}_aD^k_{x_0}\phi + \left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}\,{}_aD^k_{x_{\mu_2}}\,{}_aD^k_{x_0}\phi)}\right){}_aD^k_{x_{\mu_1}}\,{}_aD^k_{x_{\mu_2}}\,{}_aD^k_{x_0}\phi] + \ldots(-1)^{N-1}[{}_aD^k_{x_{\mu_1}}\,{}_aD^k_{x_{\mu_2}}\cdots{}_aD^k_{x_{\mu N-1}}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}\,{}_aD^k_{x_{\mu_2}}\cdots{}_aD^k_{x_{\mu N-1}}\,{}_aD^k_{x_0}\phi)}\right){}_aD^k_{x_0}\phi - {}_aD^k_{x_{\mu_1}}\cdots{}_aD^k_{x_{\mu N-2}}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}\,{}_aD^k_{x_{\mu N-1}}\,{}_aD^k_{x_0}\phi)}\right){}_aD^k_{x_{\mu N-1}}\,{}_aD^k_{x_0}\phi + (-1)^{N-1}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_{\mu_1}}\cdots{}_aD^k_{x_{\mu N-1}}\,{}_aD^k_{x_0}\phi)}\right){}_aD^k_{x_{\mu_1}}\cdots{}_aD^k_{x_{\mu N-1}}\,{}_aD^k_{x_0}\phi] \quad (9)$$

4 The fractional form of the Euler–Lagrange equation of the Extended Maxwell–Chern–Simons Lagrangian density

As an example of a Lagrangian with a higher derivative, let us consider the Lagrangian for the extended Maxwell–Chern–Simons model in 2+1 dimensions in the Lorentz gauge, which was proposed by Reyes [8]

$$\mathscr{L} = \frac{-1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g}{2}\varepsilon^{\alpha\beta\gamma}(\Box A_\alpha)\,{}_aD^k_{x_\beta}A_\gamma - \frac{1}{2}({}_aD^k_{x_\mu}A^\mu)^2 \quad (10)$$

where A_μ is the four-vector potential, $F_{\mu\nu} = {}_aD^k_{x_\mu}A^\nu - {}_aD^k_{x_\nu}A^\mu$ is a four dimension antisymmetric second rank tensor, $\eta_{\mu\nu}$=diag (1,-1,1) spacetime metric raises and lowers the indices, $\varepsilon^{\alpha\beta\gamma}$ the Levi–Civita tensor, fully antisymmetric, is given by ε^{012}=1, and the constant $g>0$ being a coupling coefficient of the Chern-Simons term.

To reformulate the extended Maxwell–Chern–Simons Lagrangian density in fractional form, we first define the relations

$${}_aD^k_{x_\mu} = ({}_aD^k_t, {}_aD^k_{x_i}), {}_aD^k_{x^\mu} = ({}_aD^k_t, -{}_aD^k_{x^i}) \quad (11)$$

$$A^\sigma = (\phi, \mathbf{A}),\; A_\sigma = (\phi, -\mathbf{A}) \quad (12)$$

In fractional form, the Euler–Lagrange equation for such a Lagrangian density is given by

$$\frac{\partial \mathscr{L}}{\partial A_\sigma} - {}_aD^k_{x_\lambda}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_\lambda}A_\sigma)}\right) + {}_aD^k_{x_\mu}\,{}_aD^k_{x_\lambda}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_\mu}\,{}_aD^k_{x_\lambda}A_\sigma)}\right) = 0 \quad (13)$$

