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# Dependency Measures For New Bivariate Models Based on Copula Function

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**Abstract:** The first part of this paper reviews the properties of bivariate dependence measures as Spearman's rho and Kendall's tau under the different copula. The second part of this paper derives the bivariate inverted Topp-Leone (BITL) distribution based on Farlie-Gumbel-Morgenstern (FGM), Ali-Mikhail-Haq (AMH), Plackett, and Clayton copula. The reliability function obtained for bivariate ITL distributions based on copula. The maximum likelihood estimation method for the parameters of the four bivariate ITL distributions has been discussed. Asymptotic confidence intervals for the model parameters are also considered. To evaluate the performance of the models, a Monte Carlo simulation study is conducted to compare the efficiency between the four models. Also, a medical real data set of diabetic nephropathy is analyzed to investigate the models and useful results are obtained for illustrative purposes. Anderson–Darling-type statistic and Cramér–von Mises statistic for copula goodness-of-fit testing are obtained.

**Keywords:** Copula function, Spearman's rho, Kendall's tau, copula goodness-of-fit, Contour plot

## 1 Introduction

As we considering bivariate distribution, measuring the dependence between different bivariate data is of particular interest in a variety of applications. Spearman's rho, Kendall's tau, Blomqvist's beta, Gini's gamma, and a few other lesser-known measures are all classic bivariate dependence measures. These measures are designed to assess dependency in bivariate data based on copula function.
In fact, traditional 'dependence' measures such as Pearson's r, Spearman's rank correlation, or Kendall's are only useful in particular circumstances such as linearity, monotonicity, concordance, which are symmetric therefore undirected by design. Implying that the random variable X is equally dependent on the random variable Y and vice versa. The famous but often misunderstood measure of dependency is linear correlation. The popularity of linear correlation stems from its ease of calculation, and it is a natural scalar measure of dependency in elliptical distributions with well-known members like the multivariate-normal and multivariate-t distributions. In many real-life cases, the relationship between random variables is asymmetric. For example, the linear correlation can result in incorrect risk assessments in stock market trades which was discussed by Okimoto [1], inaccurate gene network reconstructions which discussed by Wang and Huang [2]. Also see, Al-Sadoon [3], Bárdossy and Hórning [4], Coenen and Weitz [5] and Junker et al. [6].
In the literature review, we found additional more dependence measures including, e.g., dependence measures based on copula function which are discussed by Embrechts et al. [7], Smith et al. [8], distance correlation R which is discussed by Székely et al. [9], the maximal information coefficient (MIC) which is discussed by Reshef et al. [10,11], and robust copula dependence (RCD) discussed by Chang [12].
A copula is a simple way to describe a bivariate distribution with a dependency structure. A copula is a function that joins bivariate distribution functions with uniform [0, 1] margins, as defined by Nelsen [13].

**Theorem 1(Sklar theorem).** *Assuming that the two random variables X and Y have distribution functions* $F(x)$ *and* $F(Y)$. *Let* $C(u,v)$ *is the copula cdf and* $c(u,v)$ *is the copula pdf. Then the cdf and pdf for bivariate*

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*distributions based on copula are given as*

$$F(x,y) = C(F(x),F(y)), \tag{1}$$

*and*

$$f(x,y) = f(x)f(y)c(F(x),F(y)). \tag{2}$$

Many copulas based on Theorem 1 have been used to define the bivariate distributions. For more information see Nelsen [13]. For more examples see Elaal and Jarwan [14], Almetwally et al. [15], Almetwally and Muhammed [16] and El-Sherpieny et al. [17, 18, 19].

In the univariate cases, the inverted distributions are important in a variety of fields, including biological sciences, life test issues, medicine, and so on. The inverted conformation distributions have a different structure than non-inverted conformation distributions in terms of density and hazard ratio. The cdf and pdf of the inverted Topp-Leone distribution (ITL) with shape parameter $\lambda > 0$ were proposed by Hassan et al. [20] as follows:

$$F(x;\lambda) = 1 - \frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}; \quad x \geq 0 > 0 \tag{3}$$

and,

$$f(x;\lambda) = 2\lambda x \frac{(1+2x)^{\lambda-1}}{(1+x)^{2\lambda+1}}; \quad x > 0 \tag{4}$$

Many authors introduced a new distribution based on ITL distribution such as Almetwally et al. [21] introduced modified Kies inverted Topp-Leone distribution. Muhammed [22] introduced the inverted Topp Leone distribution with different forms. Almetwally [23] introduced odd Weibull ITL distribution with applications of COVID-19 data. Abushal et al. [24] introduced power ITL distribution in acceptance sampling plans. Hassan et al. [25] introduced Kumaraswamy ITL distribution with applications of COVID-19 data. Ibrahim et al. [26] alpha power ITL distribution with different applications.

Kendall's tau and Spearman's rho are two essential indicators of dependency (concordance) that we will address. They are possibly the best alternatives to the linear correlation coefficient as a measure of dependency for non-elliptical distributions, in which the linear correlation coefficient is mostly ineffective and misleading.

In this paper, we study the bivariate extension of the ITL distribution based on different copula functions and obtain their statistical properties of dependence measure using Kendall's tau and Spearman's rho. Point and interval estimation of the parameters of BITL distribution are discussed using maximum likelihood estimation (MLE).

The present paper is organized as follows. After this introduction Section 2 reviews the original copula function with dependence measures. In Section 3, we introduced different bivariate extension of the ITL distribution based on different copula functions. In Section 4 parameter estimation are obtained. Furthermore in Section 5, the potentiality of the four new models are illustrated by simulation study. In Section 6, application of a diabetic nephropathy real data set is discussed. Finally, Section 7 is devoted to the research perspective.

