Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/090301

Λ-Fractional Logistic Equation

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Received: 2 Sep. 2021, Revised: 6 Oct. 2021, Accepted: 6 Feb. 2022
Published online: 1 Jul. 2023

Abstract: Λ-fractional logistic differential equation is considered, providing explicit solution in the Λ-space. Comparison between the solution to the classical logistic differential equation and the solution to the Λ-fractional logistic differential equation is presented.

Keywords: Λ-fractional logistic differential equation, fractional calculus, logistic differential equation.

1 Introduction

Let $T > 0$ and $u \in \mathrm{AC}[0,T]$. The Riemann-Liouville derivative of order $\alpha \in (0,1)$ is defined as [1]

$$D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}u(s)ds, \quad t \in [0,T].$$

We recall that for $n \in \mathbf{N}$,

$$D^{\alpha}t^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}. \tag{1}$$

In general, for $\alpha > 0$ and $\gamma > -1$,

$$D^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha}. \tag{2}$$

For $\gamma = 2$, we have

$$D^{\gamma}t^2 = \frac{\Gamma(3)}{\Gamma(3-\gamma)}t^{2-\gamma} = \frac{2t^{2-\gamma}}{\Gamma(3-\gamma)}. \tag{3}$$

The Λ-fractional derivative is defined as [2,3]

$${}^{\Lambda}D^{\alpha}u(t) = \frac{D^{\alpha}u(t)}{D^{\alpha}t}. \tag{4}$$

Therefore, from (1)

$${}^{\Lambda}D^{\alpha}t^n = \frac{\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}}{\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}} = \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n-\alpha+1)}t^{n-1},$$

which in the limit as $\alpha \to 1^-$ converges to nt^{n-1}.

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In particular

$$^{\Lambda}D^{\alpha}t = 1, \qquad ^{\Lambda}D^{\alpha}1 = \frac{1-\alpha}{t}.$$

The classical logistic differential equation

$$x'(t) = x(t)(1-x(t)), \tag{5}$$

is a first order non-linear differential equation that can be explicitly solved. The constant solutions are $x(t) = 0$ and $x(t) = 1$. If we impose as initial condition $x(t) = x_0$, then the explicit solution is

$$x(t) = \frac{x_0}{x_0 + (1-x_0)\exp(-t)}.$$

For example, for the initial condition $x_0 = 1/2$ one obtains the classical logistic function

$$x(t) = \frac{1}{1+e^{-t}}.$$

The logistic differential equation has been generalized to fractional calculus in several directions: by considering the Liouville–Caputo fractional derivative [4,5,6,7,8,9,10], providing explicit results in some particular cases [5,6,8], or by using the Prabhakar fractional calculus [1,11,12] as in [13].

The main aim of this work is to present the Λ-fractional logistic differential equation, as well as give an explicit solution to the equation.

2 Methodology

Let us consider the transformation

$$t \mapsto T = \frac{t^{2-\gamma}}{\Gamma(3-\gamma)}$$

so that (see Figure 1)

$$u(t) \mapsto U(T),$$

and

$$t^b \mapsto T^{\tilde{B}}.$$

Notice that

$$T = \frac{1/2}{\Gamma(1-\gamma)} \int_0^t \frac{s^2}{(t-s)^{\gamma}} ds = \frac{t^{2-\gamma}}{\Gamma(3-\gamma)},$$

so that

$$t = (\Gamma(3-\gamma)T)^{1/(2-\gamma)}.$$

Moreover

$$T^{\tilde{B}} = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{s^b}{(t-s)^{\gamma}} ds = \frac{\Gamma(1+b)t^{1+b-\gamma}}{\Gamma(2+b-\gamma)},$$

which implies

$$T^{\tilde{B}} = \frac{\Gamma(1-b)(\Gamma(3-\gamma)T)^{(1+b-\gamma)/(2-\gamma)}}{\Gamma(2+b-\gamma)}.$$

