Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/100304

A Bivariate Conditional Fréchet Distribution with Application

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Received: 22 May 2021, Revised: 12 Jun. 2021, Accepted: 22 Aug. 2021
Published online: 1 Sep. 2021

Abstract: In this paper the probability density function of the bivariate conditional Fréchet distribution are obtained from conditional and marginal Fréchet distribution. Cumulative distribution function, survival function, hazard function and reversed hazard function of the bivariate conditional Fréchet distribution are discussed. Also, we determined marginal distribution function, joint moments, marginal moments and covariance matrix for the bivariate conditional Fréchet distribution. Estimation the unknown parameters is studied and we applicable bivariate conditional Fréchet distribution with the real date.

Keywords: Bivariate conditional Fréchet distribution; Moments; Maximum likelihood; Fisher information matrix

1 Introduction

The Fréchet distribution (FrD) is very important in extreme value theory and it has a various real life applications in earthquakes, floods, rainfall and sea waves. Many studies have been made on FrD such as Mansour,et al [1] introduced the Kumaraswamy exponentiated Fréchet distribution. Harlow [2] studied the applications of the FrD function. Teamah, et al. [3] analyzed the Fréchet-Weibull distribution with a data collection on earthquakes. Abd-Elfattah et al. [4] studied the exponentiated generalized Fréchet Distribution. Also,Krishna, et al [5] discussed the Marshall–Olkin FrD. With applications, Teamah, et al [6] investigated the properties of the Fréchet-Weibull mixture distribution. Also, Teamah, et al [7] presented a truncated version of Fréchet-Weibull. Almetwally and Muhammed [8] discussed the bivariate Fréchet distribution.

The bivariate distributions (BDs) are absolutely necessary in many practical problems; a great number of the BDs were studied in literature, such as Arnolda and Straussa [9] studied the bivariate conditional exponential distributions. Teamah and Abd El-Bar [10], [11] introduced the compound sum of the BDs. Diawaraa and Carpenter [12] introduced the mixture of bivariate exponential distributions. Sens et al. [13] discussed the BDs with conditional gamma and its multivariate form. Vaidyanathan and Sharon [14] studied the morgenstern type bivariate Lindley distribution. Gongsin and Saporu [15] introduced the bivariate conditional Weibull distribution with application. Teamah et al [16] studied the generalized weighted exponential-Gompertz distribution. El-Sherpieny et al [17] discussed the FGM Bivariate Weibull Distribution. Almetwally et al [18] studied the properties and different methods of estimation for bivariate Weibull distribution.

Our aim is introducing the probability density function (PDF) of bivariate conditional Fréchet distribution (BCFrD). We discuss some of the mathematical properties for the BCFrD. Also, we estimate the unknown parameters using real data. In Section 2 we derived the PDF of BCFrD. Also, we plot PDF for the BCFrD for different values of the parameters. Cumulative distribution function (CDF), survival function (SF), hazard function (HF), comulative hazard function(CHF), reversed hazard function (RHF), moments and covariance matrix are obtained in section 3. In section 4, we estimated the unknown parameters of the BCFrD. In Section 5, real data are used to get the value of unknown parameters for BCFrD.

2 Bivariate conditional Fréchet distribution

let X be a random variable has a FrD with β as the scale parameter and γ as the shape parameter. The PDF of X is

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$$g_X(x) = \frac{\gamma}{\beta}(\frac{\beta}{x})^{\gamma+1}\exp[-(\frac{\beta}{x})^{\gamma}], \quad x>0,\gamma,\beta>0. \quad (1)$$

Consider Y a random variable whose value is determined by X, and the distribution of Y given X a realization at x is also FrD with x as the scale parameter and λ as the shape parameter Sens et al. [13]. The PDF of $Y \mid X = x$ is

$$g_{Y \mid X}(y \mid X = x) = \frac{\lambda}{x}(\frac{x}{y})^{\lambda+1}\exp[-(\frac{x}{y})^{\lambda}],$$
$$x>0, y>0, \lambda>0 \quad (2)$$

Hence, the PDF of the BCFrD is

$$g\ (x,y) = \frac{\gamma\lambda\beta^{\gamma}x^{\lambda-\gamma-1}}{y^{\lambda+1}}\exp\{-[(\frac{\beta}{x})^{\gamma}+(\frac{x}{y})^{\lambda}]\},$$
$$x>0, y>0, \gamma,\beta,\lambda>0. \quad (3)$$

Such that,

$$g\ (x,y)>0, \iint_0^{\infty} g\ (x,y)dxdy = 1.$$

Figure 1 appears the PDF of the BCFrD for various values of the parameters.