## 2 Copula Function

The family of Archimedean copulas is one of the most common families of copulas. If a copula has the functional form

$$C(u,v) = \varphi\left(\varphi^{-1}(u) + \varphi^{-1}(v)\right)$$

for a suitable, decreasing function $\varphi$: $(0,\infty) \to [0,1]$, with $\varphi(0) = 1$ and $\lim_{x\to\infty} \varphi(x) = 0$, it is called an Archimedean copula. For more information see Genest and Rivest [27] and Charpentier and Segers [28]. Popular completely monotone generators of Archimedean copulas as Clayton, Gumbel, and Ali–Mikhail–Haq (AMH) copula. Plackett [29] suggested the Plackett family of bivariate distributions, which were based on the principle of constant cross-product ratio (or odds ratio). Bekrizadeh et al. [30] discussed and developed Farlie Gumbel Morgenstern (FGM) copula.

### *Spearman's Rho*

Let C be the copula of continuous random variables X and Y. The population variant of Spearman's rho for X and Y is then calculated as follows:

$$\rho_s = 12 \int_0^1 \int_0^1 C(u,v)dudv - 3 \tag{5}$$

It's worth noting that $\rho_s$ corresponds to the correlation coefficient $\rho$ between uniform marginal distributions.

### *Kendall's Tau*

In terms of copula, Kendall's tau $\tau_k$ is defined as

$$\tau_k = 4 \int_0^1 \int_0^1 c(u,v)C(u,v)dudv - 1 \tag{6}$$

The AMH, Clayton, FGM, and Plackett copulas, which are used to form bivariate distributions, will be discussed.

### *2.1 FGM Copula*

In FGM copula; the cdf of FGM copula with copula parameter $-1 < \theta < 1$ is given as following

$$C(u,v) = uv(\theta(1-u)(1-v)) \tag{7}$$

the pdf of FGM copula is as following

$$c(u,v) = 1 + \theta(1-2u)(1-2v) \tag{8}$$

The Spearman's rho of FGM is

$$\rho_s = \frac{\theta}{3}.$$

The Kendall's tau of FGM is

$$\tau_k = \frac{2\theta}{9}.$$

### 2.2 AMH Copula

In AMH copula; the cdf of AMH copula with copula parameter $-1 < \theta < 1$ is given as following

$$C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)} \tag{9}$$

the pdf of AMH copula is as following

$$c(u,v) = \frac{1-\theta+2\theta\frac{uv}{1-\theta(1-u)(1-v)}}{[1-\theta(1-u)(1-v)]^2} \tag{10}$$

For the AMH copula, Nelsen ([13], p. 172) provides the following formula for Spearman's rho

$$\rho_s = \frac{12(\theta+1)}{\theta^2} dilog(1-\theta) - \frac{24(1-\theta)}{\theta^2}\ln(1-\theta) - \frac{3(\theta+12)}{\theta},$$

where his "dilogarithm" $dilog(x) = polylog(1-x,2)$, and it is the usual definition of the dilogarithm as a special function.
The Kendall's tau of AMH is

$$\tau_k = 1 - 2\frac{(1-\theta)^2\ln(1-\theta)+\theta}{3\theta^2}.$$

### 2.3 Clayton Copula

In Clayton copula; the cdf of Clayton copula with copula parameter $\theta > 0$ is as following

$$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{\frac{-1}{\theta}} \tag{11}$$

the pdf of Clayton copula is given as following

$$c(u,v) = (1+\theta)(uv)^{-1-\theta}\left(u^{-\theta}+v^{-\theta}-1\right)^{\frac{-2\theta-1}{\theta}} \tag{12}$$

The Kendall's tau of Clayton is

$$\tau_k = \frac{\theta}{2+\theta}.$$

### 2.4 Plackett Copula

In Plackett copula; the cdf of Plackett copula with copula parameter $\theta > 0$ is given as following

$$C(u,v) = \frac{1+(\theta-1)(u+v)}{2(\theta-1)} - \frac{\sqrt{(1+(\theta-1)(u+v))^2 - 4uv\theta(\theta-1)}}{2(\theta-1)} \tag{13}$$

the pdf of Plackett copula is as following

$$c(u,v) = \frac{\theta\,(1+(u-2uv+v)(\theta-1))}{[(1+(\theta-1)(u+v))^2 - 4uv\theta(\theta-1)]^{1.5}} \tag{14}$$

For the Plackett copula the following formula for Spearman's rho is given as follows:

$$\rho_s = \frac{2\theta+\theta^2-2(\theta+1)\ln(\theta+1)}{\theta^2},$$

## 3 Bivariate ITL Distribution

In this section, we intorduced four bivariate ITL distributions based on FGM, AMH, Clayton and Plackett copulas.

### 3.1 FGM Bivariate ITL Distribution

By using FGM copula Equations (7,8), Sklar Theorem 1 and ITL distribution in Equations (4,3), we get the FGM bivariate ITL distribution which can be denoted by FGMBITL with cdf, pdf and reliability function as follows in Equations (15, 16,17) respectively. Figure 1 shows the joint pdf of FGM bivariate ITL distribution with different shapes.

$$F_{FGMBITL}(x,y) = \left(1-\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\right)\left(1-\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}\right)\left(1+\theta\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}\right), \tag{15}$$

$$f_{FGMBITL}(x,y) = 2\lambda_1 x\frac{(1+2x)^{\lambda_1-1}}{(1+x)^{2\lambda_1+1}}2\lambda_2 y\frac{(1+2y)^{\lambda_2-1}}{(1+y)^{2\lambda_2+1}}\left[1+\theta\left(2\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}-1\right)\left(2\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}-1\right)\right], \tag{16}$$

and

$$R_{FGMBITL}(x,y) = \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}} - 1 + \left(1+\theta\left(1-\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\right)\left(1-\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}\right)\right). \tag{17}$$

