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Study of a Continuous Fractional-Order Dynamical System of Fibrosis of Liver

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Abstract: In this manuscript, our objective is to develop a fractional-order linear model (FOLM) that characterizes the behavior of the human liver, specifically focusing on the dynamics of bromsulphthalein storage and transfer, particularly in the context of chronic alcoholic liver disease. To achieve this, we employ a novel approach utilizing non-orthogonal Bernstein polynomials to construct operational matrices. These matrices play a crucial role in describing the fractional-order coupled dynamics within the human liver system. We assessed the presence and stability of a solution through the utilization of well-established fractional calculus findings. Drawing from precise clinical data, we present numerical instances accompanied by graphical representations illustrating the fractional order dynamics inherent in our proposed model. To reinforce the credibility and efficacy of our novel approach, we conclude with a contrastive analysis against integer-order outcomes. The resultant matrix equation, stemming from the fractional operational matrices, is systematically elucidated utilizing the computational capabilities of the mathematical software Matlab. This step unveils the intricate details of the transformed system, shedding light on the underlying dynamics.

Keywords: Bernstein polynomials, FOLM, operational matrix, algebraic matrix equation.

1 Introduction

The significance of fractional calculus in mathematical modeling is undeniable. In the last few decades, it has been observed that in comparison to integer-order derivatives, many systematic approaches in science and engineering are modeled with greater precision by fractional-order. In contrast to classical calculus, arbitrary-order calculus has been found very befitting in the mathematical modeling of various real-life problems. Its dynamical aptitude has a greater degree of freedom and hence, it is tremendously applicable in day-to-day life for modeling not only biological physical phenomena, but also, a variety of applications on account of fractional order differential and integral equations (FODIEs) are found in the fields of chemistry, engineering, medicine, economics, and social sciences as well. The global dynamical decency of fractional calculus has attracted researchers, which led to the flourishing of several techniques to unravel the problems described in a fractional system. A comprehensive array of attributes linked to memory and hereditary processes is thoroughly expounded upon within this framework [1,2,3,4,5,6]. As a result, its significance has been firmly established in the recent scrutiny of diverse quandaries, intersecting with models in biophysics, biochemistry, the biological and social sciences, as well as image processing and an array of computer dynamics domains [7,8,9]. Similarly, intriguing applications of Fractional-Order Differential and Integral Equations (FODIEs) have been meticulously explored across a spectrum of disciplines, encompassing engineering, technology, and applied sciences [10,

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11,12]. Due to these applications and its gigantic success, arbitrary-order calculus has remained the center of attraction for new researchers.

In contrast with fractional calculus, mathematical modelling is an essential tool for expressing real-life problems in the language of mathematics. Whereas, fractional calculus (FC) has tremendous potential to change the way we see a mathematical model and control the results to compare with the exact data available [13,14,15]. In the present arena, FC is found in the front row, playing a leading role in the modelling of various physical, physiological, biological, and chemical phenomena. It has proven to be a landmark in the control of metabolic risk factors in people fighting obesity, hypertension, and diabetes and in examining various endocrine systems in the human body [16,17]. Since its birth, it has gained extraordinary popularity in the field of mathematical sciences. In the era of emergence, it has attracted a number of researchers in a variety of disciplines. Arbitrary order calculus has been widely used to describe numerous complex biological models in an effective manner. Various features in connection to the memory process and different hereditary processes are exclusively defined with the help of (FODIEs). It has paved the way towards a new horizon of exploration.

The newly emerging technique of generating operational matrices is of great interest, as it has omitted the discretization of data that has saved human effort, memory, and time [18,19]. Where Lebinitz was the first to introduce fractional orders, Lacroix was the first to give a written definition of fractional calculus. Many mathematicians have given their own definitions of fractional order derivative and integration. The lucky among them, gaining popularity, are the Riemman-Liouville and Caputo definitions of fractional integration and differentiation. When the exact solution is difficult to obtain, approximate solutions are tremendously taking the lead to exact solutions, and nowadays the phenomenon is running at a pace causing scholars to derive new techniques for providing approximate solutions coinciding with exact solutions with a high level of precision. In the available literature, different scientists have used different orthogonal polynomials for approximation of functions and generation of operational matrices like Legendre polynomials, Lagurre polynomials, Hermite polynomials, etc., which has remained an issue under consideration. Some authors have also used non-orthogonal polynomials for creating operational matrices and approximating results. Therefore, since these non-orthogonal polynomials cover a broader spectrum of approximation, they have gained importance in recent developments [20,21]. In this paper, we are using Bernstein polynomials, a non-orthogonal basis, for the generation of operational matrices and to approximate fractional order differentiation. In situations where exact solutions are not available, these operational matrices can provide the best approximate solutions.

