Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160302

Nadarajah Haghighi Generalised Power Weibull: Properties and Applications

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Received: 13 Nov. 2021, Revised: 2 Feb. 2022, Accepted: 2 Mar. 2022
Published online: 1 May 2022

Abstract: This study developed the Nadarajah Haghighi generalised power Weibull (NHGPW) distribution by compounding the Nadarajah Haghighi and the generalised power Weibull distribution. Various statistical properties of this new distribution are derived for the new distribution. Estimators of the parameters of the developed distribution are also derived using the maximum likelihood estimation. The hazard rate function of the NHGPW distribution has been shown to be monotonically increasing, monotonically decreasing, constant, bathtub, upside down bathtub (unimodal), modified bathtub or modified unimodal. Its probability density function also exhibits various shapes such as decreasing-increasing, decreasing, increasing, increasing-decreasing, decreasing-increasing-decreasing, increasing-decreasing-increasing, positively skewed among others. Monte Carlo simulations conducted on the estimators of the NHGPW distribution showed that, the estimators are consistent since the average bias and mean square error approaches zero as the sample size increases. The NHGPW distribution is applied to two real lifetime data and compared with existing lifetime distributions for a system connected in series. The results showed that, the NHGPW distribution provides a better fit to both data sets than the competing distributions.

Keywords: Nadarajah Haghighi, generalised power Weibull, compounding, series connection, consistent, monotonic.

1 Introduction

The quality of every parametric statistical analysis is determined greatly by the probability distribution assumed. For this reason, extensive efforts have been made in developing new and standard probability distributions with various statistical techniques for many situations including lifetime cases. Nonetheless, applications from various fields such as environment, finance/economics, biological sciences, engineering, agriculture etc, have shown that, many of the data sets do not follow these traditional distributions [1].

The practicality of any statistical distribution to a larger degree, depends on the fundamental properties and the in-build assumptions considered in deriving that statistical distribution. These properties and assumptions significantly support in distinguishing the practical circumstances in which the distributions are applied. The more clearly defined the structural properties, the better-off the scope of the distribution. Over the past decades, well known classical lifetime distributions like the exponential, Weibull, Rayleigh, linear failure rate, gamma and their extensions have been used for modeling lifetime data. However, in practice, most of these distributions are not flexible enough to accommodate different phenomena.

In order to increase the flexibility of these well-known distributions, many researchers have proposed different transformations of the models and used these extended forms in several areas. In distribution theory, generalisation of existing statistical distributions can be done by transformations, extensions or by compounding two distributions whiles introducing additional shape or scale parameter(s). In reliability and biological studies, a component or system may contain sub-systems connected in series with each of the sub-systems functioning independently and the failure rate following independent distributions. For such a system, the main component will fail if any or all of the sub-systems fail. In these situations, attempts are made to derive a single distribution that models the failure rates of the main

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system using the concept of compounding. Compounding two or more distributions have been shown to be very useful in discovering various skewed and tailed properties of these distributions and for improving the goodness-of-fit of the traditional distributions ([2]; [3]). This approach can be suitable in manufacturing, biological, medical, reliability analysis etc for modeling a component/system, with dual sub-systems working independently in successions or series at an expected time. The stochastic representation of this case is;

$$T = min(T_1, T_2), \tag{1}$$

where T_1 and T_2 are the lifetime failure rate random variables, for the two sub-systems.

Some Earlier researchers who used the concept of compounding for deriving distributions for modelling lifetime data from systems connected in series include;[4], [5], [6], [7],[8],[9], [10] among others. To fill the gap of limited distributions for systems in series, this paper developed a new lifetime probability distribution, named the Nadarajah Haghighi Generalised Power Weibull (NHGPW) distribution, via compounding the Nadarajah-Haghighi (NH) and the generalised power Weibull (GPW) distributions, in the context of a system with two of its sub-system connected in series. The statistical properties of the developed lifetime distributions are also derived. The estimators for the parameters of the new distribution are also presented. Monte Carlo simulations were also performed to assess the performance of the estimators. The developed NHGPW distribution is also applied to two lifetime data.

The NH distribution is a bi-parameter distribution proposed by [11] as a generality of the one-parameter exponential model. If a random variable T follows the NH distribution, then the cumulative distribution function (CDF), probability density function (PDF) and hazard functions of T are given respectively as;

$$F(t) = 1 - e^{[1-(1+\alpha t)^{\beta}]} \quad t > 0, \alpha > 0, \beta > 0, \tag{2}$$

$$f(t) = \alpha\beta(1+\alpha t)^{\beta-1} e^{[1-(1+\alpha t)^{\beta}]} \quad t > 0, \tag{3}$$

and

$$h(t) = \alpha\beta(1+\alpha t)^{\beta-1}, \quad t > 0, \tag{4}$$

where β is a shape/tilt parameter and α is the scale parameter.

The generalized power Weibull (GPW) model, derived by [12] is a modification of the Weibull distribution on the bases of accelerated failure time models. If T follows the GPW distribution, then its CDF, PDF and hazard functions are given respectively as;

$$F(t) = 1 - e^{[1-(1+\lambda t^{\gamma})^{\theta}]}, \quad t > 0, \gamma > 0, \theta > 0, \lambda > 0, \tag{5}$$

$$f(t) = \lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{[1-(1+\lambda t^{\gamma})^{\theta}]}, \quad t > 0, \tag{6}$$

and

$$h(t) = \lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}, \quad t > 0, \tag{7}$$

where λ is the scale parameter and γ, θ are the shape parameters.

Other sections of the paper are outlined as follows: In section 2, the survival function, CDF, PDF and hazard functions of the NHGPW distribution are derived. In section 3, the sub-distributions of the NHGPW distribution are defined. In section 4, the statistical properties of the distribution are derived. Section 5 presents the maximum likelihood estimators for the parameters of the distribution. Section 6 displays the simulation analysis whiles section 7 demonstrates the application of the NHGPW distribution to two lifetime data.