The first term $\frac{\partial \mathscr{L}}{\partial A_\sigma} = 0$. The second term in Eq. (13) may be evaluated as follows:

$$\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_\lambda} A_\sigma)}\right) = -F^{\lambda\sigma} + \frac{g}{2}\varepsilon^{\alpha\beta\gamma}(\Box A_\alpha)\delta^\lambda_\beta \delta^\sigma_\gamma - \frac{1}{2}\left(\frac{\partial}{\partial({}_aD^k_{x_\lambda} A_\sigma)}\right) [\eta^{\mu\mu\prime}{}_aD^k_{x_\mu} A_{\mu\prime}\eta^{\nu\nu\prime}{}_aD^k_{x_\nu} A_{\nu\prime}] \quad (14)$$

$$= -F^{\lambda\sigma} + \frac{g}{2}\varepsilon^{\alpha\lambda\sigma}(\Box A_\alpha) - \frac{1}{2}\eta^{\mu\mu\prime}\eta^{\nu\nu\prime}[\delta^\lambda_\mu \delta^\sigma_{\mu\prime a}D^k_{x_\nu} A_{\nu\prime} + \delta^\lambda_\nu \delta^\sigma_{\nu\prime a}D^k_{x_\mu} A_{\mu\prime}] \quad (15)$$

which can be rewritten as

$$= -F^{\lambda\sigma} + \frac{g}{2}\varepsilon^{\alpha\lambda\sigma}(\Box A_\alpha) - \frac{1}{2}[\eta^{\lambda\sigma}{}_aD^k_{x_\nu} A^\nu + \eta^{\lambda\sigma}{}_aD^k_{x_\mu} A^\mu] \quad (16)$$

$$= -F^{\lambda\sigma} + \frac{g}{2}\varepsilon^{\alpha\lambda\sigma}(\Box A_\alpha) - \eta^{\lambda\sigma}{}_aD^k_{x_\mu} A^\mu \quad (17)$$

It follows that

$${}_aD^k_{x_\lambda}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_\lambda} A_\sigma)}\right) = -{}_aD^k_{x_\lambda} F^{\lambda\sigma} + \frac{g}{2}\varepsilon^{\alpha\lambda\sigma}{}_aD^k_{x_\lambda}(\Box A_\alpha) + \eta^{\lambda\sigma}{}_aD^k_{x_\lambda}{}_aD^k_{x_\mu} A^\mu \quad (18)$$

Similarly, the third term in Eq. (13) may be obtained as follows:

$$\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_\mu}{}_aD^k_{x_\lambda} A_\sigma)}\right) = \left(\frac{\partial}{\partial({}_aD^k_{x_\mu}{}_aD^k_{x_\lambda} A_\sigma)}\right) \left(\frac{g}{2}\varepsilon^{\alpha\beta\gamma}{}_aD^k_{x_\zeta}{}_aD^k_{x^\zeta} A_\alpha\right){}_aD^k_{x_\beta} A_\gamma \quad (19)$$

which can be simplified to

$$= \frac{g}{2}\varepsilon^{\alpha\beta\gamma}\eta^{\zeta\nu}\delta^\zeta_\mu \delta^\nu_\lambda \delta^\alpha_\sigma{}_aD^k_{x_\beta} A_\gamma \quad (20)$$

$$= \frac{g}{2}\varepsilon^{\sigma\beta\gamma}\eta^{\mu\lambda}{}_aD^k_{x_\beta} A_\gamma \quad (21)$$

then we have

$${}_aD^k_{x_\mu}{}_aD^k_{x_\lambda}\left(\frac{\partial \mathscr{L}}{\partial({}_aD^k_{x_\mu}{}_aD^k_{x_\lambda} A_\sigma)}\right) = \frac{g}{2}\varepsilon^{\sigma\beta\gamma}\eta^{\mu\lambda}{}_aD^k_{x_\mu}{}_aD^k_{x_\lambda}{}_aD^k_{x_\beta} A_\gamma \quad (22)$$