The nurturing essence of the liver has captivated the scientific community's interest, prompting its modeling due to its indispensable functions within the human body [22,23,24,25]. A robust liver ensures overall bodily well-being, as it functions as a pivotal element in numerous physiological processes. This triangular organ exerts profound control over diverse glucose metabolisms. Moreover, it assumes a prominent role in overseeing the intricate workings of the human immune system. Various clinical observations have shown that the majority of people in society are facing distinct abnormalities, from mild to chronic, in their liver. The unevenness and irregularities observed in the liver due to a variety of reasons include raised levels of serum glutamic oxaloacetic transaminase (SGOT), serum alkaline phosphatase (SAP), and bromsulphthaleine retention, etc. The liver not only regulates blood flow and clotting in our bodies, but it also aids in the breakdown of various drugs and toxic compounds that we consume during various processes. One of the major roles that the liver plays in our body is controlling glucose homeostasis by commanding different metabolism alleyways.

Besides, a number of liver diseases prevail in society due to different environmental and unhealthy food intake factors. The most common viral liver infections are hepatitis B and hepatitis C, whereas, cancerous liver diseases like liver chirosis and hepatocellular carcinoma are also hitting their peak. In addition to that, different risk factors have increased the incidence of alcoholic liver disease (ALD) in the present arena. In recent developments, several diagnostic laboratory tests are used to investigate these liver diseases, like biopsy, liver CT, MRI, liver profile, etc.

In the realm of human liver modeling, Celechovska pioneered a simplistic model back in 2004 [23]. Building upon this, in 2020, Baleanu et al. elevated the simplicity by transforming the basic human liver model into a fractional variant, employing the Caputo Fabrizio derivative [24]. Furthermore, complementing Baleanu's work, Ismail Gad Ameen contributed to the field by delving into the Mittag Leffler function with fractional order in a publication from 2021 [25]. Inspired by these researchers, we are redefining the model of the human liver with the help of Caputo derivatives by using function approximation with a non-orthogonal Bernstein polynomial. The amount of bromsulphtaleine (BSP) in the blood and liver is used to investigate the retention of BSP transferred from blood to liver, liver to bile, and liver to blood [26]. To proceed further, the Caputo fractional differential model of the human liver is presented as

$$\begin{cases} {}_0^c\mathrm{D}_\mathrm{t}^\zeta \mathrm{Z(t)} = -K_1 Z(\mathrm{t}) + K_2 W(\mathrm{t}),\ 0 < \zeta \leq 1, \\ {}_0^c\mathrm{D}_\mathrm{t}^\zeta \mathrm{W(t)} = K_1 Z(\mathrm{t}) - (K_2 + K_3) W(\mathrm{t}),\ 0 < \zeta \leq 1, \\ \mathrm{Z}(0) = L,\ \mathrm{W}(0) = 0, \end{cases} \tag{1}$$

where K_1 represents the transfer rate of BSP into the liver from the blood, K_2 is the transfer rate of BSP into bile from the liver, and transportation of BSP from the liver to blood is denoted by K_3. The functions $Z(t)$ and $W(t)$ present the amount of BSP in the blood and liver, respectively. After a review of the literature, we have composed this paper in 6 sections.

Section 1 is devoted to an introduction; section 2 contains basic definitions. Section 3 is devoted to the existence and uniqueness of the numerical results. In section 4 we modelled the human liver by using fractional differential equations with the help of Bernstein basis, and an algorithm was developed for the numerical solution of the developed model of the human liver. In section 5 we have given a tabulated comparison of clinical data with some already existing integer order results with its fractional counter part. In the last section 6, we have concluded our research article with closing remarks.

2 Basic results

Here, we produced some basic results and definitions of arbitrary order differentiation and integration. For a better understanding of Bernstein polynomial, some fundamentals of Bernstein polynomial are mentioned. Moreover, some definitions are given in [1,2,3].