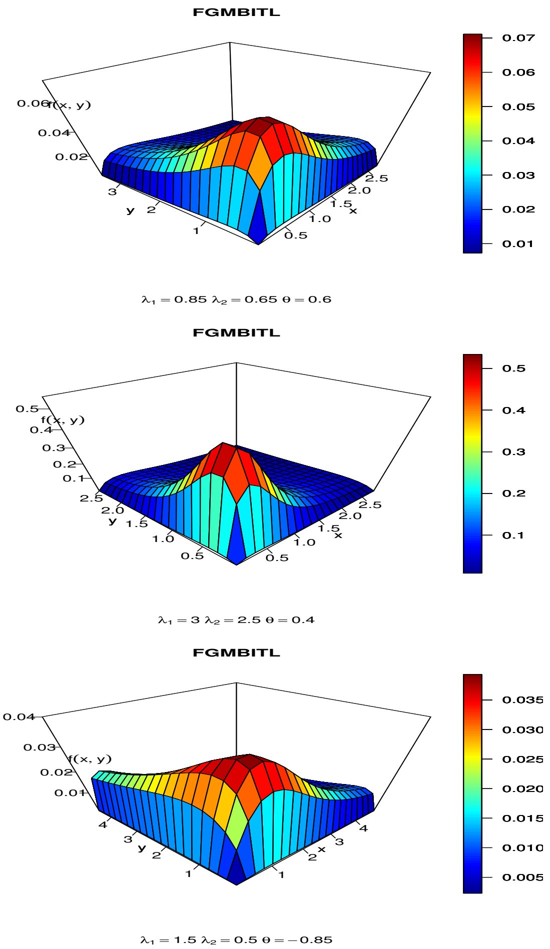


**Fig. 1:** Joint pdf of FGMBITL distribution

## 3.2 AMH Bivariate ITL Distribution

By using AMH copula Equations (9,10), Sklar theorem 1 and ITL distribution in Equations (4,3), we get the AMH bivariate ITL distribution which can be denoted by AMHBITL with cdf, pdf and reliability function given as follows in Equations (18, 19, 20) respectively. Figure 2 shows the joint pdf of AMH bivariate ITL distribution with different shapes.

$$F_{AMHBITL}(x,y)=\frac{\left(1-\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\right)\left(1-\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}\right)}{1-\theta\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}}, \quad (18)$$

$$f_{AMHBITL}(x,y)=2\lambda_1 x\frac{(1+2x)^{\lambda_1-1}}{(1+x)^{2\lambda_1+1}}2\lambda_2 y\frac{(1+2y)^{\lambda_2-1}}{(1+y)^{2\lambda_2+1}}\frac{1-\theta+2\theta\frac{\left(1-\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\right)\left(1-\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}\right)}{1-\theta\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}}}{\left[1-\theta\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}\right]^2}, \quad (19)$$

and

$$R_{AMHBITL}(x,y)=\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}+\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}-1+\frac{\left(1-\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\right)\left(1-\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}\right)}{1-\theta\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}}. \quad (20)$$

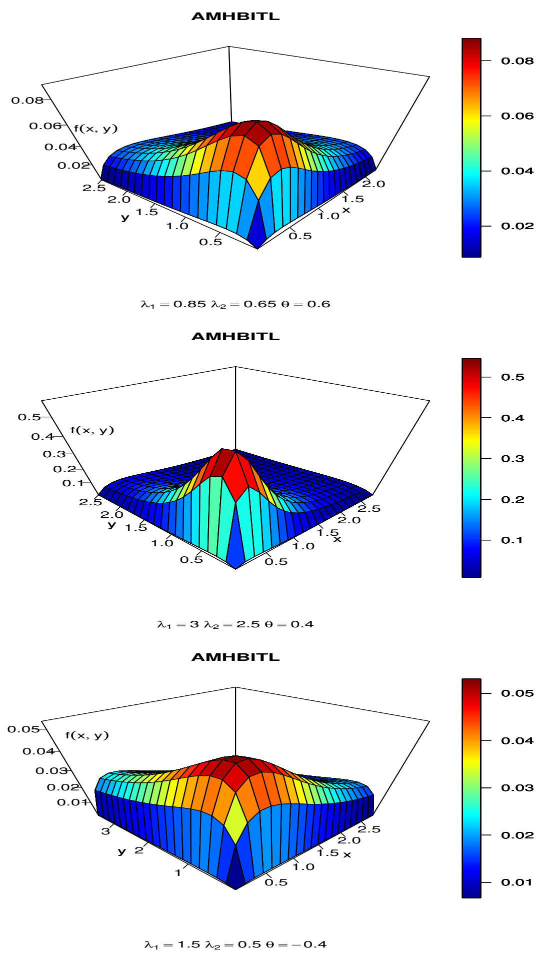


**Fig. 2:** Joint pdf of AMHBITL distribution

### 3.3 Clayton Bivariate ITL Distribution

By using Clayton copula Equations (11,12), Sklar theorem 1 and ITL distribution in Equations (4,3), to get the Clayton bivariate ITL distribution which can be denoted by CBITL with cdf, pdf and reliability function as follows in Equations (21, 22, 23) respectively. Figure 3 shows the joint pdf of Clayton Bivariate ITL distribution has different shapes.