Substituting Eqs. (18 and 22) into Eq. (13), we get

$${}_aD^k_{x_\lambda} F^{\lambda\sigma} - \frac{g}{2}\varepsilon^{\alpha\lambda\sigma}{}_aD^k_{x_\lambda}(\Box A_\alpha) + \eta^{\lambda\sigma}{}_aD^k_{x_\lambda}{}_aD^k_{x_\mu} A^\mu + \frac{g}{2}\varepsilon^{\sigma\beta\gamma}\eta^{\mu\lambda}{}_aD^k_{x_\mu}{}_aD^k_{x_\lambda}{}_aD^k_{x_\beta} A_\gamma = 0 \quad (23)$$

$$
\begin{aligned}
{}_aD^k_{x_\lambda}F^{\lambda\sigma} - \frac{g}{2}\varepsilon^{\alpha\lambda\sigma}(\Box A_\alpha)\,{}_aD^k_{x_\lambda}A_\alpha + \\
\frac{g}{2}\varepsilon^{\sigma\beta\gamma}\,{}_aD^k_{x^\lambda}\,{}_aD^k_{x_\lambda}\,{}_aD^k_{x_\beta}A_\gamma \\
+\,{}_aD^k_{x^\sigma}\,{}_aD^k_{x_\mu}A^\mu = 0
\end{aligned} \tag{24}
$$

$$
{}_aD^k_{x_\lambda}F^{\lambda\sigma} - \frac{g}{2}\Box\left(\varepsilon^{\alpha\lambda\sigma}\,{}_aD^k_{x_\lambda}A_\alpha - \varepsilon^{\sigma\beta\gamma}\,{}_aD^k_{x_\beta}A_\gamma\right) + {}_aD^k_{x^\sigma}\,{}_aD^k_{x_\mu}A^\mu = 0 \tag{25}
$$

Setting $\lambda = \gamma$ and $\alpha = \beta$ in the term $\varepsilon^{\alpha\lambda\sigma}\,{}_aD^k_{x_\lambda}A_\alpha$ in Eq. (25). This lead to following:

$$
{}_aD^k_{x_\lambda}F^{\lambda\sigma} - \frac{g}{2}\Box\left(\varepsilon^{\beta\gamma\sigma}\,{}_aD^k_{x_\gamma}A_\beta - \varepsilon^{\sigma\beta\gamma}\,{}_aD^k_{x_\beta}A_\gamma\right) + {}_aD^k_{x^\sigma}\,{}_aD^k_{x_\mu}A^\mu = 0 \tag{26}
$$

$$
{}_aD^k_{x_\lambda}F^{\lambda\sigma} - \frac{g}{2}\Box\left(\varepsilon^{\beta\gamma\sigma}\,{}_aD^k_{x_\gamma}A_\beta - \varepsilon^{\sigma\beta\gamma}\,{}_aD^k_{x_\beta}A_\gamma\right) + {}_aD^k_{x^\sigma}\,{}_aD^k_{x_\mu}A^\mu = 0 \tag{27}
$$

using the definition of the field strength tensor $F_{\beta\gamma} = {}_aD^k_{x_\beta}A_\gamma - {}_aD^k_{x_\gamma}A_\beta$, Eq. (27) can be written as:

$$
{}_aD^k_{x_\lambda}F^{\lambda\sigma} + \frac{g}{2}\varepsilon^{\sigma\beta\gamma}\Box F_{\beta\gamma} + {}_aD^k_{x^\sigma}\,{}_aD^k_{x_\mu}A^\mu = 0 \tag{28}
$$

The above equation represents the fractional form of the modified Maxwell equations. It's worth pointing out that for $k \to 1$, Eq. (28) simplifies to the usual Maxwell equation:

$$
\partial_\lambda F^{\lambda\sigma} + \frac{g}{2}\varepsilon^{\sigma\beta\gamma}\Box F_{\beta\gamma} + \partial^\sigma(\partial.A) = 0 \tag{29}
$$

Equation (28) can be rewritten as follows, see appendix A.

$$
\left(\eta^{\sigma\alpha} + g\varepsilon^{\sigma\beta\alpha}\right)(\Box A_\alpha) = 0 \tag{30}
$$