Definition 1.*[27] The arbitrary-order differentiation and integration of order ζ of a function $w(\mathrm{t})$ converging in R^+ where, $w:[0,\infty)\to R$, is defined as*

$$I^{\zeta}w(\mathrm{t})=\frac{1}{\Gamma(\zeta)}\int_0^{\mathrm{t}}(\mathrm{t}-\mu)^{\zeta-1}w(\mu)d\mu,\ n-1<\zeta\leq n, \tag{2}$$

further, Expressing the generalized arbitrary integral of a function $w:\mathbb{R}^+\to\mathbb{R}$, specifically for the form $w(t)=(t-p)^j$, involves:

$$I^{\zeta}(\mathrm{t}-r)^{\mathrm{j}}=\frac{\Gamma(\mathrm{j}+1)}{\Gamma(\mathrm{j}+\zeta+1)}(\mathrm{t}-r)^{\mathrm{j}+\zeta}. \tag{3}$$

Definition 2.*[1] For an absolute continuous function $w(\mathrm{t})$ fractional order derivative of a function $w:R^+\to R$ in the Caputo sense is defined as under*

$$D^{\zeta}\mathrm{w}(\mathrm{t})=\begin{cases}\dfrac{1}{\Gamma(n-\zeta)}\displaystyle\int_0^{\mathrm{t}}(\mathrm{t}-\mu)^{n-\zeta-1}\mathrm{w}^{(n)}(\mu)d\mu,\ n-1<\zeta<n,\\ \dfrac{d^n s}{d\mathrm{t}^n},\end{cases} \tag{4}$$

where $n=[\zeta]+1$, moreover

$$D^{\zeta}(\mathrm{t}-p)^{\mathrm{j}}=\frac{\Gamma(\mathrm{j}+1)}{\Gamma(\mathrm{j}-\zeta+1)}(\mathrm{t}-p)^{\mathrm{j}-\zeta}. \tag{5}$$

Lemma 1.*The solution of the problem with $\mathrm{y}\in L[0,T]$*

$$D^{\zeta}\mathrm{w}(\mathrm{t})=\mathrm{h}(t),\ \zeta\in(0,1]$$
$$\mathrm{w}(0)=\mathrm{w}_0$$

is given by

$$\mathrm{w}(\mathrm{t})=\mathrm{w}_0+\frac{1}{\Gamma(\zeta)}\int_0^{\mathrm{t}}\frac{\mathrm{h}(\mu)}{(t-\mu)^{1-\zeta}}d\mu.$$

Definition 3.*Bernstein Polynomials: Bernstein polynomials, formulated by the Russian mathematician Sergei Natanovich Bernstein, constitute a class of non-orthogonal polynomials. Though the BPs are non-orthogonal, but it has a wide use in function approximation of real valued function. Bps are formed by the linear combination of Bernstein basis [28,29,30,31,32]. The Bernstein definition is given as*

$$\mathrm{B}_{a,b}(\mathrm{t})=\sum_{a=0}^{b}\binom{\mathrm{b}}{\mathrm{a}}\mathrm{t}^{\mathrm{a}}(1-\mathrm{t})^{\mathrm{b}-a},\ a=0,1,2,\cdots,\mathrm{b},$$

where b presents order of the polynomial. Expanding the given polynomial by using binomial expansion, we get

$$\mathrm{B}_{a,b}(\mathrm{t})=\sum_{a=0}^{b}\sum_{l=0}^{b-a}(-1)^b\binom{b}{a}\binom{b-a}{l}\mathrm{t}^{l+a},\ a=0,1,2,\cdots,b,$$

where

$$B_{i,l,b}=(-1)^b\binom{b}{a}\binom{b-a}{l} \tag{6}$$

Approximation methods are applied to those problems where we fail to affirm the accuracy at the given point. One of the scintillating polynomials that has attained significance is BPs. The following approximation can be used to approximate any function ϖ in $L^2[0,1]$ in BPs.

$$\varpi(\mathrm{t}) = \sum_{a=0}^{b} s_a B_{a,b}(\mathrm{t}). \tag{7}$$

In same line product of matrices can be computed as in [34].

Here we recalled some operational matrices for integral and differential as given in [33].

Lemma 2.

$$\begin{aligned} R_{1\times b} &= \int_0^1 \varpi(\mathrm{t})\mathrm{B}_{a,b}(\mathrm{t})dt \\ &= \int_0^1 \sum_{a=0}^{b} s_a \mathrm{B}_{a,b}(\mathrm{t})\mathrm{B}_{j,b}(\mathrm{t})d\mathrm{t}, \\ &= \sum_{a=0}^{b} s_a \int_0^1 \mathrm{B}_{a,b}(\mathrm{t})\mathrm{B}_{j,b}(\mathrm{t})d\mathrm{t}, \end{aligned}$$