### 3.4 Plackett Bivariate ITL Distribution

By using Plackett copula Equations (13,14), Sklar theorem 1 and ITL distribution in Equations (4,3), to get the Plackett bivariate ITL distribution which can be denoted by PBITL with cdf, pdf and reliability function as follows in Equations (25, 24, 26) respectively. Figure 4 shows the joint pdf of Plackett Bivariate ITL distribution with different shape parameters.

$$F_{CBITL}(x,y) = \{\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right)^{-\theta} + \left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)^{-\theta} - 1\}^{\frac{-1}{\theta}}, \tag{21}$$

$$\begin{aligned} f_{CBITL}(x,y) =& 2\lambda_1 x\frac{(1+2x)^{\lambda_1-1}}{(1+x)^{2\lambda_1+1}} 2\lambda_2 y\frac{(1+2y)^{\lambda_2-1}}{(1+y)^{2\lambda_2+1}}(1+\theta)\left[\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right)\left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)\right]^{-1-\theta} \\ & \left\{\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right)^{-\theta} + \left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)^{-\theta} - 1\right\}^{\frac{-2\theta-1}{\theta}}, \end{aligned} \tag{22}$$

and

$$R_{CBITL}(x,y) = \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}} + \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}} - 1 + \left\{\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right)^{-\theta} + \left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)^{-\theta} - 1\right\}^{\frac{-1}{\theta}}. \tag{23}$$

$$\begin{aligned} f_{PBITL}(x,y) =& 2\lambda_1 x\frac{(1+2x)^{\lambda_1-1}}{(1+x)^{2\lambda_1+1}} 2\lambda_2 y\frac{(1+2y)^{\lambda_2-1}}{(1+y)^{2\lambda_2+1}} \\ & \frac{\theta\left(1+\left[\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right) - 2\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right)\left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right) + \left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)\right](\theta-1)\right)}{\left[\left(1+(\theta-1)(\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right) + \left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right))\right)^2 - 4\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right)\left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)\theta(\theta-1)\right]^{1.5}}, \end{aligned} \tag{24}$$

$$\begin{aligned} F_{PBITL}(x,y) =& \frac{1+(\theta-1)\left[\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right) + \left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)\right]}{2(\theta-1)} - \\ & \frac{\sqrt{\left(1+(\theta-1)\left[\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right) + \left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)\right]\right)^2 - 4\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right)\left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)\theta(\theta-1)}}{2(\theta-1)}, \end{aligned} \tag{25}$$

$$\begin{aligned} R_{PBITL}(x,y) =& \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}} + \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}} - 1 + \\ & \frac{\sqrt{\left(1+(\theta-1)\left[\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right) + \left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)\right]\right)^2 - 4\left(1-\left(\frac{1+2x}{(1+x)^2}\right)^{\lambda_1}\right)\left(1-\left(\frac{1+2y}{(1+y)^2}\right)^{\lambda_2}\right)\theta(\theta-1)}}{2(\theta-1)}. \end{aligned} \tag{26}$$