3 Properties of bivariate conditional Fréchet distribution

In this section we discuss some of statistical measures for BCFrD

3.1 Cumulative distribution function

The CDF of the BCFrD is

$$G(x,y) = \sum_{w=0}^{\infty}\frac{(-1)^{w+1}\beta^{\lambda(w+1)}y^{-\lambda(w+1)}}{w!(w+1)} \times\Gamma[\frac{\gamma-\lambda(w+1)}{\gamma},\left(\frac{\beta}{x}\right)^{\alpha}].$$

3.2 Survival function

The SF of the BCFrD is

$$S(x,y) = \sum_{w=0}^{\infty}\frac{(-1)^{w+1}\beta^{\lambda(w+1)}y^{-\lambda(w+1)}}{w!(w+1)} \{\Gamma[\frac{\gamma-\lambda(w+1)}{\gamma}]-\Gamma[\frac{\gamma-\lambda(w+1)}{\gamma},\left(\frac{\beta}{x}\right)^{\alpha}]\}.$$

3.3 Hazard function

The HF of the BCFrD is

$$h(x,y) = \frac{\gamma\lambda\beta^{\gamma}x^{\lambda-\gamma-1}y^{-\lambda-1}\exp\{-[(\frac{\beta}{x})^{\gamma}+(\frac{x}{y})^{\lambda}]\}}{\sum_{w=0}^{\infty}\frac{(-1)^{w+1}\beta^{\lambda(w+1)}y^{-\lambda(w+1)}B}{w!(w+1)}},$$

where $B = \{\Gamma[\frac{\gamma-\lambda(w+1)}{\gamma}]-\Gamma[\frac{\gamma-\lambda(w+1)}{\gamma},\left(\frac{\beta}{x}\right)^{\alpha}]\}$.

3.4 Reversed hazard function

The RHF of the BCFrD is

$$r(x,y) = \frac{\gamma\lambda\beta^{\gamma}x^{\lambda-\gamma-1}y^{-\lambda-1}\exp\{-[(\frac{\beta}{x})^{\gamma}+(\frac{x}{y})^{\lambda}]\}}{\sum_{w=0}^{\infty}\frac{(-1)^{w+1}y^{-\lambda(w+1)}\beta^{\lambda(w+1)}}{(w+1)w!}\Gamma[\frac{\gamma-\lambda(w+1)}{\gamma},(\frac{\beta}{x})^{\gamma}]}.$$

3.5 Marginal distribution of Y

The PDF of the marginal distribution of Y is given by

$$g_Y(y) = \frac{\lambda\beta^{\lambda}}{y^{\lambda+1}}\sum_{w=0}^{\infty}\frac{(-1)^{w}}{w!}(\frac{\beta}{y})^{\lambda w}\frac{\pi}{\Gamma(\frac{\lambda(w+1)}{\gamma})\sin(\pi\frac{\lambda(w+1)}{\gamma})}.$$

3.6 Moments of bivariate conditional Fréchet distribution

3.6.1 Joint moment

The (r,s) joint moments of (X,Y) are obtained as

$$E(X^r,Y^s) = \beta^{r+s}\Gamma(1-\frac{s}{\lambda})\Gamma(1-\frac{r+s}{\gamma}). \quad (4)$$