5 The Hamiltonian formulation of the Extended Maxwell–Chern–Simons Model with the Riemann–Liouville fractional derivative

Consider $A_\mu(x)$ and ${}_aD^k_tA_\mu$ as two independent configuration field variables with their corresponding conjugate momenta defined by

$$
P^\mu = \frac{\partial\mathscr{L}}{\partial({}_aD^k_tA_\mu)} - {}_aD^k_t\Pi^\mu, \quad \Pi^\mu = \frac{\partial\mathscr{L}}{\partial({}_aD^k_t\,{}_aD^k_tA_\mu)} \tag{31}
$$

By using Eq. (17), we can write the conjugate momenta Π^μ as

$$
\Pi^\mu = \frac{\partial\mathscr{L}}{\partial({}_aD^k_t\,{}_aD^k_tA_\mu)} = \frac{g}{2}\varepsilon^{\mu\beta\gamma}\,{}_aD^k_{x_\beta}A_\gamma \tag{32}
$$

which reads in components

$$
\Pi^0 = \frac{\partial\mathscr{L}}{\partial({}_aD^k_t\,{}_aD^k_tA_\mu)} = \frac{g}{2}\varepsilon^{0ij}\,{}_aD^k_{x_i}A_j \tag{33}
$$

$$
\Pi^i = \frac{g}{2}\varepsilon^{ij}\left({}_aD^k_tA_j - {}_aD^k_jA_0\right) \tag{34}
$$

and the conjugate momenta P^{μ} is given by

$$P^{\mu}=\frac{\partial\mathscr{L}}{\partial({}_aD_t^kA_\mu)}-{}_aD_t^k\left(\frac{\partial\mathscr{L}}{\partial({}_aD_t^k{}_aD_t^kA_\mu)}\right) \tag{35}$$

By making use of Eq. (17), we obtain

$$\frac{\partial\mathscr{L}}{\partial({}_aD_t^kA_\mu)}==-F^{0\mu}+\frac{g}{2}\varepsilon^{\alpha0\mu}(\Box A_\alpha)-\eta^{0\mu}\left({}_aD_{x_\gamma}^kA^\gamma\right) \tag{36}$$

or

$$\frac{\partial\mathscr{L}}{\partial({}_aD_t^kA_\mu)}==-F^{0\mu}-\frac{g}{2}\varepsilon^{\mu0\alpha}(\Box A_\alpha)-\eta^{0\mu}\left({}_aD_{x_\gamma}^kA^\gamma\right) \tag{37}$$

Setting $\alpha=\gamma$ in the second term of Eq. (37), we get

$$\frac{\partial\mathscr{L}}{\partial({}_aD_t^kA_\mu)}==-F^{0\mu}-\frac{g}{2}\varepsilon^{\mu0\gamma}(\Box A_\gamma)-\eta^{0\mu}\left({}_aD_{x_\gamma}^kA^\gamma\right) \tag{38}$$

Substituting Eqs. (38 and 32) into Eq. (35), we get

$$P^{\mu}=-F^{0\mu}-\frac{g}{2}\varepsilon^{\mu0\gamma}(\Box A_\gamma)-\eta^{0\mu}\left({}_aD_{x_\gamma}^kA^\gamma\right)-\frac{g}{2}\varepsilon^{\mu\beta\gamma}{}_aD_{x_\beta}^k{}_aD_{x_t}^kA_\gamma \tag{39}$$

which read in components

$$P^{0}=-F^{00}-\frac{g}{2}\varepsilon^{00\gamma}(\Box A_\gamma)-\eta^{00}\left({}_aD_{x_\gamma}^kA^\gamma\right)-\frac{g}{2}\varepsilon^{0\beta\gamma}{}_aD_{x_\beta}^k{}_aD_{x_t}^kA_\gamma \tag{40}$$