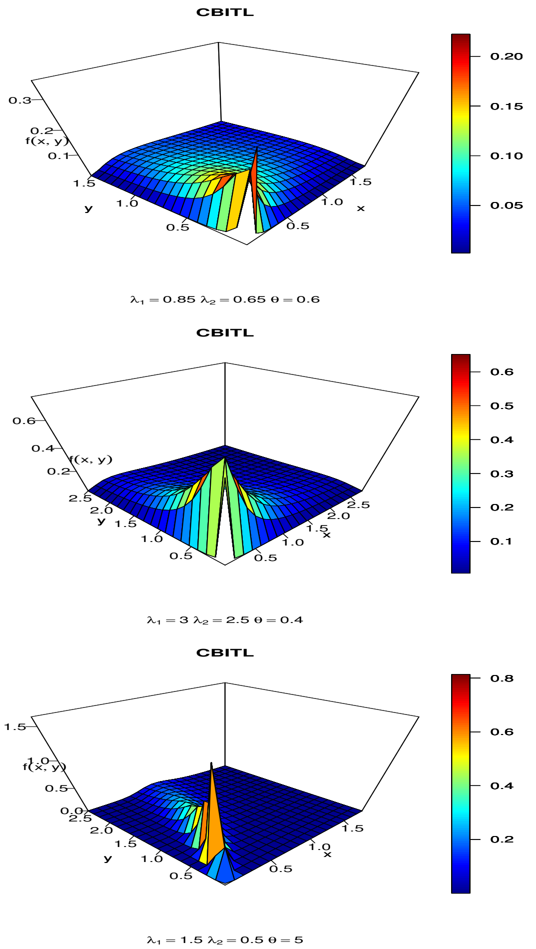


**Fig. 3:** Joint pdf of CBITL distribution

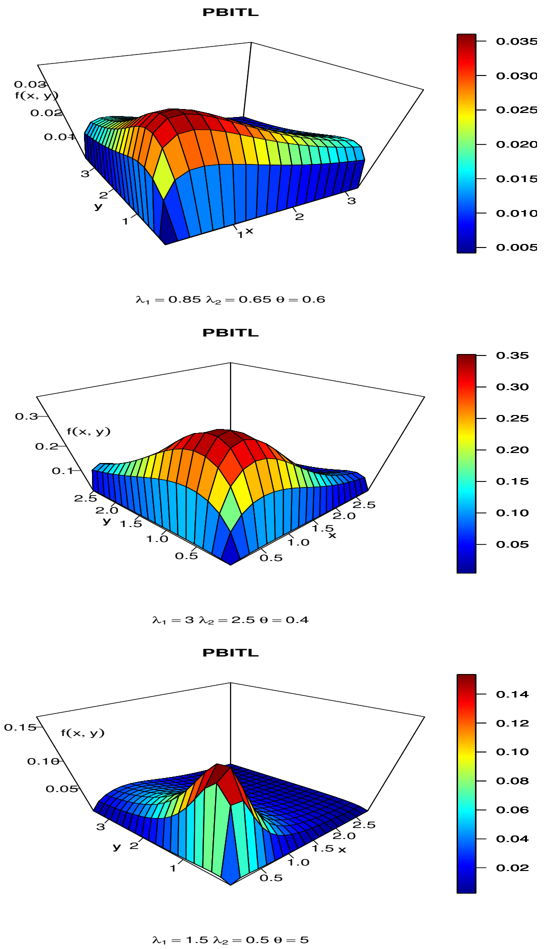


**Fig. 4:** Joint pdf of PBITL distribution

## 4 Maximum Likelihood Estimation (MLE)

In this section, we estimate the unknown parameters of the bivariate ITL distribution using the MLE method. Suppose that $(x_1,y_1)$, $(x_2,y_2),\ldots,(x_n,y_n)$ is a sample of size $n$. The likelihood function of bivariate ITL distribution with vector of parameters $\Theta = (\lambda_1,\lambda_2,\theta)$ based on different copulas are as follows:

The log-likelihood function of FGMBITL distribution can be written in Eqution (27). The log-likelihood function of AMHBITL distribution can be written in Eqution (28). The log-likelihood function of CBITL distribution can be written in Eqution (29). The log-likelihood function of PBITL distribution can be written in Equation (30). By obtaining the first derivatives with respect to the unknown parameters for each log-likelihood function separately, and by equating them to zero. we can get the likelihood equations and hence solving them numerically to obtain the MLEs for each distribution.

In asymptotic confidence intervals (ACI), the asymptotic normal distribution of the MLE is the most used to determine parameter confidence intervals of bivariate ITL distribution. Fisher information matrix $I(\Theta)$, which is composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at $\hat{\Theta} = (\hat{\lambda}_1,\hat{\lambda}_2,\hat{\theta})$, is related to the asymptotic variance-covariance matrix of the MLE of the parameters of bivariate ITL distribution. Suppose the asymptotic variance-covariance matrix of the parameter vector $\Theta$ is

$$I(\hat{\Theta}) = -\begin{bmatrix} I_{\hat{\lambda}_1\hat{\lambda}_1} & & \\ I_{\hat{\lambda}_2\hat{\lambda}_1} & I_{\hat{\lambda}_2\hat{\lambda}_2} & \\ I_{\hat{\theta}\hat{\lambda}_1} & I_{\hat{\theta}\hat{\lambda}_2} & I_{\hat{\theta}\hat{\theta}} \end{bmatrix}$$

where $V(\hat{\Theta}) = I^{-1}(\hat{\Theta})$

A $100(1-\alpha)\%$ confidence interval for the vector parameter $\Theta$ can be constructed based on the asymptotic normality of the MLE. $\hat{\lambda}_j \pm Z_{0.025}\sqrt{I_{\hat{\lambda}_j\hat{\lambda}_j}}; j = 1,2$ and $\hat{\theta}_j \pm Z_{0.025}\sqrt{I_{\hat{\theta}_j\hat{\theta}_j}}$, where $Z_{0.025}$ is the percentile of the standard normal distribution with right tail probability $\frac{\alpha}{2}$.

$$l(\Theta) = n\left[\ln(4)+\ln(\lambda_1)+\ln(\lambda_2)\right] + \sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln(y_i) + \sum_{i=1}^{n}\ln\left(\frac{(1+2x_i)^{\lambda_1-1}}{(1+x_i)^{2\lambda_1+1}}\right) + \sum_{i=1}^{n}\ln\left(\frac{(1+2y_i)^{\lambda_2-1}}{(1+y_i)^{2\lambda_2+1}}\right)$$
$$+\sum_{i=1}^{n}\ln\left[1+\theta\left(2\frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^{2\lambda_1}}-1\right)\left(2\frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^{2\lambda_2}}-1\right)\right], \tag{27}$$

$$l(\Theta) = n\left[\ln(4)+\ln(\lambda_1)+\ln(\lambda_2)\right] + \sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln(y_i) + \sum_{i=1}^{n}\ln\left(\frac{(1+2x_i)^{\lambda_1-1}}{(1+x_i)^{2\lambda_1+1}}\right) + \sum_{i=1}^{n}\ln\left(\frac{(1+2y_i)^{\lambda_2-1}}{(1+y_i)^{2\lambda_2+1}}\right)$$
$$+\sum_{i=1}^{n}\ln\left(\frac{1-\theta+2\theta\frac{\left(1-\frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^{2\lambda_1}}\right)\left(1-\frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^{2\lambda_2}}\right)}{1-\theta\frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^{2\lambda_1}}\frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^{2\lambda_2}}}}{\left[1-\theta\frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^{2\lambda_1}}\frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^{2\lambda_2}}\right]^2}\right), \tag{28}$$

$$l(\Theta) = n\left[\ln(4)+\ln(\lambda_1)+\ln(\lambda_2)\right] + \sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln(y_i) + \sum_{i=1}^{n}\ln\left(\frac{(1+2x_i)^{\lambda_1-1}}{(1+x_i)^{2\lambda_1+1}}\right) + \sum_{i=1}^{n}\ln\left(\frac{(1+2y_i)^{\lambda_2-1}}{(1+y_i)^{2\lambda_2+1}}\right)$$
$$+\sum_{i=1}^{n}\ln\left(\left\{\left(1-\left(\frac{1+2x_i}{(1+x_i)^2}\right)^{\lambda_1}\right)^{-\theta}+\left(1-\left(\frac{1+2y_i}{(1+y_i)^2}\right)^{\lambda_2}\right)^{-\theta}-1\right\}^{\frac{-2\theta-1}{\theta}}\right), \tag{29}$$