3.6.2 Marginal moments

We get the rth marginal moment of X as follows

$$E(X^r) = \beta^r\Gamma(1-\frac{r}{\gamma}).$$

The sth marginal moment of Y

$$E(Y^s) = \beta^s\Gamma(1-\frac{s}{\lambda})\Gamma(1-\frac{s}{\gamma}).$$

3.6.3 Covariance matrix

The covariance matrix for X and Y can be given by

$$
\begin{aligned}
Cov(X,Y) &= \beta^2\Gamma(1-\frac{1}{\lambda})\Gamma(1-\frac{2}{\gamma}) \\
&\quad -\beta\,\Gamma(1-\frac{1}{\gamma})\beta\,\Gamma(1-\frac{1}{\lambda})\Gamma(1-\frac{1}{\gamma}) \\
&= \beta^2\Gamma(1-\frac{1}{\lambda})\left\{\Gamma(1-\frac{2}{\gamma})-\left[\Gamma(1-\frac{1}{\gamma})\right]^2\right\},
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= \beta^2\,\Gamma(1-\frac{2}{\gamma})-\left[\beta\,\Gamma(1-\frac{1}{\gamma})\right]^2 \\
&= \beta^2\left\{\Gamma(1-\frac{2}{\gamma})-\left[\Gamma(1-\frac{1}{\gamma})\right]^2\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
Var(Y) &= \beta^2\,\Gamma(1-\frac{2}{\lambda})\Gamma(1-\frac{2}{\gamma})-\left[\beta\,\Gamma(1-\frac{1}{\lambda})\Gamma(1-\frac{1}{\gamma})\right]^2 \\
&= \beta^2\left\{\Gamma(1-\frac{2}{\lambda})\Gamma(1-\frac{2}{\gamma})-\left[\Gamma(1-\frac{1}{\lambda})\Gamma(1-\frac{1}{\gamma})\right]^2\right\}.
\end{aligned}
$$

Thus, the random vactor $X' = (X,Y)$ having BCFrD with a mean vector

$$
\mu = \begin{pmatrix} \beta\,\Gamma(1-\frac{1}{\gamma}) \\ \beta\,\Gamma(1-\frac{1}{\lambda})\Gamma(1-\frac{1}{\gamma}) \end{pmatrix}
$$

and the variance-covariance matrix of X is

$$
\sum = \beta^2\begin{pmatrix} A & \Gamma(1-\frac{1}{\lambda})A \\ A\Gamma(1-\frac{1}{\lambda}) & \left\{\Gamma(1-\frac{2}{\lambda})\Gamma(1-\frac{2}{\gamma})-\left[\Gamma(1-\frac{1}{\lambda})\Gamma(1-\frac{1}{\gamma})\right]^2\right\} \end{pmatrix}.
$$

where $A = \left\{\Gamma(1-\frac{2}{\gamma})-\left[\Gamma(1-\frac{1}{\gamma})\right]^2\right\}$.

The correlation between X and Y is

$$
\rho_{XY} = \frac{\left\{\Gamma(1-\frac{2}{\gamma})-\left[\Gamma(1-\frac{1}{\gamma})\right]^2\right\}}{\sqrt{\Gamma(1-\frac{2}{\gamma})-\left[\Gamma(1-\frac{1}{\gamma})\right]^2}} \frac{\Gamma(1-\frac{1}{\lambda})}{\sqrt{\left[\Gamma(1-\frac{2}{\lambda})\Gamma(1-\frac{2}{\gamma})-\left[\Gamma(1-\frac{1}{\lambda})\Gamma(1-\frac{1}{\gamma})\right]^2\right]}}.
$$

4. Maximum likelihood estimation

Let $(X_1,Y_1),...,(X_n,Y_n)$ is a random sample from $(X,Y)\sim$BCFrD(λ,γ,β), then

$$
\begin{aligned}
L(\theta) &= n\log\gamma + n\log\lambda + n\gamma\log\beta + (\lambda-\gamma-1)\sum_{i=1}^{n}\log x_i \\
&\quad -(\lambda+1)\sum_{i=1}^{n}\log y_i - \sum_{i=1}^{n}(\frac{\beta}{x_i})^{\gamma} - \sum_{i=1}^{n}(\frac{x_i}{y_i})^{\lambda}. \qquad (5)
\end{aligned}
$$

Such that $\theta = (\lambda,\gamma,\beta)$, maximizing equation(5) with respect to the unknown parameters yields the MLEs. They aren't available in explicit ways. To compute the MLEs, we must solve three non linear equations. then, we should used the real data to get the values for the unknown parameters.