2 The Nadarajah Haghighi Generalised Power Weibull Distribution

Assuming we have system in series with its two components having independently distributed lifetime random variables T_1 and T_2 with independently distributed failure rate. Then the stochastic representation of their failure rate distribution is defined in Equation (1). Since the sub-components are independent, and the two components must be working for the system's success, the main system's reliability (survival) function is the product of the marginal reliability of the sub-components ([13]; [14]; [15]). Thus;

$$S(t) = e^{-\int_0^x h_1(t)dt} \times e^{-\int_0^x h_2(t)dt}$$

$$= e^{-[\int_0^x (h_1(t)+h_2(t))dt]}. \tag{8}$$

Assuming for these two components operating in series, the failure rate of component one follows the NH distribution and that of component two follows the GPW distribution, then a new distribution can be developed to model the joint reliability (survival) function of this system. The survival function of this new distribution (NHGPW) is given as;

$$s(t) = e^{-\left[\int_0^x (((\alpha\beta(1+\alpha t)^{\beta-1}) + (\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}))dt\right]}. \tag{9}$$

We represent

$$H_1(t) = \int_0^x \alpha\beta(1+\alpha t)^{\beta-1} dt.$$

and

$$H_2(t) = \int_0^x \lambda\theta\gamma t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} dt.$$

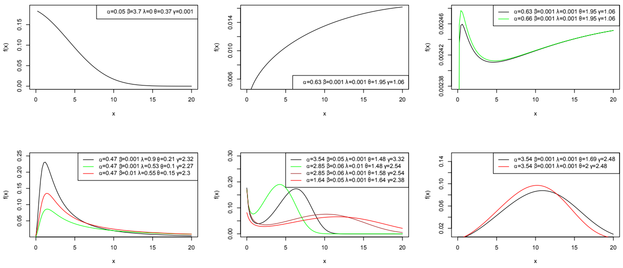

Fig. 1: PDF plots of the NHGPW distribution

Solving $H_1(t)$ and $H_2(t)$ using integration by substitution, we have;

$$H_1(t) = (1+\alpha x)^{\beta} - 1,$$

and

$$H_2(t) = (1+\lambda t^{\gamma})^{\theta} - 1.$$

Hence the survival function of the NHGPW distribution is given by;

$$s(x) = e^{-\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}. \tag{10}$$

The CDF of the NHGPW distribution is given as;

$$F(x) = 1 - e^{-\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}, x>0, \tag{11}$$

where $\beta > 0, \gamma > 0, \theta > 0$ shape parameters and $\alpha > 0, \lambda > 0$ are scale parameters.

The PDF of the NHGPW distribution is given as;

$$f(x) = \left\{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\} e^{-\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}, x>0. \tag{12}$$

The plots of the PDF of the NHPGW distribution is shown in Figure (1). It is evident that, for various parameter values, the PDF of the NHPGW distribution can be decreasing, increasing, increasing-decreasing, increasing-decreasing-increasing, decreasing-increasing-decreasing, right skewed and symmetric. The PDF also showed various values of skewness and kurtosis based on different combination of the parameter values.

The hazard rate function of the NHGPW distribution is;

$$h(x) = \alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}. \tag{13}$$

For various parameter combinations as shown in Figure (2), the hazard function can be constant, monotonically increasing, monotonically decreasing, bathtub, unimodal (upside down bathtub), modified bathtub (bathtub followed by unimodal). Hence the NHGPW distribution can adequately model both monotonic and non-monotonic failure rates.

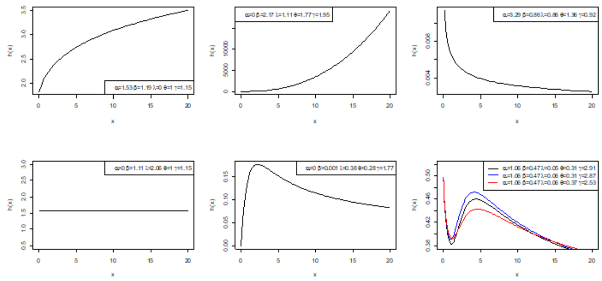

Fig. 2: Hazard plots of the NHGPW distribution

2.1 Sub-distributions of the NHGPW Distribution

The NHGPW distribution has, as sub-distributions, a number of existing and new lifetime distributions for modeling lifetime dataset. Some of these distributions are;

1. the generalised power Weibull distribution.
   The NHGPW distribution reduces to the GPW distribution if either $\beta = 0$ or $\alpha = 0$ with the CDF of the GPW distribution given as;
   $$F(x) = 1 - e^{\left[1-(1+\lambda x^{\gamma})^{\theta}\right]}, x, \lambda, \gamma, \theta > 0. \tag{14}$$
2. the Nadarajah Haghighi Distribution.
   If $\lambda = 0$, or $\theta = 0$ the NHGPW distribution reduces to the NH distribution with CDF given as;
   $$F(x) = 1 - e^{\left[1-(1+\alpha x)^{\beta}\right]}, x, \alpha, \beta > 0. \tag{15}$$
3. the Exponential distribution
   For $\beta = 1$ and $\lambda = 0$ (or $\theta = 0$), the NHGPW distribution reduces to an exponential distribution with CDF defined as;
   $$F(x) = 1 - e^{-\alpha x}, \alpha > 0. \tag{16}$$
   or
   For $\alpha = 0$ (or $\beta = 0$), $\gamma = 1$, and $\theta = 1$, the NHGPW distribution reduces to an exponential distribution with CDF defined as;
   $$F(x) = 1 - e^{-\lambda x}, \lambda > 0. \tag{17}$$
4. the Weibull distribution.
   For $\alpha = 0$ and $\theta = 1$, the NHGPW distribution reduces to a two parameter Weibull distribution with its CDF given as;
   $$F(x) = 1 - e^{-\lambda x^{\gamma}}, x, \lambda, \gamma > 0. \tag{18}$$
5. the Linear failure rate distribution.
   For $\beta = 1, \gamma = 2$ and $\theta = 1$, the NHGPW distribution reduces to the linear failure rate distribution with CDF defined as;
   $$F(x) = 1 - e^{-\left[\alpha x + \lambda x^{2}\right]}, x, \alpha, \lambda > 0. \tag{19}$$

6.the Rayhigh distribution.
For $\beta = 1, \alpha = 0, \gamma = 2$, and $\theta = 1$, the NHGPW distribution is equivalent to the Rayhigh distribution with CDF defined as;

$$F(x) = 1 - e^{\lambda x^2}, x, \lambda > 0. \quad (20)$$

7.the Generalized power Rayhigh distribution.
For $\alpha = 0$ and $\gamma = 2$, the NHGPW distribution equates to the Generalized power Rayhigh distribution with the following CDF;

$$F(x) = 1 - e^{[1-(1+\lambda x^2)^\theta]}, x, \lambda > 0, \gamma > 0. \quad (21)$$

8.The NH-NH distribution (New distribution)
For, $\gamma = 1$, a new distribution called the NH-NH distribution can be obtained from the NHGPW distribution with its CDF given as;

$$F(x) = 1 - e^{-[1-(1+\gamma x)^\beta + (1+\lambda x)^\theta - 2]}, x, \gamma >, \beta, \lambda, \theta > 0. \quad (22)$$

9.The Exponential-Exponential Distribution (New distribution)
For $\beta = 1, \gamma = 1$ and $\theta = 1$, another new distribution termed the exponential exponential distribution can be derived from the NHGPW distribution with CDF defined as;

$$F(x) = 1 - e^{-[\alpha x + \lambda x]}, x, \alpha, \lambda > 0. \quad (23)$$

2.2 Quantiles of the NHGPW distribution

The quantile function of the NHGPW distribution is obtained by inverting its CDF specified in Equation (11). The quantile function can be used for generating random numbers from a given distribution for Monte Carlo studies and various statistical applications.