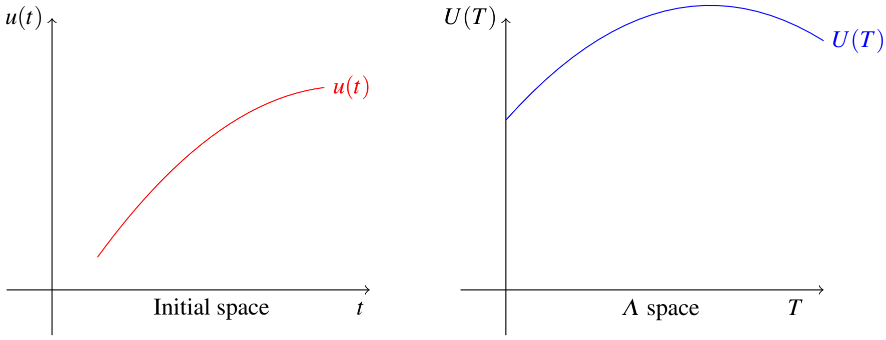

Fig. 1: Transformation of the variables.

3 The Λ-fractional logistic differential equation

Let us consider the Λ-fractional logistic differential equation

$$ {}^{\Lambda}D^{\gamma}u = t^{b}u(1-u), \tag{6}$$

with $0 < \gamma < 1$ and in the initial space. Then, we have

$$\frac{dU(T)}{dT} = AT^{B}(U(T) - U^{2}(T)), \tag{7}$$

with

$$B = \frac{1+b-\gamma}{2-\gamma},$$

and

$$A = \frac{\Gamma(1-b)(\Gamma(3-\gamma))^{(1+b-\gamma)/(2-\gamma)}}{\Gamma(2+b-\gamma)}.$$

The general solution of (7) is (see Figure 2, obtained by using Mathematica [14])

$$U(T) = \frac{1}{1 + C\exp\left(-\frac{AT^{B+1}}{B+1}\right)},$$

where C is a constant, giving the solution $U(T)$ in the Λ-space.

We finally (see Figure 3) obtain

$$\tilde{U}(t) = U\left(T \mapsto \frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\right).$$

so that the solution in the initial space is given by

$$u(t) = D^{1-\gamma}\tilde{U}(t) = \frac{1}{\Gamma(\gamma)}\frac{d}{dt}\int_{0}^{t}\frac{\tilde{U}(s)}{(t-s)^{1-\gamma}}ds$$

$$= \frac{e^{\frac{A\left(\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\right)^{B+1}}{B+1}}\left(\gamma t^{\gamma}\Gamma(3-\gamma)\left(e^{\frac{A\left(\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\right)^{B+1}}{B+1}} + C\right) - AC(\gamma-2)t^{2}\left(\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\right)^{B}\right)}{\gamma t\Gamma(3-\gamma)\Gamma(\gamma)\left(e^{\frac{A\left(\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\right)^{B+1}}{B+1}} + C\right)^{2}}. \tag{8}$$