$$l(\Theta) = n\left[\ln(4)+\ln(\lambda_1)+\ln(\lambda_2)\right] + \sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln(y_i) + \sum_{i=1}^{n}\ln\left(\frac{(1+2x_i)^{\lambda_1-1}}{(1+x_i)^{2\lambda_1+1}}\right) + \sum_{i=1}^{n}\ln\left(\frac{(1+2y_i)^{\lambda_2-1}}{(1+y_i)^{2\lambda_2+1}}\right)$$
$$+\sum_{i=1}^{n}\ln\left(\frac{\theta\left(1+\left[\left(1-\left(\frac{1+2x_i}{(1+x)^2}\right)^{\lambda_1}\right)-2\left(1-\left(\frac{1+2x}{(1+x_i)^2}\right)^{\lambda_1}\right)\left(1-\left(\frac{1+2y_i}{(1+y_i)^2}\right)^{\lambda_2}\right)+\left(1-\left(\frac{1+2y_i}{(1+y_i)^2}\right)^{\lambda_2}\right)\right](\theta-1)\right)}{\left[\left(1+(\theta-1)\left[\left(1-\left(\frac{1+2x_i}{(1+x_i)^2}\right)^{\lambda_1}\right)+\left(1-\left(\frac{1+2y_i}{(1+y_i)^2}\right)^{\lambda_2}\right)\right]\right)^2-4\left(1-\left(\frac{1+2x_i}{(1+x_i)^2}\right)^{\lambda_1}\right)\left(1-\left(\frac{1+2y_i}{(1+y_i)^2}\right)^{\lambda_2}\right)\theta(\theta-1)\right]^{1.5}}\right) \tag{30}$$

## 5 Simulation and Generation of Bivariate ITL Sample

In this section; a Monte Carlo simulation is done for comparison between bivariate ITL under different copulas by using point, interval estimation and dependence measures. To generate random variables: Nelsen [13] discussed generating a sample from a specified joint distribution by using the conditional distribution method. we will discuss the generation of bivariate ITL under different copulas, using the conditional distribution and iterative algorithms.
Firstly: We generate $Q$ and $W$ independently from a uniform distribution with lower is 0 and upper is 1.
Secondly: We obtain the conditional distribution of X given Y for each distribution based on copulas:
In FGM copula

$$C(v|u) = v(1+\theta(1-v)(1-2u)), \quad (31)$$

and

$$C(u|v) = u(1+\theta(1-u)(1-2v)). \quad (32)$$

In Clayton copula

$$C(v|u) = \left(u^{-\theta}+v^{-\theta}-1\right)^{\frac{-1}{\theta}-1} u^{-\theta-1}, \quad (33)$$

and

$$C(u|v) = \left(u^{-\theta}+v^{-\theta}-1\right)^{\frac{-1}{\theta}-1} v^{-\theta-1}. \quad (34)$$

In AMH copula

$$C(v|u) = \frac{v}{1-\theta(1-u)(1-v)} - \frac{\theta uv(1-v)}{(1-\theta(1-u)(1-v))^2}, \quad (35)$$

and

$$C(u|v) = \frac{u}{1-\theta(1-u)(1-v)} - \frac{\theta uv(1-u)}{(1-\theta(1-u)(1-v))^2}. \quad (36)$$

In Plackett copula

$$C(v|u) = 0.5\left[1-(1+(\theta-1)(u+v))\right]-v\theta \frac{\left((1+(\theta-1)(u+v))^2-4uv\theta(\theta-1)\right)^{-0.5}}{4(\theta-1)}, \quad (37)$$

and

$$C(u|v) = 0.5\left[1-(1+(\theta-1)(u+v))\right]-u\theta \frac{\left((1+(\theta-1)(u+v))^2-4uv\theta(\theta-1)\right)^{-0.5}}{4(\theta-1)}. \quad (38)$$

where $u$ is a cdf of ITL distribution with parameter $\lambda_1$ and $v$ is a cdf of ITL distribution with parameter $\lambda_2$.
A simulation algorithm: Simulation experiments were carried out based on the following data generated from bivariate ITL distributions under different copulas, where the values of the parameters $\lambda_1, \lambda_2$, and $\theta$ are chosen as in the following cases for the random variables generation:
Case 1: ($\lambda_2 = 2.5, \theta = 0.4$) and $\lambda_1$ are change from 0.75 to 3.
Case 2: ($\lambda_2 = 0.5, \theta = 0.4$) and $\lambda_1$ are change from 0.75 to 3.
Case 3: ($\lambda_1 = 0.75, \lambda_2 = 0.5$) and $\theta$ are change from 0.1 to 0.9.
For different sample size $n = 35, 70$ and 150. The simulation results are compared using parameter estimation criteria, which include measuring the Bias, MSE, length of ACI, and dependence measure for each model as follows:
$Bias = (\hat{\Theta} - \Theta)$
Where $\hat{\Theta}$ is the estimated value of $\Theta$.
$MSE = Mean(\hat{\Theta} - \Theta)^2$.
and L.CI=Upper.CI($\Theta$)-Lower.CI($\Theta$)
The number of repeated samples in simulation was limited to 5000.
Figure 5 Kendall's Tau, and Spearman's Rho coefficients limiting for BITL distribution. The simulation results of the methods discussed in this paper for point estimation, interval estimation, and dependency calculation are summarized in Tables 1, 2, and 3. The Bias, MSE, and L.CI for parameter of bivariate ITL distributions under different copula, Kendall's Tau, and Spearman's Rho values are used to perform the requisite comparison between different point estimation methods. The following conclusions can be drawn from these tables:

1. The Bias, MSE, and L.CI decrease as $n$ increases for actual parameters of bivariate ITL distributions under different copulas, Kendall's Tau, and Spearman's Rho coefficients.