where $j = 0,1,2,\cdots$. Hence, we get

$$R_{1\times b} = \begin{bmatrix} s_0 & s_1 & \cdots & s_b \end{bmatrix} \begin{bmatrix} \mathrm{G}_{0,0} & \mathrm{G}_{0,1} & \cdots & \mathrm{G}_{0,r} & \cdots & \mathrm{G}_{0,b} \\ \mathrm{G}_{1,0} & \mathrm{G}_{1,1} & \cdots & \mathrm{G}_{1,r} & \cdots & \mathrm{G}_{1,b} \\ \mathrm{G}_{2,0} & \mathrm{G}_{2,1} & \cdots & \mathrm{G}_{2,r} & \cdots & \mathrm{G}_{2,b} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \mathrm{G}_{3,0} & \mathrm{G}_{3,1} & \cdots & \mathrm{G}_{3,r} & \cdots & \mathrm{G}_{3,b} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \mathrm{G}_{b,0} & \mathrm{G}_{b,1} & \cdots & \mathrm{G}_{b,r} & \cdots & \mathrm{G}_{b,b} \end{bmatrix},$$

where $S_{1\times b} = \begin{bmatrix} s_0 & s_1 & \cdots & _b \end{bmatrix}$ is the array of coefficients , hence

$$S_{1\times b} = R_{1\times b}\mathrm{G}_{i,j}^{-1}. \tag{8}$$

Lemma 3.*[33] Let a vector function $\mathrm{B}_{a,b}(\mathrm{t})$ mentioned in (7), subsequently, the integration of the function with a fractional order is expressed as:*

$$I^{\varkappa}P_{i,\varsigma}(t) = V^{\varkappa}_{\varsigma\times\varsigma}P_{i,\varsigma}(t), \tag{9}$$

here $V^{\varkappa}_{\varsigma\times\varsigma}$ represent the Operational Matrix of integral, next

$$V^{\varkappa}_{\varsigma\times\varsigma} = \Upsilon_{\varsigma\times\varsigma}G^{-1}_{\varsigma\times\varsigma},$$

Furthermore,

$$\Upsilon^{\varkappa}_{c\times c} = \begin{bmatrix} \Lambda_{0,0} & \Lambda_{0,1} & \cdots & \Lambda_{0,r} & \cdots & \Lambda_{0,\varsigma} \\ \Lambda_{1,0} & \Lambda_{1,1} & \cdots & \Lambda_{1,r} & \cdots & \Lambda_{1,\varsigma} \\ \Lambda_{2,0} & \Lambda_{2,1} & \cdots & \Lambda_{2,r} & \cdots & \Lambda_{2,\varsigma} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \Lambda_{r,0} & \Lambda_{r,1} & \cdots & \Lambda_{r,r} & \cdots & \Lambda_{r,\varsigma} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \Lambda_{\varsigma,0} & \Lambda_{\varsigma,1} & \cdots & \Lambda_{\varsigma,r} & \cdots & \Lambda_{\varsigma,\varsigma} \end{bmatrix}, \tag{10}$$

where

$$\Lambda_{i,j} = \sum_{l=0}^{\varsigma-i}\sum_{u=0}^{\varsigma-j} P_{i,l,k}P_{j,u,\varsigma}\frac{\Gamma(\varsigma+i+1)}{(i+j+\varsigma+l+\varkappa+1)\Gamma(\varsigma+i+\varkappa+1)} \tag{11}$$

Definition 4.*Operational Matrices of differential: [33] Taking into account the vector function $P_{i,k}(t)$, the fractional order differentiation of the function can be formulated as follows:*

$$D^{\varkappa}P_{i,\varsigma}(t) = W^{\varkappa}_{\varsigma\times\varsigma}P_{i,\varsigma}(t), \tag{12}$$

and

$$W^{\varkappa}_{\varsigma\times\varsigma} = \P^{\varkappa}_{\varsigma\times\varsigma}G^{-1}_{\varsigma\times\varsigma}, \tag{13}$$

and $\P^{\varkappa}_{\varsigma\times\varsigma}$ is defined as:

$$\P^{\varkappa}_{\varsigma\times\varsigma} = \begin{bmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,r} & \cdots & \alpha_{0,\varsigma} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,r} & \cdots & \alpha_{1,\varsigma} \\ \alpha_{2,0} & \alpha_{2,1} & \cdots & \alpha_{2,r} & \cdots & \alpha_{2,\varsigma} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_{r,0} & \alpha_{r,1} & \cdots & \alpha_{r,r} & \cdots & \alpha_{r,\varsigma} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_{\varsigma,0} & \alpha_{\varsigma,1} & \cdots & \alpha_{\varsigma,r} & \cdots & \alpha_{\varsigma,\varsigma} \end{bmatrix}, \tag{14}$$