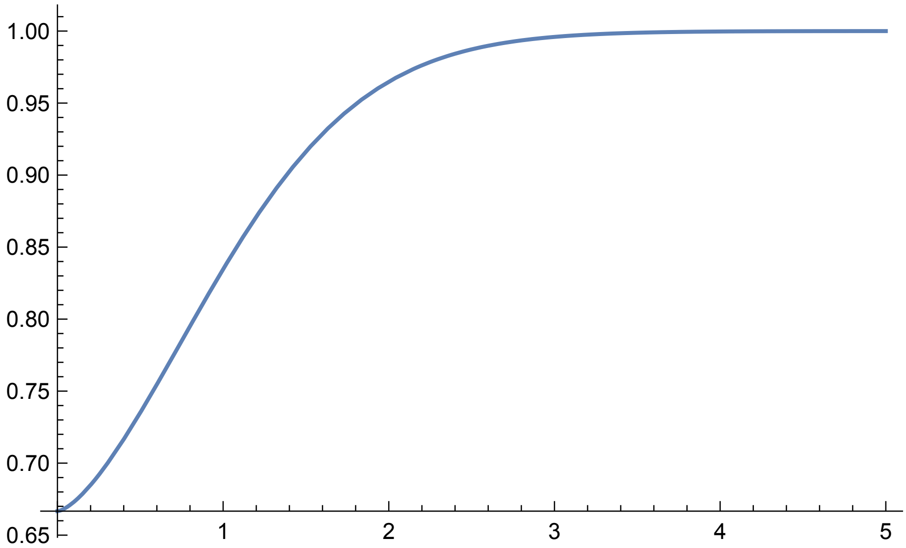

Fig. 2: Solution $U(T)$ of the logistic differential equation (7). This is in the $\Lambda-$space and we have taken $A = 1$, $B = 1/2$ and the initial condition $U(0) = 1/2$.

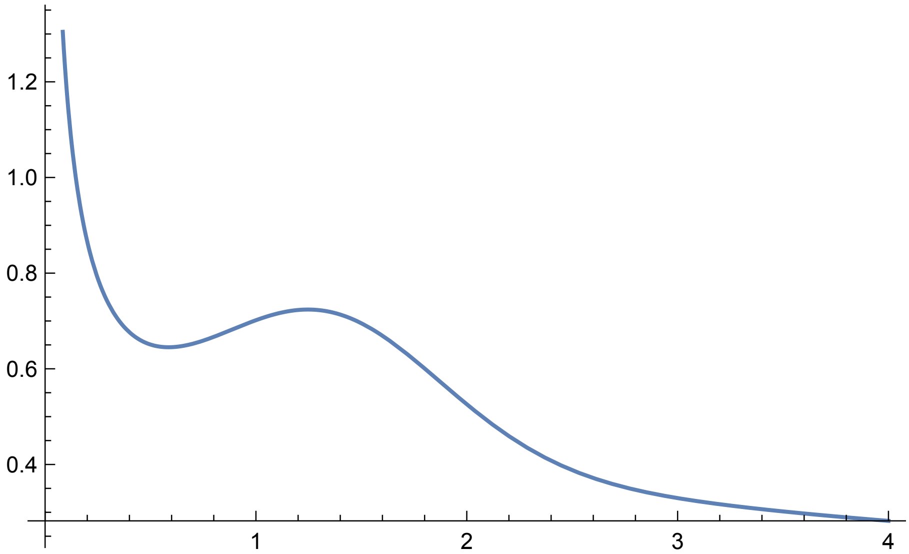

Fig. 3: Solution $u(t)$ of the Λ-logistic differential equation in the initial space. The values of the parameters are $b = 1/4$, $\gamma = 1/2$ and $C = 1/2$.

Observe that the qualitative behaviour of this solution is quite different from the solution of the fractional logistic equation. Indeed,

$$\lim_{t \to 0^+} u(t) = \infty, \qquad \lim_{t \to \infty} u(t) = 0.$$

We finally present the graphic of the solution for other values of the order of derivation: $\gamma = 0.1$ in Figure 4 and $\gamma = 0.99$ in Figure 5.