Proposition 1. The Quantile function of the NHGPW distribution is obtained by solving the equation (24).

$$(1+\alpha x_p)^\beta + (1+\lambda x_p^\gamma)^\theta + log(1-p) - 2 = 0, p\varepsilon[0,1]. \quad (24)$$

Proof. Using the CDF of the NHPGW distribution in equation (11), we have;

$$1 - e^{-\left[(1+\alpha x_p)^\beta + (1+\lambda x_p^\gamma)^\theta - 2\right]} = p.$$

Making the exponent of the expression the subject, we have;

$$e^{-\left[(1+\alpha x_p)^\beta + (1+\lambda x_p^\gamma)^\theta - 2\right]} = 1 - p$$
$$(1+\alpha x_p)^\beta + (1+\lambda x_p^\gamma)^\theta = 2 - \log(1-p),$$

which can also be written as;

$$(1+\gamma x_p)^\beta + (1+\lambda x_p^\gamma)^\theta + log(1-p) - 2 = 0, p\varepsilon[0,1].$$

Using the Newton Raphson estimation approach, some random numbers are generated using the quantile function of the NHGPW distribution. The parameter value combinations are then used to obtain the quantiles as presented in Table 1. The Booleys skewness (B.Sk) and Moors kurtosis (M.Ku) values are also calculated. The Bowleys skewness and Moors kurtosis measures based on the quantile function, as proposed by Kenney and Keeping (1962) is;

$$B.Sk = \frac{Q(\frac{3}{4}) - 2Q(\frac{1}{2}) + Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}. \quad (25)$$

$$M.Ku = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}. \quad (26)$$

In order to obtain other statistical properties of the NHGPW distribution, further expansion of its PDF is necessary. The PDF of the NHGPW distribution in equation (12) distribution can also be written as;

$$f(x) = f_1(x) + f_2(x), \quad (27)$$

where

$$f_1(x) = \alpha\beta(1+\alpha x)^{\beta-1} e^{[1-(1+\alpha x)^\beta]} e^{[1-(1+\lambda x^\gamma)^\theta]}, \quad (28)$$

and

$$f_2(x) = \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1} e^{-[(1+\alpha x)^\beta + (1+\lambda x^\gamma)^\theta - 2]}. \quad (29)$$

Using Taylor series, and the generalized binomial expansion theorem, $f_1(x)$ and $f_2(x)$ can be expanded into;

$$f_1(x) = \alpha\beta(1+\alpha x)^{\beta-1} \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} x^{\gamma\theta(i-j)-\gamma k} w_{ijk} e^{[1-(1+\alpha x)^\beta]}, \quad (30)$$

and

$$f_2(x) = \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1} \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} x^{\beta(i-j)-k} w_{ijk}^{*} e^{[1-(1+\lambda x^\gamma)^\theta]}, \quad (31)$$

where $w_{ijk} = \frac{(-1)^{i+j}\lambda^{\theta(i-j)-k}}{i!}\binom{i}{j}\binom{\theta(i-j)}{k}$ and $w_{ijk}^{*} = \frac{(-1)^{i+j}\alpha^{\beta(i-j)-k}}{i!}\binom{i}{j}\binom{\beta(i-j)}{k}$.

2.3 Moments of the NHGPW distribution

Moments of a random variable are important in statistical analysis. They can be used in measuring central tendency, variation, skewness, kurtosis and other statistical procedures.

Proposition 2. The r^{th} non-central moment of NHGPW distribution is given as;

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[w_{ijk} A\Gamma\left(\frac{m}{\beta}+1,1\right) + w_{ijk}^{*} B\Gamma\left(\frac{m}{\theta}+1,1\right)\right], \quad (32)$$

Table 1: NHGPW Quantiles for Selected Parameter Values $(\alpha,\beta,\lambda,\theta,\gamma)$

p	(2.5, 10.3,4.5, 0.1,0.5)	(3.54, 5.3, 6.5, 2.1,8.5)	(2.0, 2.0, 3.5, 0.5,2.0)	(4.0, 3.0, 5.0, 4.0,0.8)	(3.54, 0.05,0.01, 1.46,3.32)
0.1	0.0031	0.0054	0.0254	0.0011	1.1259
0.2	0.0068	0.0109	0.0519	0.0027	1.8596
0.3	0.0108	0.0167	0.0799	0.0046	2.2988
0.4	0.0151	0.0229	0.1103	0.0069	2.6402
0.5	0.0198	0.0295	0.1437	0.0095	2.9386
0.6	0.0250	0.0369	0.1819	0.0126	3.2203
0.7	0.0307	0.0454	0.2275	0.0166	3.5053
0.8	0.0380	0.0560	0.2866	0.0218	3.8198
0.9	0.0482	0.0714	0.3773	0.0302	4.2253
B.Sk	0.1358	-6.1056	1.3831	0.2390	-0.0790
M.Ku	1.1435	1.1604	1.2028	1.2297	1.2905

where
$A = e\alpha^{-r-r\theta(i-j)+\gamma k}(-1)^{r+\gamma\theta(i-j)-\gamma k-m}\binom{r+\gamma\theta(i-j)-\gamma k}{m}$
and
$B = e\lambda^{\frac{-\gamma-\beta(i-j)+k}{\gamma}}(-1)^{\frac{r+\beta(i-j)-k}{\gamma}-m}\binom{\frac{r+\beta(i-j)-k}{\gamma}}{m}$.