$$P^{0}=-\left({}_aD_{x_\gamma}^kA^\gamma\right)-\frac{g}{2}\varepsilon^{ij}{}_aD_{x_i}^k{}_aD_{x_t}^kA_j \tag{41}$$

$$\begin{aligned}P^{i}=-F^{0\mu}-\frac{g}{2}\varepsilon^{i0\gamma}(\Box A_\gamma)-\eta^{0i}\left({}_aD_{x_\gamma}^kA^\gamma\right)-\\ \frac{g}{2}\varepsilon^{i\beta\gamma}{}_aD_{x_\beta}^k{}_aD_{x_t}^kA_\gamma\end{aligned} \tag{42}$$

$$P^{i}=F_{0i}+\frac{g}{2}\varepsilon^{ij}(\Box A_\gamma)-\frac{g}{2}\varepsilon^{i\beta\gamma}{}_aD_{x_\beta}^k{}_aD_t^kA_\gamma \tag{43}$$

Eq. (43) can be rewritten as

$$P^{i}=F_{0i}+\frac{g}{2}\varepsilon^{ij}(\Box A_j)-\frac{g}{2}\varepsilon^{i0\gamma}{}_aD_t^k{}_aD_t^kA_\gamma-\frac{g}{2}\varepsilon^{ij\gamma}{}_aD_{x_j}^k{}_aD_t^kA_\gamma \tag{44}$$

$$P^{i}=F_{0i}+\frac{g}{2}\varepsilon^{ij}(\Box A_j)+\frac{g}{2}\varepsilon^{0ij}{}_aD_t^k{}_aD_t^kA_\gamma-\frac{g}{2}\varepsilon^{ij0}{}_aD_{x_j}^k{}_aD_t^kA_0 \tag{45}$$

or

$$P^i = F_{0i} + \frac{g}{2}\varepsilon^{ij}(\Box A_j) + \frac{g}{2}\varepsilon^{ij}{}_aD_t^kF_{jo} \tag{46}$$

The Hamiltonian density is given by

$$\mathscr{H} = P^\mu{}_aD_t^kA_\mu + \Pi^\mu{}_aD_t^k{}_aD_t^kA_\mu - \mathscr{L} \tag{47}$$

Taking into account Eqs. (10 and 32), then the Hamiltonian density in fractional form is

$$\mathscr{H} = P^0{}_aD_t^kA_0 + P^i{}_aD_t^kA_i + \frac{g}{2}\varepsilon^{\mu\beta\gamma}{}_aD_{x_\beta}^kA_{\gamma a}D_t^k{}_aD_t^kA_\mu - \frac{1}{2}(F_0j)^2 + \frac{1}{4}(F_ij)^2 - \frac{g}{2}\varepsilon^{\alpha\beta\gamma}(\Box A_\alpha)_aD_{x_\beta}^kA_\gamma + \frac{1}{2}\left({}_aD_{x_\mu}^kA^\mu\right)^2 \tag{48}$$

By using the relation $\Box A = {}_aD_t^k{}_aD_t^k - \bigtriangledown^2$ Eq. (48), simplified to

$$\mathscr{H} = P^0{}_aD_t^kA_0 + P^i{}_aD_t^kA_i - \frac{1}{2}(F_0j)^2 + \frac{1}{4}(F_ij)^2 + \frac{g}{2}\varepsilon^{\alpha\beta\gamma}\left(\bigtriangledown^2A_\alpha\right){}_aD_{x_\beta}^kA_\gamma + \frac{1}{2}\left({}_aD_{x_\mu}^kA^\mu\right)^2 \tag{49}$$