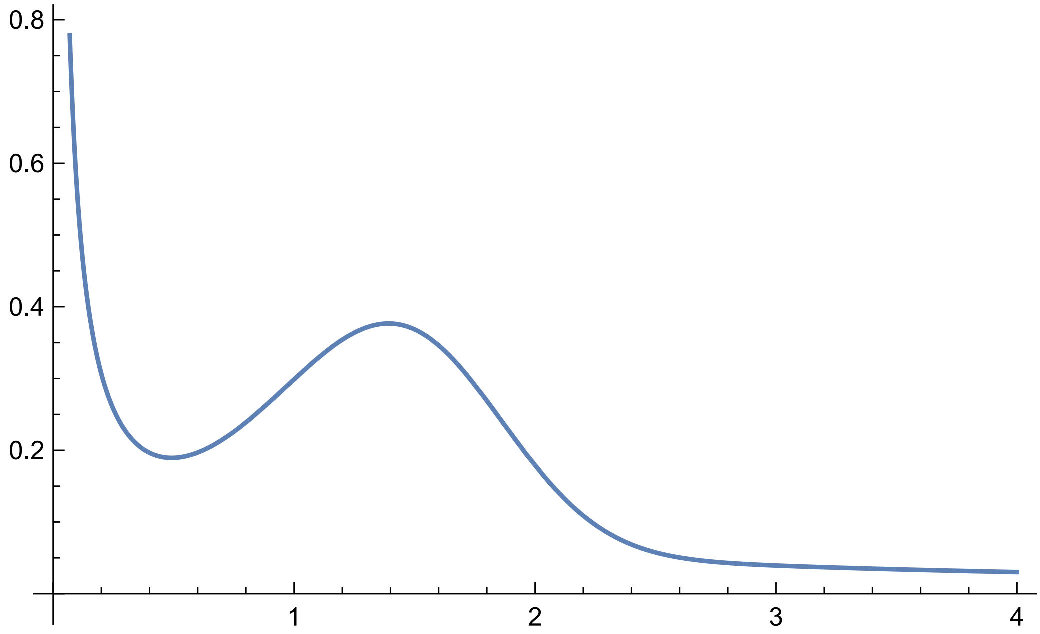

Fig. 4: Solution $u(t)$ of the Λ-logistic differential equation in the initial space. The values of the parameters are $b = 1/4$, $\gamma = 0.1$ and $C = 1/2$.

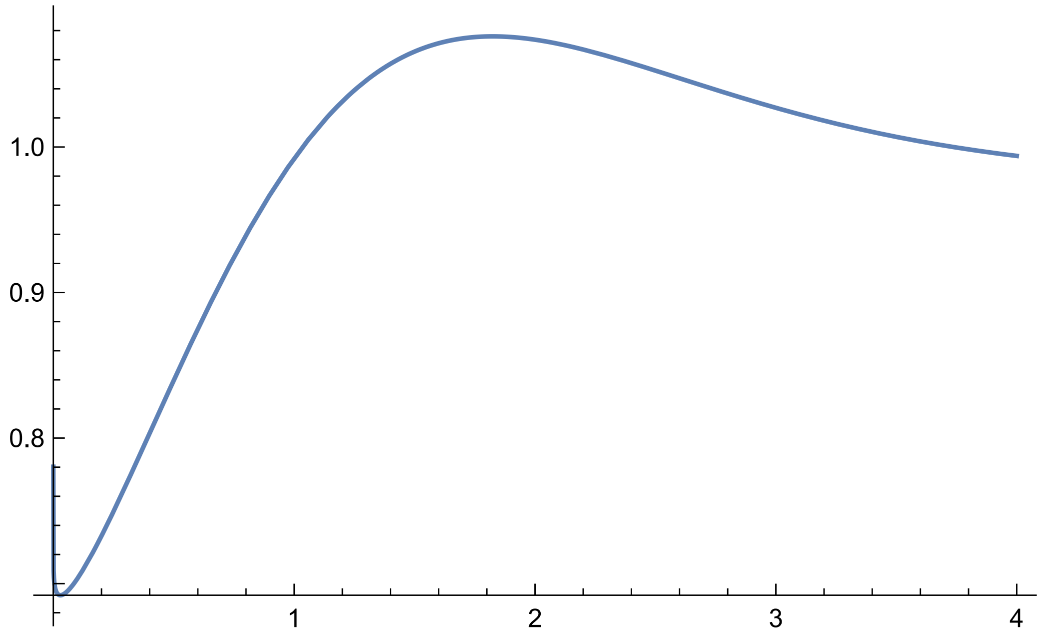

Fig. 5: Solution $u(t)$ of the Λ-logistic differential equation in the initial space. The values of the parameters are $b = 1/4$, $\gamma = 0.99$ and $C = 1/2$.

Acknowledgments

This work has been partially supported by the Agencia Estatal de Investigación (AEI) of Spain under Grant PID2020-113275GB-I00, cofinanced by the European Community fund FEDER, and Xunta de Galicia, grant ED431C 2019/02 for Competitive Reference Research Groups (2019–2022).

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