Proof. For the NHGPW distribution, the r^{th} non-central moment is defined as;

$$\mu_r^{'} = \int_0^\infty x^r f_1(x)dx + \int_0^\infty x^r f_2(x)dx.$$

Using the expanded form of $f_1(x)$ defined in equation (30) we get;

$$\int_0^\infty x^r f_1(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w_{ijk}\alpha\beta e \times \int_0^\infty (1+\alpha x)^{\beta-1}x^{r+\gamma\theta(i-j)-\gamma k} \times e^{-(1+\alpha x)^\beta}dx.$$

Using integration by substitution, we have;

$$\int_0^\infty x^r f_1(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w_{ijk}e\alpha^{-r-\gamma\theta(i-j)+\gamma k} \times \int_1^\infty \left(y^{\frac{1}{\beta}}-1\right)^{r-\gamma\theta(i-j)-\gamma k} e^{-y}dy.$$

Using the generalised form of binomial expansion, $(x+y)^i = \sum_{j=o}^{i}\binom{i}{j}y^{i-j}x^j, (\mid y\mid > \mid x\mid)$ with $y=(-1)$ and $x = y^{\frac{1}{\beta}}$ to expand the expression above we have;

$$\int_0^\infty x^r f_1(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} w_{ijk}e\alpha^{-r-\gamma\theta(i-j)+\gamma k} \times(-1)^{r+\gamma\theta(i-j)-\gamma k-m}\binom{r+\gamma\theta(i-j)-\gamma k}{m} \times \int_1^\infty y^{[(\frac{m}{\beta}+1)-1]}e^{-y}dy.$$

but $\int_1^\infty y^{(\frac{m}{\beta}+1)-1}e^{-y}dy$ is a complementary gamma function given as $\Gamma(b,a)$, where b and a are the parameters. Therefore,

$$\int_0^\infty x^r f_1(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty} w_{ijk}e\alpha^{-r-\gamma\theta(i-j)+\gamma k} \times(-1)^{r+\gamma\theta(i-j)-\gamma k-m}\binom{r+\gamma\theta(i-j)-\gamma k}{m} \times\Gamma\left(\frac{m}{\beta}+1,1\right).$$

Also,

$$\int_0^\infty x^r f_2(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w^*_{ijk}\lambda\gamma\theta \times\int_0^\infty x^{r+\gamma-1+\beta(i-j)-k}(1+\lambda x^\gamma)^{\theta-1} \times e^{-(1+\lambda x^\gamma)^\theta}dx.$$

Using integration by substitution and the generalised form of binomial expansion to solve $f_2(x)$, we have;

$$\int_0^\infty x^r f_2(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} w^*_{ijk} e\lambda^{\frac{-r-\beta(i-j)+k}{\gamma}} \times(-1)^{\frac{r+\beta(i-j)-k}{\gamma}-m}\binom{\frac{r+\beta(i-j)-k}{\gamma}}{m} \times\int_1^\infty y^{[(\frac{m}{\theta}+1)-1]}e^{-y}dy.$$

But $\int_1^\infty y^{[(\frac{m}{\theta}+1)-1]}e^{-y}dy = \int_1^\infty y^{b-1}e^{-y}dy$ is a complementary gamma function. Therefore,

$$f_2(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} w^*_{ijk} e\lambda^{\frac{-r-\beta(i-j)+k}{\gamma}} \times(-1)^{\frac{r+\beta(i-j)-k}{\gamma}-m}\binom{\frac{r+\beta(i-j)-k}{\gamma}}{m}\Gamma\left(\frac{m}{\theta}+1,1\right).$$

Hence,

$$\mu_r' = \int_0^\infty x^r f_1(x)dx + \int_0^\infty x^r f_2(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[w_{ijk}A\Gamma\left(\frac{m}{\beta}+1,1\right) + w^*_{ijk}B\Gamma\left(\frac{m}{\theta}+1,1\right)\right].$$

The first five non-central moments obtained by numerical integration of the NHGPW distribution for selected parameter values are presented in Table 2. The standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and kurtosis (CK) calculated using these non-central moments are also presented.

2.4 Moment Generating function of the NHGPW distribution

Proposition 3. The moment generating function (MGF) of the NHPGW distribution is given as;

$$M_x(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{\gamma!}\left[w_{ijk}A\Gamma\left(\frac{m}{\beta}+1,1\right) + w^*_{ijk}B\Gamma\left(\frac{m}{\theta}+1,1\right)\right]. \quad (33)$$

Proof. By definition, the MGF is given as;

$$M_X(t) = E\left[e^{tX}\right] = \int_0^\infty e^{tx}f(x)dx.$$

Using Taylor series to expand $\int_0^\infty e^{tx}f(x)dx$, we get;

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'.$$

Inputting, μ_r' in equation (32), the MGF is obtained.

2.5 Order Statistics of the NHGPW Distribution

Order statistics are used among other things to identify the maximum and minimum values (extreme values) of a random variable. They are mostly used in extreme value theory. Let X_1 and X_2 denote the smallest and second smallest value of $(X_1,X_2,....,X_n)$ and X_p denote the p^{th} smallest value of $(X_1,X_2,....,X_n), 1 \leq p \leq n$, then the random variables $X_1,X_2,....,X_n$, are called the order statistics of the sample $(X_1,X_2,....,X_n)$ and has pdf of the pth order, given as;

$$f_{p:n}(x) = \frac{n!}{(n-p)!(p-1)!}[F(x)]^{p-1}[1-F(x)]^{n-p}f(x). \quad (34)$$

Proposition 4. The PDF of the first order statistics of the NHGPW distribution is given as;

$$f_{x(1)}(x) = n\left\{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}\right\} \times e^{-n\left[(1+\alpha x)^\beta+(1+\lambda x^\gamma)^\theta-2\right]}. \quad (35)$$

Proof. Using Equation (34), the PDF of the first order statistic is defined as;

$$f_{x(1)}(x) = n[1-F(x)]^{n-1}f(x).$$

Imputing the CDF and PDF of the NHGPW distribution from equation (11) and (12) into $f_{x(1)}(x)$, we obtain the PDF of the first order statistics.

Proposition 5. The PDF of the largest order statistics for the NHGPW distribution is also given as;

$$f_{x(n)}(x) = n\left\{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}\right\} \times\sum_{i=1}^{n-1}(-1)^i\binom{n-1}{i}e^{-(i+1)\left[(1+\alpha x)^\beta+(1+\lambda x^\gamma)^\theta-2\right]}. \quad (36)$$

Proof. Using equation (34), the PDF of the largest order statistics, $(p = n)$ is also expressed as;

$$f_{x(n)}(x) = n[F(x)]^{n-1}f(x).$$

Inputting the CDF and PDF of the NHPGW distribution from equation (11) and (12), $f_x(n)(x)$ yields;

$$f_{x(n)}(x) = n\left\{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}\right\} \times\sum_{i=1}^{n-1}(-1)^i\binom{n-1}{i}e^{-(i+1)\left[(1+\alpha x)^\beta+(1+\lambda x^\gamma)^\theta-2\right]},$$

since $[1 - e^{-\left[(1+\alpha x)^\beta+(1+\lambda x^\gamma)^\theta-2\right]}]^{n-1} = \sum_{i=1}^{n-1}(-1)^i\binom{n-1}{i}[e^{-i\left[(1+\alpha x)^\beta+(1+\lambda x^\gamma)^\theta-2\right]}]$, $f_{x(n)}(x)$.