where

$$\alpha i,j = \begin{cases} \sum_{w=\varkappa}^{\varsigma-i}\sum_{z=0}^{\varsigma-j} P_{i,w,\varsigma}P_{j,z,\varsigma}\dfrac{\Gamma(w+i-\varkappa)}{\Gamma(w+i-\varkappa+1)\Gamma(w+z+i+j-\varkappa+1)}, i<[\varkappa] \\ \sum_{w=0}^{\varsigma-i}\sum_{z=0}^{\varsigma-j} P_{i,w,\varsigma}P_{j,z,\varsigma}\dfrac{\Gamma(w+i-\varkappa)}{\Gamma(w+i-\varkappa+1)\Gamma(w+z+i+j-\varkappa+1)}, i>=[\varkappa]. \end{cases} \tag{15}$$

Definition 5.*Product of a function and Bernstein Basis: The following lemma defines the product of a function with a Bernstein approximation:*

Lemma 4.*[33] Suppose $P_{i,c}$ represents an arbitrary function vector and A_M stands for any coefficient matrix. Let $f(t)$ denote a function within the space $L^2[0,1]$, which is approximated as:*

$$f(t) \approx A_M P_{i,\varsigma}(t),$$

consequently, the Operational Matrix corresponding to the specified boundary condition is defined as:

$$B^{\varkappa}_{\varsigma\times\varsigma} = \begin{bmatrix} \aleph_{0,0} & \nabla_{0,1} & \cdots & \aleph_{0,r} & \cdots & \aleph_{0,\varsigma} \\ \aleph_{1,0} & \aleph_{1,1} & \cdots & \aleph_{1,r} & \cdots & \aleph_{1,\varsigma} \\ \aleph_{2,0} & \aleph_{2,1} & \cdots & \aleph_{2,r} & \cdots & \aleph_{2,\varsigma} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \aleph_{r,0} & \aleph_{r,1} & \cdots & \aleph_{r,r} & \cdots & \aleph_{r,\varsigma} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \aleph_{\varsigma,0} & \aleph_{\varsigma,1} & \cdots & \aleph_{\varsigma,r} & \cdots & \aleph_{\varsigma,\varsigma} \end{bmatrix}, \tag{16}$$

where ever $\aleph_{i,j}$ is presented as

$$\aleph_{i,j} = \int_0^1 \wp_{i,\varsigma}\phi(\xi)P_{j,\varsigma}(t)d\xi, i,j = 0,1,2,\cdots,\varsigma, \tag{17}$$

with

$$\wp i,\varsigma = \sum_{d=0}^{\varsigma-i} A\frac{\Gamma(d+i+1)}{(d+i+\varkappa+1)}$$

3 Convergence Analysis

This section delves into the convergence analysis of the preceding segment. Given that the Hilbert matrices acquired are resolved using diverse techniques such as QR factorization and comparable methods, they inherently encompass errors. Thus, we are poised to assess the convergence of our methodology by juxtaposing it with established approaches from the past.
The concept of "best approximation," holds profound significance, serving as a methodology to transform complex problems into more manageable counterparts, which yield solutions that closely approximate the exact solution. The origins of this concept can be traced back to Taylor's pioneering work, where he employed polynomial approximations to transcendental functions. Approximation theory constitutes the foundational framework for determining the most optimal approximation to a function or a given problem.
Assuming $\mathcal{X}$ is a normed linear space and $\mathcal{Y}$ represents a finite-dimensional subspace of $\mathcal{X}$, the following condition holds: for any $\chi \in \mathcal{X}$, there exists $y' \in \mathcal{Y}$ satisfying:

$$\|\chi - y'\| \leq \|\chi - y\|,$$

when compared to every $y \in \mathcal{Y}$, y' is denoted as the optimal approximation for χ within $\mathcal{Y}$. Leveraging these findings, function approximation is employed to achieve solutions that closely approximate the exact solutions [35].

Lemma 5.*Consider $\varphi(t) \in L^2[0,1]$, a function possessing $n+1$ differentiable properties, and let $P_M = P_0, P_1, P_3, \cdots, P_m$ denote the Bernstein Basis. If we designate $\omega(t) \in P_M$ as the optimal approximation for $\varphi(t)$, then the relationship that follows is:*

$$\|\varphi(t) - \omega(t)\| \leq \frac{\sqrt{2}RH^{\frac{2n+1}{H}}}{\sqrt{(2n+3)}\Gamma(n+2)}, \tag{18}$$

Furthermore, the provided inequality establishes the upper limit of the error, wherein:
$\mathrm{R} = \max_{t\in[0,1]} \mid \varphi^{\mathrm{n+1}}(\mathrm{t}) \mid$ *and* $\mathrm{H} = \max[1-t_0, t_0]$.