2. For fixed sample size and case of actual values, we note that the Bias, MSE, and L.CI for parameters of bivariate ITL distributions under different copulas, Kendall's Tau, and Spearman's Rho coefficients increase when $\lambda_1$ increases.

3. In case 1 and fixed sample size, we note that the PBITL has the smallest Bias and MSE for $\lambda_1, \lambda_2$, $\theta$, while the shortest CI in case of AMHBITL distribution.

4. For fixed sample size and case of actual values, we note the Bias, MSE, and L.CI for parameters of bivariate ITL distributions under different copulas, Kendall's Tau, and Spearman's Rho coefficients increase when $\lambda_2$ increases.

5. For fixed sample size and case of actual values, we note that the Bias, MSE, and L.CI for parameters of bivariate ITL distributions under different copulas, Kendall's Tau, and Spearman's Rho coefficients

increase when $\theta$ increases.

## 6 Application

In this section, we used medical data related to diabetic nephropathy. We focused at the duration of diabetes and serum creatinine levels (SrCr). We are measuring the complications that may arise because of diabetics using SrCr values and non-diabetic nephropathy (SrCr $<$ 1.4mg/dl). Pathological reports for these patients were collected from Dr. Lal's path lab database between January 2012 and August 2013. Grover et al. [31] looked at these findings, which included the average diabetes cycle of 132 patient Type-II diabetic nephropathy patients over time intervals. The parameters estimation for each distribution have been done, see Table 4. And it is through that we gain access to a model that is well-suited to the study of fragile relationships and the degree to which they have an influence and effectiveness. Table 5 shows the model criterion selection as Akaike information criterion (AIC), corrected AIC (CAIC), Hannan-Quinn information criterion (HQIC) and goodness of fit test of copulas as Anderson–Darling-type statistics (ADTS) with p-value and Cramer-von Mises functional (CVMF) wiht p-value for FGMBITL, AMHBITL, CBITL and PBITLF distributions. According to Tables 4 and 5, we can conclude that the CBITL model is better than FGMBITL, AMHBITL and PBITL distributions. For more information of goodness of fit test of copula see Genest et al. [32].

The plot of the bivariate distribution's density and cumulative is more interesting. However, keep in mind that the scatter plot and density plot may be deceiving. This is one of the reasons why, if contour plots are open, we tend to look at them in most cases. As shown in Figures 6, 7, these four copulas have correlation properties of various characteristics. The Clayton copula correlates better than the others copulas. The correlation structure of the FGM copula is weaker than the others copulas.

## 7 Conclusion

In this paper, we have proposed the use of the bivariate models based on copula functions for medical data related to diabetic nephropathy. We showed that the duration of diabetes and SrCr levels are closely related in a probabilistic way to each other by analyzing medical data. These empirical facts authorize our endeavor to bivariate model diabetic nephropathy based on different copulas. Four copula models were employed and their accuracy was examined through fitting to the real data. We have proposed a new class of bivariate ITL distributions based on FGM, AMH, Plackett, and Clayton copulas. Moreover, we obtained the reliability functions for bivariate ITL distributions based on different copulas understudy, therefore, it can be used quite effectively in life testing data. We studied dependence measures as Kendall's Tau and Spearman's Rho. Monte Carlo simulation is used to compare distributions. The CBITL distribution is the best model to fit this data.