The second term in Eq. (49) can be written as

$$\Pi^i = \frac{g}{2}\varepsilon^{i\beta\gamma}{}_aD_{x_\beta}^kA_\gamma = \frac{g}{2}\varepsilon^{ij}\left({}_aD_{x_j}^kA_0 - {}_aD_t^kA_j\right) \tag{50}$$

or

$${}_aD_t^kA_j = -\frac{2}{g}\varepsilon^{ij}\Pi^i + {}_aD_{x_j}^kA_0 \tag{51}$$

$${}_aD_t^kA_i = \frac{2}{g}\varepsilon^{ij}\Pi^j + {}_aD_{x_i}^kA_0 \tag{52}$$

Multiply both sides of Eq. (52) by P^i, we get

$$P^i{}_aD_t^kA_i = \frac{2}{g}\varepsilon^{ij}P^i\Pi^j + P^i{}_aD_{x_i}^kA_0 \tag{53}$$

or

$$P^i{}_aD_t^kA_i = \frac{2}{g}\varepsilon^{ij}P_i\Pi_j - P_i{}_aD_{x_i}^kA_0 \tag{54}$$

The third term in Eq. (49) can be expressed as

$$\Pi^i = \frac{g}{2}\varepsilon^{ij}\left({}_aD_t^kA_j - {}_aD_{x_j}^kA_0\right) = \frac{g}{2}\varepsilon^{ij}F_0j \tag{55}$$

It follows that

$$(F_{0i})^2 = 4(\Pi_i)^2 \setminus (g^2) \tag{56}$$

The fourth term in Eq. (49) can be written in the following way:

$$\frac{g}{2}\varepsilon^{\alpha\beta\gamma}\left(\bigtriangledown^2A_\alpha\right){}_aD_{x_\beta}^kA_\gamma = \frac{g}{2}\varepsilon^{0\beta\gamma}\left(\bigtriangledown^2A_0\right){}_aD_{x_\beta}^kA_\gamma + \frac{g}{2}\varepsilon^{i\beta\gamma}\left(\bigtriangledown^2A_i\right){}_aD_{x_\beta}^kA_\gamma \tag{57}$$

$$= \frac{g}{2}\varepsilon^{ij}\left(\bigtriangledown^2A_0\right){}_aD_{x_i}^kA_j + \frac{g}{2}\left(\varepsilon^{i0\gamma}{}_aD_{x_0}^kA_\gamma + \varepsilon^{i\beta 0}{}_aD_{x_\beta}^kA_0\right)\left(\bigtriangledown^2A_i\right) \tag{58}$$

$$= \frac{g}{2}\varepsilon^{ij}\left(\bigtriangledown^2 A_0\right) {}_aD^k_{x_i}A_j + \frac{g}{2}\varepsilon^{ij}\left({}_aD^k_{x_j}A_0 - {}_aD^k_{x_0}A_j\right)\left(\bigtriangledown^2 A_i\right) \quad (59)$$

$$= \frac{g}{2}\varepsilon^{ij}\left(\bigtriangledown^2 A_0\right) {}_aD^k_{x_i}A_j - \frac{g}{2}\varepsilon^{ij}(F_0 j)\left(\bigtriangledown^2 A_i\right) \quad (60)$$

Inserting Eq. (55) into Eq. (60), we get

$$\frac{g}{2}\varepsilon^{\alpha\beta\gamma}\left(\bigtriangledown^2 A_\alpha\right) {}_aD^k_{x_\beta}A_\gamma = \frac{g}{2}\varepsilon^{ij}\left(\bigtriangledown^2 A_0\right) {}_aD^k_{x_i}A_j - \Pi_i\left(\bigtriangledown^2 A_i\right) \quad (61)$$

Substituting Eqs. (54, 56, and 61) into Eq. (49), the Hamiltonian density can be written as

$$\mathscr{H} = P^0 {}_aD^k_t A_0 + \frac{2}{g}\varepsilon^{ij}P_i\Pi_j - P_i {}_aD^k_{x_i}A_j - \frac{2\left(\Pi_i\right)^2}{(g^2)} + \frac{1}{4}\left(F_{ij}\right)^2 + \frac{g}{2}\varepsilon^{ij}\left(\bigtriangledown^2 A_0\right) {}_aD^k_{x_i}A_j - \Pi_i\left(\bigtriangledown^2 A_i\right) + \frac{1}{2}\left({}_aD^k_{x_\mu}A^\mu\right)^2 \quad (62)$$