4 Main Work

If in the blood and liver the quantity of BSP, in certain time t, is presented by the function $\mathrm{Z(t)} : [0,\infty) \to R$ and $\mathrm{W(t)} : [0,\infty) \to R$, respectively, then the differential model of integer order is given by the following coupled system

$$\begin{cases} \dfrac{d\mathrm{Z(t)}}{dt} = -\mathrm{K_1 Z(t)} + \mathrm{K_2 W(t)}, \\ \dfrac{d\mathrm{W(t)}}{dt} = \mathrm{K_1 Z(t)} - (\mathrm{K_2} + \mathrm{K_3})\mathrm{W(t)}, \\ \mathrm{Z}(0) = z_0,\ \mathrm{W}(0) = 0, \end{cases} \tag{19}$$

where $\mathrm{z_0} > 0$ and $\mathrm{K_1, K_2, K_3}$ are constants representing the rates of transfer. The Fractional model of the above system is represented as

$$\begin{cases} {}_0^c\mathrm{D_t^{\zeta}Z(t)} = -\mathrm{K_1 Z(t)} + \mathrm{K_2 W(t)},\ 0 < \zeta \leq 1, \\ {}_0^c\mathrm{D_t^{\zeta}W(t)} = \mathrm{K_1 Z(t)} - (\mathrm{K_2} + \mathrm{K_3})\mathrm{W(t)},\ 0 < \zeta \leq 1, \\ \mathrm{Z}(0) = z_0,\ \mathrm{W}(0) = 0, \end{cases} \tag{20}$$

consider,

$${}_0^c\mathrm{D_t^{\zeta}Z(t)} = \mathrm{A}_M \mathrm{B}_M^T(\mathrm{t})$$

Performing a ζ-order integral on both sides yields:

$$I^{\zeta}\,{}_0^c\mathrm{D_t^{\zeta}Z(t)} = I^{\zeta}\mathrm{A}_M\mathrm{B}_M^T(\mathrm{t})$$

$$\mathrm{Z(t)} - c_0 = I^{\zeta}\mathrm{A}_M\mathrm{B}_M^T(\mathrm{t}),$$
$$\mathrm{Z(t)} = c_0 + I^{\zeta}\mathrm{A}_M\mathrm{B}_M^T(\mathrm{t}),$$

By the initial condition $Z(0) = z_0$ we get $Z(t) = z_0 + I^{\zeta} A_M B_M^T(t)$. Besides, we will Approximate $v_0 \approx D_M B_M^T(t)$ and by Lemma 1 $A_M I^{\zeta} B_M^T(t) \approx A_M V_{m\times m}^{\zeta} B_M^T(t)$. Hence the function $Z(t)$ become

$$Z(t) = D_M B_M^T(t) + A_M V_{m\times m}^{\zeta} B_M^T(t). \tag{21}$$

Further, let us assume

$$ {}_0^c D_t^{\zeta} w(t) = C_M B_M^T(t)$$

Upon applying a ζ-order integral on both sides, the result is:

$$I^{\zeta} {}_0^c D_t^{\zeta} W(t) = I^{\zeta} C_M B_M^T(t)$$

$$W(t) - c_0 = I^{\zeta} C_M B_M^T(t),$$
$$W(t) = c_0 + C_M I^{\zeta} B_M^T(t),$$

Utilizing the initial condition $W(0) = 0$, we deduce: $c_0 = 0$ and hence $W(t) = I^{\zeta} C_M B_M^T(t)$. Besides, by Lemma **??** $C_M I^{\zeta} B_M^T(t) \approx C_M V_{m\times m}^{\zeta} B_M^T(t)$. Hence the function $W(t)$ become

$$W(t) = C_M V_{m\times m}^{\zeta} B_M^T(t), \tag{22}$$

Now by using equation (21) and (22) the given couple system can be shaped as

$$\begin{aligned} A_M B_M^T(t) &= -K_1 \left[D_M B_M^T(t) + A_M V_{m\times m}^{\zeta} B_M^T(t)\right] + K_2 \left[C_M V_{m\times m}^{\zeta} B_M^T(t)\right] \\ A_M B_M^T(t) &+ K_1 \left[D_M B_M^T(t) + A_M V_{m\times m}^{\zeta} B_M^T(t)\right] - K_2 \left[C_M V_{m\times m}^{\zeta} B_M^T(t)\right] = 0 \\ A_M &+ K_1 D_M + K_1 A_M V_{m\times m}^{\zeta} - C_M V_{m\times m}^{\zeta} = 0 \end{aligned} \tag{23}$$