Table 2: Moments of the NHGPW Distribution

p	(0.1, 0.8,0.8, 0.5,2.4)	(3.54, 0.05, 0.01, 1.4,3.32)	(3.54, 5.3, 6.5, 2.1,8.5)	(2, 2, 3.5, 0.5,2)	(4, 3,5, 4,0.8)
$\mu_1^{'}$	1.476	2.813	0.035	0.178	1.307
$\mu_2^{'}$	3.002	9.280	0.002	0.052	3.217
$\mu_3^{'}$	7.724	33.130	0.001	0.020	1.085
$\mu_4^{'}$	24.049	125.117	1×10^{-05}	0.009	4.502
$\mu_5^{'}$	87.858	494.022	8.9×10^{-07}	0.005	2.190
SD	0.907	1.170	0.026	0.143	1.229
CV	0.614	0.416	0.760	0.807	0.940
CS	1.160	-0.422	0.750	1.203	-3.810
CK	5.086	2.737	3.502	3.813	8.145

3 Methods of Estimation

This study employed the maximum likelihood estimation (MLE) method to obtain estimators of the NHGPW Distribution. For the NHGPW distribution, with PDF given in equation (12), the likelihood function is given as;

$$L = \prod_{i=1}^{n} \left\{ \alpha\beta(1+\alpha x_i)^{\beta-1} + \lambda\gamma\theta x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1} \right\} \times e^{-\left[(1+\alpha x_i)^{\beta} + (1+\lambda x_i^{\gamma})^{\theta} - 2\right]}. \tag{37}$$

We take logarithm of equation (37) to obtain,

$$l = \sum_{i=1}^{n} log\left\{ \alpha\beta(1+\alpha x_i)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1} \right\} - \sum_{i=1}^{n} \left[(1+\alpha x_i)^{\beta} + (1+\lambda x_i^{\gamma})^{\theta} - 2 \right]. \tag{38}$$

We obtain the score function by taking the partial differential of the log-likelihood (l) with respect to the individual parameters. These are given as;

$$\frac{\partial l}{\partial \alpha} = -\sum_{i=1}^{n} \beta x_i(1+\alpha x_i)^{\beta-1} + \sum_{i=1}^{n} \frac{\alpha(\beta-1)\beta x_i(1+\alpha x_i)^{\beta-2} + \beta(1+\alpha x_i)^{\beta-1}}{\alpha\beta(1+\alpha x_i)^{\beta-1} + \gamma\theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}. \tag{39}$$

$$\frac{\partial l}{\partial \beta} = -\sum_{i=1}^{n} log[1+\alpha x_i](1+\alpha x_i)^{\beta} + \sum_{i=1}^{n} \frac{\alpha(1+\alpha x_i)^{\beta-1} + \alpha\beta log[1+\alpha x_i](1+\alpha x_i)^{\beta-1}}{\alpha\beta(1+\alpha x_i)^{\beta-1} + \gamma\theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}. \tag{40}$$

$$\frac{\partial l}{\partial \theta} = -\sum_{i=1}^{n} log[1+\lambda x_i^{\gamma}](1+\lambda x_i^{\gamma})^{\theta} + \sum_{i=1}^{n} \frac{\gamma\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1} + B}{\alpha\beta(1+\alpha x_i)^{\beta-1} + \gamma\theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}, \tag{41}$$

where $B = \gamma\theta\lambda log[1+\lambda x_i^{\gamma}]x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}$

$$\frac{\partial l}{\partial \lambda} = -\sum_{i=1}^{n} \theta x_i^{\gamma}(1+\lambda x_i^{\gamma})^{\theta-1} + \sum_{i=1}^{n} \frac{\gamma(\theta-1)\theta\lambda x_i^{2\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-2} + C}{\alpha\beta(1+\alpha x_i)^{\beta-1} + \gamma\theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}, \tag{42}$$

$C = \gamma\theta x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}$

$$\frac{\partial l}{\partial \gamma} = -\sum_{i=1}^{n} \theta\lambda log[x_i]x_i^{\gamma}(1+\lambda x_i^{\gamma})^{\theta-1} + \sum_{i=1}^{n} \frac{(\gamma(\theta-1)\theta\lambda^2 log[x_i]x_i^{2\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-2} + \theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1} + A}{\alpha\beta(1+\alpha x_i)^{\beta-1} + \gamma\theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}} \tag{43}$$

where $A = \gamma\theta\lambda log[x_i]x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}$

4 Monte Carlo Simulation

Monte Carlo simulations are conducted to assess the performance of the maximum likelihood estimators for the parameters of the NHGPW distribution. Five different combinations of parameter values of the NHGPW distribution are specified. The quantile function of the NHGPW distribution is then used to generate five different random samples of sizes, n=40, 80, 120, 160, 200. These are further used to obtain the maximum likelihood estimates of the parameters of the distribution. The simulation is replicated for N=1000 times. The average bias (ABias) and mean square error (MSE) are calculated for the estimators of the parameters of the NHGPW distribution. From Table 3, the results showed that, the maximum likelihood estimates of the parameters of the NHGPW distribution converges to the true parameter value since the mean square errors decay to zero and the biases of each parameter also decrease as the sample size increases.

5 Applications

The NHGPW distribution is applied to failure times of 36 appliances subjected to an automatic life test and the Aircraft Windshield failure rate data. The goodness-of-fit of the NHGPW distribution was compared with some of its sub-models and other five parameter distributions. Thus; the NH distribution, the GPW distribution, Weibull NH (WNH) distribution, Kumaraswamy log-logistic Weibull (KLLoGW) distribution, exponentiated Kumaraswamy Dagum (EKD) distribution, exponentiated generalised Fisk (EGFD) distribution, exponentiated generalised exponential Fisk Dagum (EGEFD) distribution, exponentiated generalised exponential Dagum (EGED) distribution. The comparison are done using the Kolmogorov Smirnov statistic, Cramér-Von Mises statistic, log-likelihood and model selection criteria such as the AIC, AICc and BIC. The CDFs of the distributions, the NHGPW distribution is compared with are given as;

$$F_{KLLoGW}(x) = 1 - \left[1-(1-(1+x^{c})^{-1}e^{-\alpha x^{\beta}})^{a}\right]^{b}, \quad (44)$$

$a,b,c,\alpha,\beta > 0$ and $x > 0$.

$$F_{EKD}(x,\alpha,\lambda,\sigma,\phi,\theta) = \left\{1-\left[1-(1+\lambda x^{-\sigma})^{-\alpha}\right]^{\phi}\right\}^{\theta}, \quad (45)$$