6 Conclusion

In this work, the Hamiltonian formulation for higher derivatives has been reformulated using fractional derivatives. For higher order derivatives, the extended Maxwell–Chern–Simons Lagrangian density is refomulated in fractional form using the Riemann–Liouville fractional derivative. The fractional form of the Euler–Lagrange equation and the Hamiltonian density of these systems are obtained. The classical results are obtained when fractional derivatives are replaced by integer order derivatives.

Appendix A: The Derivation of Equation (30):
We can write Eq. (28) as

$${}_aD^k_{x_\lambda}\left({}_aD^k_{x^\lambda}A^\sigma - {}_aD^k_{x^\sigma}A^\lambda\right) + {}_aD^k_{x^\sigma}\, {}_aD^k_{x_\mu}A^\mu + \frac{g}{2}\varepsilon^{\sigma\beta\gamma}\Box\left({}_aD^k_{x_\beta}A_\gamma - {}_aD^k_{x_\gamma}A_\beta\right) = 0 \quad (A.1)$$

The above equation can be rearranged as:

$$\Box A^\sigma - {}_aD^k_{x^\sigma}\, {}_aD^k_{x_\lambda}A^\lambda + {}_aD^k_{x^\sigma}\, {}_aD^k_{x_\mu}A^\mu + \frac{g}{2}\varepsilon^{\sigma\beta\gamma}\Box\left({}_aD^k_{x_\beta}A_\gamma - {}_aD^k_{x_\gamma}A_\beta\right) = 0 \quad (A.2)$$

The second and third terms cancel after changing a dummy index from $\mu$to λ. This leaves us with

$$\eta^{\sigma\alpha}\Box A_\alpha + \frac{g}{2}\varepsilon^{\sigma\beta\gamma}\Box\left({}_aD^k_{x_\beta}A_\gamma - {}_aD^k_{x_\gamma}A_\beta\right) = 0 \quad (A.3)$$

using $\varepsilon^{\sigma\beta\zeta}\varepsilon_{\sigma\beta\alpha} = \delta^\zeta_\alpha$, we rewrite Eq. (A.3) as:

$$\eta^{\sigma\alpha}\Box A_\alpha + \frac{g}{2}\varepsilon^{\sigma\beta\gamma}\varepsilon^{\sigma\beta\alpha}\varepsilon_{\sigma\beta\alpha}\Box\left({}_aD^k_{x_\beta}A_\gamma - {}_aD^k_{x_\gamma}A_\beta\right) = 0 \quad (A.4)$$

or

$$\eta^{\sigma\alpha}\Box A_\alpha + \frac{g}{2}\left(\varepsilon^{\sigma\beta\alpha}{}_a D^k_{x_\beta} A_\alpha - \varepsilon^{\sigma\beta\alpha}{}_a D^k_{x_\alpha} A_\beta\right) = 0 \tag{A.5}$$

let $\alpha \to \beta$ and $\beta \to \alpha$ in the term $\varepsilon^{\sigma\beta\alpha}{}_a D^k_{x_\alpha} A_\beta$ of the above equation, we get

$$\eta^{\sigma\alpha}\Box A_\alpha + \frac{g}{2}\Box\left(\varepsilon^{\sigma\beta\alpha}{}_a D^k_{x_\beta} A_\alpha + \varepsilon^{\sigma\beta\alpha}{}_a D^k_{x_\beta} A_\alpha\right) = 0 \tag{A.6}$$

It follows that

$$\left(\eta^{\sigma\alpha} + g\varepsilon^{\sigma\beta\alpha}{}_a D^k_{x_\beta}\right)\Box A_\alpha = 0 \tag{A.7}$$

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