The first derivatives of equation (5) for (λ,γ,β) are given as

$$
\frac{\partial l(\theta)}{\partial\gamma} = \frac{n}{\gamma} + n\log\beta - \sum_{i=1}^{n}\log x_i - \log(\frac{\beta}{x_i})\sum_{i=1}^{n}(\frac{\beta}{x_i})^{\gamma} = 0,
$$

$$
\frac{\partial l(\theta)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\log y_i - \log(\frac{x_i}{y_i})\sum_{i=1}^{n}(\frac{x_i}{y_i})^{\lambda} = 0,
$$

$$
\frac{\partial l(\theta)}{\partial\beta} = \frac{n\gamma}{\beta} - \frac{\gamma}{\beta}\sum_{i=1}^{n}(\frac{\beta}{x_i})^{\gamma} = 0.
$$

We want to calculate the Fisher information matrix, which can be derived from the second derivatives of equation (5) for (γ,λ,β) , to get the asymptotic confidence intervals for the unknown parameters as seen in the next equations

$$
I_{\gamma^2} = \frac{\partial^2 l(\theta)}{\partial\gamma^2} = -\frac{n}{\gamma^2} - [\log(\frac{\beta}{x_i})]^2\sum_{i=1}^{n}(\frac{\beta}{x_i})^{\gamma},
$$

$$
I_{\lambda^2} = \frac{\partial^2 l(\theta)}{\partial\lambda^2} = -\frac{n}{\lambda^2} - [\log(\frac{x_i}{y_i})]^2\sum_{i=1}^{n}(\frac{x_i}{y_i})^{\lambda},
$$

$$
I_{\beta^2} = \frac{\partial^2 l(\theta)}{\partial\beta} = -\frac{n\gamma}{\beta^2} + \frac{\gamma(1-\gamma)}{\beta^2}\sum_{i=1}^{n}(\frac{\beta}{x_i})^{\gamma},
$$

$$
I_{\gamma\lambda} = \frac{\partial^2 l(\theta)}{\partial\gamma\partial\lambda} = I_{\lambda\gamma} = \frac{\partial^2 l(\theta)}{\partial\lambda\partial\gamma} = 0,
$$

$$
I_{\gamma\beta} = \frac{\partial^2 l(\theta)}{\partial\gamma\partial\beta} = \frac{n}{\beta} - \frac{1}{\beta}\sum_{i=1}^{n}(\frac{\beta}{x_i})^{\gamma} - \frac{\gamma}{\beta}\log(\frac{\beta}{x_i})\sum_{i=1}^{n}(\frac{\beta}{x_i})^{\gamma},
$$

$$I_{\lambda\beta} = \frac{\partial^2 l(\theta)}{\partial\lambda\partial\beta} = I_{\beta\lambda} = \frac{\partial^2 l(\theta)}{\partial\beta\partial\lambda} = 0.$$

Then, the Fisher information matrix $I_n(\theta) = I_n(\gamma, \lambda, \beta)$ can be written as shown in the following matrix

$$I_n(\theta) = \begin{bmatrix} I_{\gamma^2} \ldots I_{\gamma\lambda} \ldots I_{\gamma\beta} \\ I_{\lambda\gamma} \ldots I_{\lambda^2} \ldots I_{\lambda\beta} \\ I_{\beta\gamma} \ldots I_{\beta\lambda} \ldots I_{\beta^2} \end{bmatrix}$$

5. Real Data Application

The BCFrD must be applied to a real data set. Table 1 shows the data set representing the economic data set such that (X_1 Exports of goods and services) and (X_2 GDP growth) for 31 annual time series observations [1980: 2010] on the response variable. We cannot obtain analytical solution for the three equations hence, we can get the values of the unknown parameters as seen in table 2. The economics real data set was collected by World Bank National Accounts data and OECD National Accounts data.

Table 1: Data of Economics.

No	years	X_1	X_2	No	years	X_1	X_2
1	1980	30.51	10.01	16	1995	22.55	5.15
2	1981	33.37	3.76	17	1996	20.75	4.64
3	1982	27.03	9.91	18	1997	18.84	4.99
4	1983	25.48	7.40	19	1998	16.21	5.49
5	1984	22.35	6.09	20	1999	15.05	4.04
6	1985	19.91	6.60	21	2000	16.20	6.11
7	1986	15.73	2.65	22	2001	17.48	5.37
8	1987	12.56	2.52	23	2002	18.32	3.54
9	1988	17.32	7.93	24	2003	21.80	2.37
10	1989	17.89	4.97	25	2004	28.23	3.19
11	1990	20.05	5.70	26	2005	30.34	4.09
12	1991	27.82	1.08	27	2006	29.95	4.48
13	1992	28.40	4.43	28	2007	30.25	6.85
14	1993	25.84	2.90	29	2008	33.04	7.09
15	1994	22.57	3.97	30	2009	24.96	7.16
				31	2010	21.35	4.67

In Table 2. The MLEs for the parameters of BCFrD(γ, λ, β), bivariate independent Fréchet distribution (BIFrD)(γ, β) and bivariate conditional weibull distribution (BCWD) (γ, λ, β) are obtained. The value of the negative log likelihood function (-LOG), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Second Order of AKaike Information Criterion(AICc) and Hanan-Quinn Information Criterion (HQIC) are obtained in Table 3.