$\alpha,\lambda,\sigma,\phi,\theta > 0$ and $x > 0$.

$$F_{EGEBD}(x) = 1 - \left\{1-\left[1-(1-(1+x^{-\theta})^{-\beta})^{d}\right]^{c}\right\}^{\lambda}, \quad (46)$$

$\beta,\lambda,c,d,\theta > 0$ and $x > 0$.

$$F_{EGED}(x) = 1 - \left\{1-\left[1-(1-(1+\alpha x^{-\theta})^{-\beta})^{d}\right]^{c}\right\}^{\lambda}, \quad (47)$$

$\alpha,\theta,\beta,d,c,\lambda > 0$ and $x > 0$.

$$F_{EGFD}(x) = \left[1-\left(1-(1+\alpha x^{-\theta})^{-1}\right)^{d}\right]^{c}, \quad (48)$$

$\alpha,\theta,d,\phi,c > 0$ and $x > 0$.

$$F_{EGEFD}(x) = 1 - \left\{1-\left[1-(1-(1+\alpha x^{-\theta})^{-1})^{d}\right]^{c}\right\}^{\lambda}, \quad (49)$$

$\alpha,\lambda,\theta,d,c > 0$ and $x > 0$.

$$F_{WNH}(x) = 1 - e^{\left[-ae^{((1+\lambda x)^{\alpha}-1)}\right]^{b}}, \quad (50)$$

$\alpha,\lambda,a,b > 0$ and $x > 0$.

5.1 Application I: Failure times of 36 appliances

The first application used failure times of 36 appliances subjected to an automatic life test. This data was obtained by [16]. Table 4 gives this data set.

Table 3: Monte Carlo Simulation Results of parameters of the NHGPW distribution

n	Parameter value					ABiase					MSE				
	α	β	λ	θ	γ	α	β	λ	θ	γ	α	β	λ	θ	γ
40	0.4	0.4	0.5	0.3	2.7	0.43	0.21	0.28	0.150	0.35	0.24	0.06	0.10	0.07	0.17
80	0.4	0.4	0.5	0.3	2.7	0.41	0.20	0.26	0.10	0.34	0.23	0.06	0.08	0.04	0.16
120	0.4	0.4	0.5	0.3	2.7	0.41	0.20	0.20	0.10	0.34	0.22	0.05	0.06	0.03	0.16
160	0.4	0.4	0.5	0.3	2.7	0.40	0.20	0.19	0.08	0.34	0.22	0.06	0.06	0.02	0.16
200	0.4	0.4	0.5	0.3	2.7	0.39	0.20	0.17	0.07	0.33	0.20	0.05	0.05	0.02	0.15
40	0.3	0.5	0.7	0.5	2.5	0.53	0.28	0.34	0.41	0.52	0.35	0.10	0.13	0.20	0.28
80	0.3	0.5	0.7	0.5	2.5	0.41	0.28	0.28	0.35	0.53	0.24	0.10	0.10	0.15	0.28
120	0.3	0.5	0.7	0.5	2.5	0.36	0.28	0.27	0.32	0.54	0.19	0.07	0.08	0.13	0.28
160	0.3	0.5	0.7	0.5	2.5	0.32	0.23	0.23	0.29	0.53	0.16	0.01	0.07	0.11	0.28
200	0.3	0.5	0.7	0.5	2.5	0.30	0.27	0.20	60.28	0.52	0.14	0.01	0.06	0.10	0.27
40	0.32	0.5	0.8	0.6	2.7	0.62	0.35	0.28	0.39	0.66	0.41	0.13	0.09	0.16	0.44
80	0.32	0.5	0.8	0.6	2.7	0.59	0.34	0.25	0.38	0.66	0.38	0.13	0.07	0.15	0.43
120	0.32	0.5	0.8	0.6	2.7	0.57	0.34	0.25	0.38	0.66	0.36	0.13	0.07	0.15	0.44
160	0.32	0.5	0.8	0.6	2.7	0.57	0.34	0.24	0.38	0.66	0.36	0.13	0.07	0.15	0.43
200	0.32	0.5	0.8	0.6	2.7	0.57	0.33	0.24	0.38	0.66	0.36	0.13	0.06	0.15	0.43

The TTT transformed plot of the failure times of the 36 appliances is shown in Figure 3. The data have a modified bathtub shaped hazard function since its hazard function is first convex in shape, followed by a concave shape and then another convex shape.

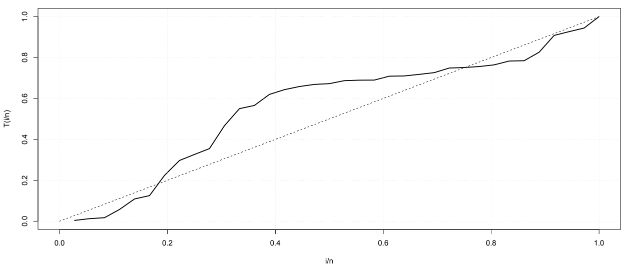

Fig. 3: TTT plot Failure times of 36 appliances data set

The detailed parameter estimates of the NHGPW distribution and the competitive distributions considered for the failure times of the 36 appliances are shown in Table 5. By using the standard errors of the NHGPW distribution, it is seen that all the parameters are significant at 5 percent significance level since their standard errors are less than half of their parameter estimates.

Table 6 presents the goodness-of-fit measures and the information criteria for the considered distribution for the failure times of the 36 appliances. As compared to the other distributions, the developed NHGPW distribution has the highest log-likelihood value with the smallest values of the Kolmogorov Smirnov (K-S) and Cramér-Von Mises (W*) statistics. By using the model selection criteria, the NHGPW distribution has the smallest AIC, AICC, and BIC values as compared to the other distributions. These indicate that, the NHPGW distribution provides a better fit to the failure times of the 36 appliances data as compared to the existing distributions.

The asymptotic variance-covariance matrix for the estimated parameters of the NHGPW distribution for the 36 appliance data is given by;

$$A^{-1}=\begin{bmatrix} 2.5474\times10^{-6} & -7.8642\times10^{-5} & -1.9109\times10^{-6} & -7.3224\times10^{-6} & 1.2569\times10^{-5} \\ -7.8642\times10^{-5} & 1.3349\times10^{-3} & -1.1343\times10^{-4} & 4.4359\times10^{-3} & 2.9433\times10^{-3} \\ -1.9109\times10^{-6} & -1.1343\times10^{-4} & 5.0988\times10^{-5} & 9.1186\times10^{-5} & -4.8622\times10^{-4} \\ -7.3224\times10^{-6} & 4.4359\times10^{-3} & 9.1186\times10^{-5} & 1.2669\times10^{-2} & -1.0107\times10^{-2} \\ 1.2569\times10^{-5} & 2.9433\times10^{-3} & -4.8622\times10^{-4} & -1.0107\times10^{-2} & 8.3314\times10^{-3} \end{bmatrix}$$

The approximate 95 percent confidence interval (CI) for the five parameters of the NHGPW distribution are; $\alpha : [0.0161;0219], \beta : [0.1399;0.2833], \lambda : [0810;1089], \theta : [0.1831;0.6245]$; and $\gamma : [0.4602;0.8178]$. The estimated CI of the parameters of the NHGPW distribution also showed that, its parameters were all significant at 5 percent significance level since their estimated confidence intervals do not contain zero.