similarly

$$\begin{aligned} C_M B_M^T(t) &= K_1 \left[D_M B_M^T(t) + A_M V_{m\times m}^{\zeta} B_M^T(t)\right] - (K_2 + K_3) \left[C_M V_{m\times m}^{\zeta} B_M^T(t)\right] \\ C_M B_M^T(t) &- K_1 \left[D_M B_M^T(t) + A_M V_{m\times m}^{\zeta} B_M^T(t)\right] + (K_2 + K_3) \left[C_M V_{m\times m}^{\zeta} B_M^T(t)\right] = 0 \\ C_M &- K_1 D_M + K_1 A_M V_{m\times m}^{\zeta} + (K_2 + K_3) C_M V_{m\times m}^{\zeta} = 0 \end{aligned} \tag{24}$$

Hence by using (23) and (24) the aforementioned coupled model is transformed into following algebraic Sylvester matrix equation

$$\begin{bmatrix} A_M \\ C_M \end{bmatrix} + \begin{bmatrix} K_1 V_{m\times m}^{\zeta} & -K_2 V_{m\times m}^{\zeta} \\ K_1 V_{m\times m}^{\zeta} & (K_2 + K_3) V_{m\times m}^{\zeta} \end{bmatrix} \begin{bmatrix} A_M \\ C_M \end{bmatrix} + \begin{bmatrix} K_1 \\ -K_1 \end{bmatrix} \begin{bmatrix} D_M \end{bmatrix} = 0. \tag{25}$$

The presented matrix equation describes the required model of human liver with the bernstein type operational matrices.

5 Numerical Results and Discussion

Consider the the model (1) as

$$\begin{cases} {}_0^c D_t^{\zeta} Z(t) = -0.09 Z(t) + 0.038 W(t), \ 0 < \zeta \leq 1, \\ {}_0^c D_t^{\zeta} W(t) = 0.066 Z(t) - 0.038 W(t), \ 0 < \zeta \leq 1, \\ Z(0) = 220, \ W(0) = 0, \end{cases} \tag{26}$$

The exact solution at $\zeta = 1$ is computed as

$$\begin{aligned} Z(t) &= -K_1 220 \exp(-0.1204t) + 0.2696 w_0 \exp(-0.0076t) \\ W(t) &= 0.2696 w_0 \exp(-0.1204t) + 0.7304 w_0 \exp(-0.0076t). \end{aligned}$$

Now we first present the numerical solution corresponding to different fractional orders for both compartments in Figures 1 and 2 respectively.

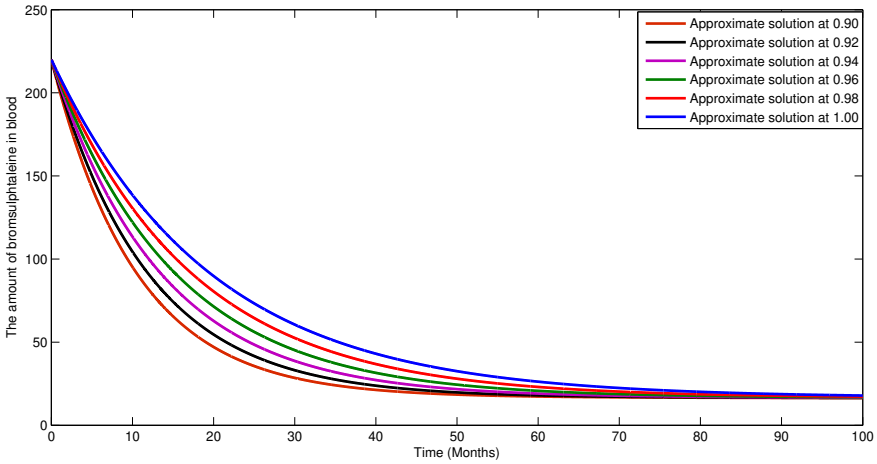

Fig. 1: Numerical solutions of the amount of BSP in blood at various fractional order taking scale level =10 for system 26.