Table 2: MLEs for the data set.

Distribution	γ	λ	β	α
BCFrD	3.99213	0.610385	19.546	-
BIFrD	3.66813	1.7854	19.5423	3.99614
BCWD	25.2073	0.859608	4.50309	-

Table 3: The statistics-Log likelihood, AIC, BIC, AICc and HQC for the data set.

Distribution	-Log likelihood	AIC	BIC	AICc	HQIC
BCFrD	93.48591	192.972	197.274	193.861	194.374
BIFrD	174.76576	357.532	363.267	359.07	359.401
BCWD	200.7994	407.599	411.901	408.488	409.001

Tables (2) and (3) show that the model BCFrD was applied to a real data set to demonstrate the superiority of this new distribution by comparing its fitness to that of BIFrD and BCWD.

6. Conclusions

In this paper We obtained the PDF of BCFrD. CDF, SF,HF, RHF, marginal distribution of Y are calculated. Also, we found joint moment, marginal moments, covariance matrix and correlation between X and Y. MLEs for parameters of the BCFrD are given to estimate the unknown parameters and the fisher information matrix is obtained . Finally, data analysis is discussed to illustrate the behavior of BCFrD in practice and proofed that BCFrD is better than BIFrD and BCWD.

Acknowledgments

We appreciate the reviewers' thorough reading and insightful remarks, which improved the work.

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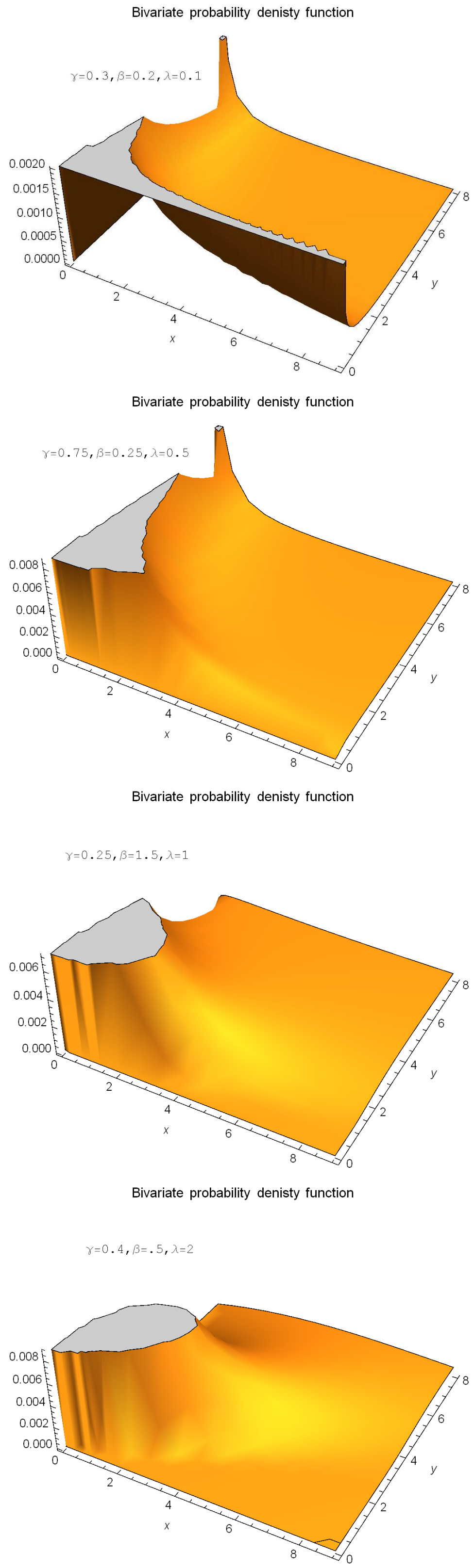

Fig. 1: The PDF of BCFrD