Table 4: Failure times of 36 appliances

11	35	49	170	329	381	708	958	1062	1167	1594	1925	1990
2223	2327	2400	2451	2471	2551	2565	2568	2694	2702	2761	2831	3034
3059	3112	3214	3478	3504	4329	6367	6976	7846	13403			

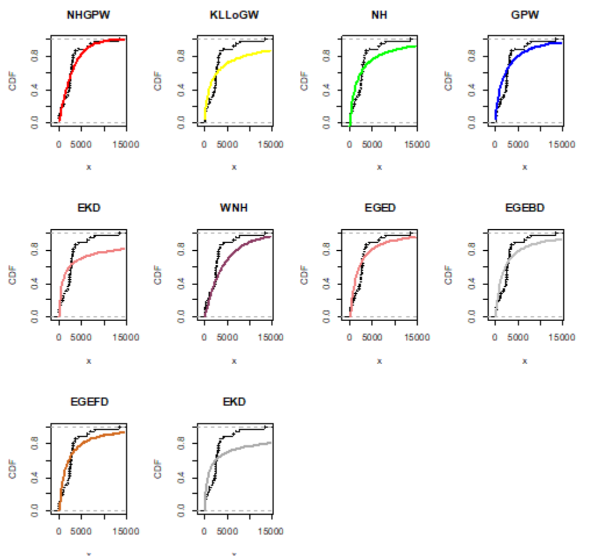

Fig. 4: Plots of CDFs of the failure times of the 36 appliancesdata set

Figure 4 gives the plot of the empirical CDF of the failure times of the 36 appliances, the CDF of the NHGPW distribution and the CDFs of the comparison distributions. As it is seen from the figure, the NHGPW distribution fits better to this data set as compared to the considered models since its CDF approximates the empirical CDF.

5.2 Application II: Air Craft Windshield Failure Data Set

The NHPGW was also applied on failure times of 84 aircraft windshield data. This data was given in Murthy et al. (2004). The failure rate data for the 84 aircraft windshield is given in Table 8.
The TTT transform plot of this data in Figure 5 indicated that, the data set has an increasing failure rate.

The parameter estimates and standard errors of the NHGPW distribution and the candidate distributions for the aircraft windshield failure rate data are shown in Table 8. It is seen that parameters α, β, λ and γ are significant at 5 percent significant level since their standard errors are less than half of their parameter estimates.

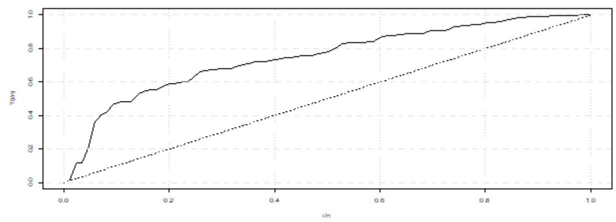

Fig. 5: TTT Plot of 84 Aircraft Windshield data set

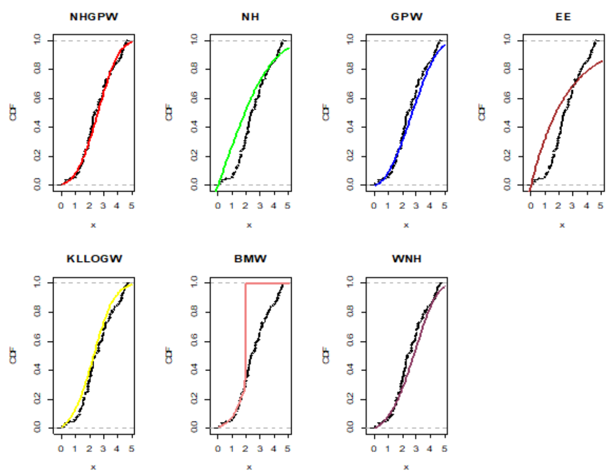

Fig. 6: Empirical CDF and CDF plots of Aircraft Windshield Failure data set

The goodness-of-fit and information criteria for the competitive distributions are presented in Table 9. Among the competitive distributions, the developed NHGPW is shown to be the best distribution for the aircraft windshield data set since it has the minimum value of all the information criteria as well as the minimum value of the goodness-of-fit statistics and the largest log likelihood value.
The asymptotic variance covariance matrix for the parameter estimates of the NHGPW distribution for the air craft windshield failure data is given by;

$$A^{-1}=\begin{bmatrix} 8.126\times10^{-5} & 1.324\times10^{-5} & 3.415\times10^{-4} & -2.712\times10^{-2} & 4.981\times10^{-3} \\ 1.324\times10^{-5} & 4.076\times10^{-6} & 4.283\times10^{-5} & -4.358\times10^{-3} & 1.060\times10^{-3} \\ 3.415\times10^{-4} & 4.283\times10^{-5} & 1.690\times10^{-6} & -1.145\times10^{-1} & 1.869\times10^{-2} \\ -2.712\times10^{-2} & -4.358\times10^{-3} & -1.145\times10^{-1} & 9.052 & -1.651 \\ 4.981\times10^{-3} & 1.060\times10^{-3} & 1.869\times10^{-2} & -1.651 & 0.352 \end{bmatrix}$$

Table 5: Maximum Likelihood Parameter Estimates of the failure times of the 36 appliances