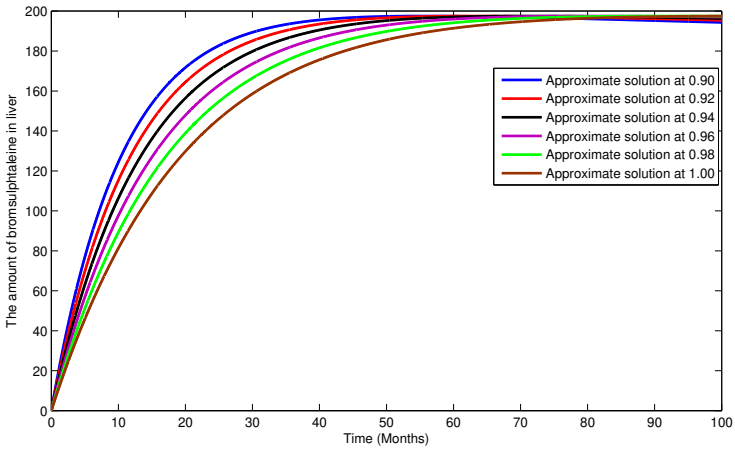

Fig. 2: Numerical solutions of the amount of BSP in liver at various fractional order taking scale level =10 for system 26.

Here we see that as the $\zeta \to 1$, the corresponding solution approaches to the exact solution. Figure 1 shows the decay at different fractional order while Figure 2 shows increase in the BSP amount in liver. Next we compared our exact and numerical solution in Figures 3 and 4 respectively. We see that both have close agreement which shows the efficiency of the proposed method.

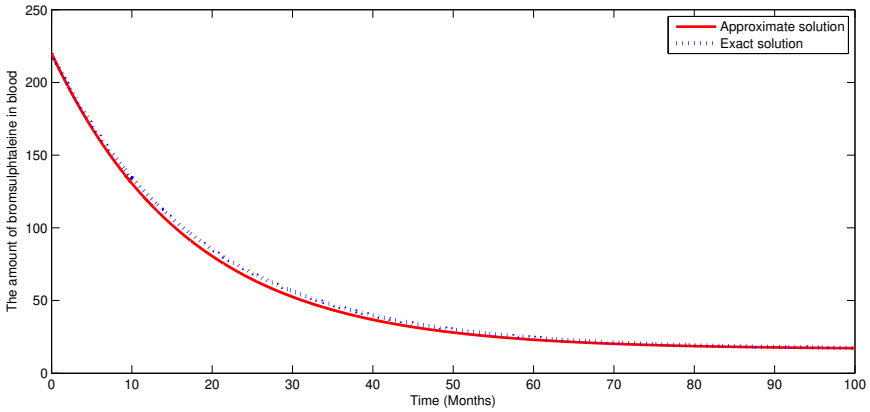

Fig. 3: Comparison between Numerical and exact solutions of the amount of BSP in blood taking scale level =10 for system 26.

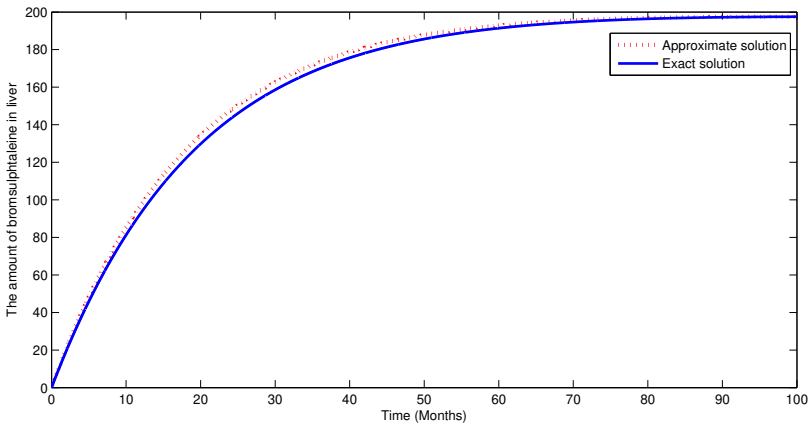

Fig. 4: Comparing the numerical and exact solutions of the amount of BSP in liver at various fractional order taking scale level =10 for system 26.

6 Conclusion

A mathematical model of the liver has been treated numerically by using a spectral method based on Bernstein polynomials. Without discretizing we have formed a numerical scheme for the solution of the proposed system. The method is rapidly convergent as well as stable. Further omitting collocation and discretization save enough memory and time during such a process. The graphical presentations have been given against various fractional orders. Also,

comparison between exact and approximate solutions is also given at scale level =10. Also, as fractional order is approaching integer-order the solution is also converging to the exact solution at integer order.

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