Distribution						
Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$	
NHGPW	0.019	0.2116	0.095	0.4038	0.6390	
	(0.0015)	(0.0366)	(0.0071)	(0.1126)	(0.0912)	
NH	$\hat{\alpha}$	$\hat{\beta}$				
	0.0021	0.3642				
	(0.0011)	(0.0794)				
GPW			$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$	
			0.0047	0.7072	0.8951	
			(0.0044)	(0.1163)	(0.2599)	
KLLoGW	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$	
	13.7743	41.0088	0.0528	0.0159	0.4332	
	(3.7281)	(0.2302)	(0.0721)	(0.0096)	(0.0571)	
EGED	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$
	0.001	27.198	4.560	2.838	20.866	0.070
	(0.0001)	(0.001)	(0.847)	(0.123)	(0.010)	(0.003)
EKD	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\phi}$	$\hat{\theta}$	
	5.561	12.683	3.716	0.128	11.609	
	(1.517)	(2.158)	(3.716)	(0.128)	(3.922)	
EGEBD	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$	
	25.705	14.152	3.412	8.332	0.047	
	(0.514)	(0.110)	(0.247)	(1.934)	(0.009)	
EGFD	$\hat{\alpha}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$		
	8.4843	3.429	16.533	0.143		
	(1.550)	(0.711)	(5.833)	(0.034)		
EGEFD	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$	
	13.048	27.555	3.561	9.084	0.047	
	(1.817)	(0.071)	(0.392)	(2.186)	(0.009)	
WNH	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$		
	0.0410	7.8923	1.7341	5.7122		
	(0.0010)	(0.0567)	(0.1325)	(0.7444)		

For the aircraft windshield data, the approximate 95 percent CI of the five parameters of the NHGPW distribution are; α : $[101.8083; 101.8437], \beta$: $[0.0006; 0.0086], \lambda$: $[0.00045; 0.0055], \theta$: $[-3.8530; 7.9410]$; and $\gamma: [1.1401; 3.4659]$.

The plot of the empirical CDF, the CDF of the NHGPW distribution and the CDFs of the competitive distributions are shown in Figure 6. From the plots, the CDF of the NHGPW distribution approximates the empirical CDF of the aircraft windshield failure data set hence provides a

Table 6: Goodness of fit and Information Criteria of failure times of the 36 appliances

Distribution	LL	AIC	AICc	BIC	CVM	KS
NHGPW	−302.600	615.202	617.202	622.833	0.255	0.169
NH	−309.850	623.705	623.705	626.758	0.495	0.256
GPW	−307.040	661.805	662.555	666.385	0.351	0.590
KLLoGW	−312.850	635.705	637.705	643.336	0.502	0.236
EGED	−328.870	669.740	670.957	679.241	0.569	0.569
EKD	−341.650	693.295	694.198	701.213	0.925	0.269
EGEBD	−330.910	671.823	672.726	679.741	0.272	0.634
EGFD	−341.030	690.054	690.692	696.388	0.907	0.269
EGEFD	−330.730	671.460	672.363	679.377	0.625	0.269
WNH	−322.200	652.392	653.682	658.726	0.343	0.204

Table 7: Failure times data of 84 Aircraft Windshield

4.167	1.281	3.00	4.035	2.3	3.344	4.602	1.757	2.324	2.265	3.578	0.943	4.121
1.303	2.089	2.632	2.135	2.962	2.688	2.902	0.557	1.911	1.568	3.595	1.07	4.255
1.899	2.61	3.478	1.248	2.01	1.194	1.505	2.154	2.964	4.278	1.056	0.309	1.281
1.912	3.924	2.19	3	4.305	3.376	2.246	3.699	1.432	2.097	2.934	4.24	1.48
2.194	3.103	4.376	1.615	2.223	0.04	1.866	2.385	3.443	0.301	1.876	2.481	3.467
4.663	2.085	2.89	2.038	2.82	1.124	1.981	2.661	3.779	3.114	4.449	1.619	2.224
3.117	4.485	1.652	2.229	3.166	4.57	1.652						

better fit as compared to the other distributions considered.

6 Conclusion

In this study, the NHGPW distribution is developed based on the concept of compounding for systems connected in series. The statistical properties of this distribution are derived. From the analysis, we conclude that, the CDF and PDF of the developed NHGPW distribution are well defined and meet all necessary condition of a probability distribution. The plot of the PDF and hazard functions of the NHGPW distribution indicated that this distribution can adequately model both monotonic and non-monotonic failure rate data set since its PDF can be decreasing, increasing, bathtub, unimodal, modified bathtub and symmetric and its hazard function can also be constant, monotonically increasing, decreasing, bathtub, unimodal and modified bathtub. The NHGPW distribution is also very flexible as compared to existing distributions since it contains several well-known distributions as sub-distributions hence contains the desirable properties of these sub-distributions. Monte Carlo simulation analysis on the maximum likelihood estimators showed that, the estimators of the NHGPW distribution are consistent since they converges to the true parameter value as the sample size increases. Based on the log-likelihood value, Kolmogorov Smirnov (KS), Cramér-Von Mises (W*), AIC, AICc, and BIC statistics, the NHGPW distribution provides a better fit to the failure times of 36 appliances data set and the aircraft windshield failure rate data set when compared with the competitive distributions for systems connected in series. Also, the parameter estimates of the distribution are all significant at 5 percent significant level.

Conflicts of Interests

The authors declare that they have no conflicts of interests

Table 8: Maximum Likelihood Parameter Estimate for Aircraft Windshield Failure Data

Distribution					
NHGPW	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$
	101.826	0.0046	0.003	2.044	2.303
	(0.009)	(0.000)	(0.000)	(3.009)	(0.594)
NH	$\hat{\alpha}$	$\hat{\beta}$			
	0.008	33.695			
	(0.000)	(0.000)			
GPW			$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$
			0.01	1.757	10.051
			(0.002)	(0.172)	(0.001)
Exp-Exp	0.345		0.045		
	(0.021)		(0.021)		
BetaMW	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$
	0.286	0.009	6.224	2.543	0.057
	(0.184)	(0.003)	(0.012)	(1.062)	(0.341)
KLLoGW	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$
	7.934	12.501	0.129	0.15	1.286
	(5.499)	(43.587)	(0.143)	(0.114)	(0.651)
WNH	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$	
	0.562	4.22	0.023	1.088	
	(0.226)	(9.357)	(0.043)	(0.547)	

Table 9: Goodness of Fit and Information Criteria of Air Craft Windshield Data

Distribution	LL	AIC	AICc	BIC	CVM	K-S(p-value)
NHGPW	-127.230	264.059	264.828	276.272	0.062	0.085
NH	-145.550	295.099	295.248	299.985	0.082	0.258
Exp-Exp	-164.990	333.975	334.123	338.861	0.166	0.303
GPW	-128.960	686.572	686.872	693.900	0.307	0.915
BetaMW	-128.260	264.517	265.286	277.730	3.129	0.682
KLLoGW	-127.990	265.971	266.740	278.184	0.077	0.086
WNH	-128.180	264.355	264.861	274.125	0.105	0.088

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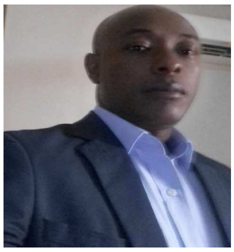

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