**Table 1:** Point, interval and dependence measure of the bivariate ITL distributions under different copulas: Case 1

| $\lambda_2=2.5,\theta=0.4$ | | | FGM | | | Clayton | | | AMH | | | Plackett | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $n$ | | Bias | MSE | L.CI | Bias | MSE | L.CI | Bias | MSE | L.CI | Bias | MSE | L.CI |
| 3 | 35 | $\lambda_1$ | 0.0900 | 0.2782 | 2.0384 | -0.2574 | 0.4908 | 2.5554 | 0.0434 | 0.2709 | 2.0343 | 0.0017 | 0.2412 | 2.0218 |
| | | $\lambda_2$ | 0.0719 | 0.2056 | 1.7557 | -0.1581 | 0.2815 | 1.9862 | 0.0298 | 0.1902 | 1.7063 | 0.0042 | 0.1886 | 1.7304 |
| | | $\theta$ | 0.0040 | 0.2210 | 1.8437 | 0.1486 | 0.1508 | 1.4070 | 0.2264 | 0.1308 | 1.1060 | -0.1802 | 0.0426 | 1.1566 |
| | | $\tau_k$ | 0.0009 | 0.0109 | 0.4097 | 0.0339 | 0.0124 | 0.4162 | 0.0815 | 0.0148 | 0.3534 | -0.1421 | 0.0278 | 0.8729 |
| | | $\rho_s$ | 0.0013 | 0.0246 | 0.6146 | 0.0448 | 0.0245 | 0.5880 | 0.1179 | 0.0309 | 0.5118 | -0.1899 | 0.0489 | 1.2319 |
| | 70 | $\lambda_1$ | 0.0506 | 0.1302 | 1.4014 | -0.3834 | 0.4689 | 2.2251 | 0.0090 | 0.1230 | 1.3753 | -0.0312 | 0.1190 | 1.4015 |
| | | $\lambda_2$ | 0.0505 | 0.1041 | 1.2499 | -0.2498 | 0.2425 | 1.6641 | 0.0151 | 0.0987 | 1.2308 | -0.0128 | 0.0960 | 1.2503 |
| | | $\theta$ | -0.0053 | 0.1283 | 1.4049 | 0.0939 | 0.0653 | 0.9321 | 0.2406 | 0.0950 | 0.7556 | -0.1941 | 0.0417 | 0.9371 |
| | | $\tau_k$ | -0.0012 | 0.0063 | 0.3122 | 0.0244 | 0.0061 | 0.2918 | 0.0804 | 0.0106 | 0.2514 | -0.1463 | 0.0250 | 0.7705 |
| | | $\rho_s$ | -0.0018 | 0.0143 | 0.4683 | 0.0332 | 0.0123 | 0.4156 | 0.1175 | 0.0225 | 0.3653 | -0.1976 | 0.0450 | 1.0949 |
| | 150 | $\lambda_1$ | 0.0178 | 0.0595 | 0.9541 | -0.4029 | 0.3272 | 1.5923 | -0.0240 | 0.0587 | 0.9454 | -0.0490 | 0.0556 | 0.9498 |
| | | $\lambda_2$ | 0.0155 | 0.0450 | 0.8298 | -0.2751 | 0.1670 | 1.1849 | -0.0196 | 0.0438 | 0.8167 | -0.0394 | 0.0414 | 0.8199 |
| | | $\theta$ | -0.0012 | 0.0632 | 0.9857 | 0.0688 | 0.0287 | 0.6070 | 0.2521 | 0.0804 | 0.5095 | -0.1958 | 0.0402 | 0.7872 |
| | | $\tau_k$ | -0.0003 | 0.0031 | 0.2190 | 0.0200 | 0.0030 | 0.2007 | 0.0812 | 0.0085 | 0.1729 | -0.1441 | 0.0225 | 0.6957 |
| | | $\rho_s$ | -0.0004 | 0.0070 | 0.3286 | 0.0280 | 0.0062 | 0.2885 | 0.1194 | 0.0184 | 0.2519 | -0.1956 | 0.0412 | 0.9936 |
| 0.75 | 35 | $\lambda_1$ | 0.0287 | 0.0203 | 0.5468 | -0.0204 | 0.0162 | 0.4925 | 0.0178 | 0.0196 | 0.5453 | -0.0124 | 0.0167 | 0.5554 |
| | | $\lambda_2$ | 0.1075 | 0.2270 | 1.8203 | -0.0698 | 0.2170 | 1.8063 | 0.0634 | 0.2129 | 1.7925 | 0.0878 | 0.2115 | 1.7570 |
| | | $\theta$ | -0.0141 | 0.2431 | 1.9328 | 0.0817 | 0.1121 | 1.2732 | 0.2152 | 0.1406 | 1.2042 | -0.1792 | 0.0433 | 1.2035 |
| | | $\tau_k$ | -0.0031 | 0.0120 | 0.4295 | 0.0146 | 0.0101 | 0.3909 | 0.0791 | 0.0154 | 0.3740 | -0.1419 | 0.0280 | 0.8829 |
| | | $\rho_s$ | -0.0047 | 0.0270 | 0.6443 | 0.0177 | 0.0204 | 0.5557 | 0.1142 | 0.0322 | 0.5424 | -0.1896 | 0.0492 | 1.2468 |
| | 70 | $\lambda_1$ | 0.0129 | 0.0088 | 0.3646 | -0.0359 | 0.0084 | 0.3304 | 0.0029 | 0.0084 | 0.3595 | -0.0251 | 0.0084 | 0.3805 |
| | | $\lambda_2$ | 0.0408 | 0.0913 | 1.1744 | -0.1381 | 0.1112 | 1.1907 | 0.0055 | 0.0862 | 1.1510 | 0.0309 | 0.0866 | 1.1238 |
| | | $\theta$ | -0.0079 | 0.1188 | 1.3516 | 0.0272 | 0.0442 | 0.8173 | 0.2366 | 0.0914 | 0.7377 | -0.1900 | 0.0405 | 0.9256 |
| | | $\tau_k$ | -0.0018 | 0.0059 | 0.3004 | 0.0036 | 0.0046 | 0.2643 | 0.0788 | 0.0102 | 0.2468 | -0.1426 | 0.0240 | 0.7648 |
| | | $\rho_s$ | -0.0026 | 0.0132 | 0.4505 | 0.0036 | 0.0094 | 0.3790 | 0.1152 | 0.0216 | 0.3583 | -0.1927 | 0.0433 | 1.0873 |
| | 150 | $\lambda_1$ | 0.0046 | 0.0040 | 0.2473 | -0.0446 | 0.0052 | 0.2230 | -0.0061 | 0.0038 | 0.2415 | -0.0314 | 0.0045 | 0.2617 |
| | | $\lambda_2$ | 0.0185 | 0.0413 | 0.7941 | -0.1541 | 0.0701 | 0.8447 | -0.0176 | 0.0403 | 0.7840 | 0.0202 | 0.0405 | 0.7467 |
| | | $\theta$ | 0.0122 | 0.0622 | 0.9770 | 0.0145 | 0.0221 | 0.5805 | 0.2564 | 0.0825 | 0.5080 | -0.1876 | 0.0372 | 0.7862 |
| | | $\tau_k$ | 0.0027 | 0.0031 | 0.2171 | 0.0020 | 0.0024 | 0.1929 | 0.0828 | 0.0088 | 0.1735 | -0.1361 | 0.0203 | 0.6898 |
| | | $\rho_s$ | 0.0041 | 0.0069 | 0.3257 | 0.0021 | 0.0050 | 0.2778 | 0.1216 | 0.0189 | 0.2526 | -0.1852 | 0.0373 | 0.9869 |

**Table 2:** Point, interval and dependence measure of the bivariate ITL distributions under different copulas: Case 2

| $\lambda_2 = 0.5, \theta = 0.4$ | | | FGM | | | Clayton | | | AMH | | | Plackett | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $n$ | | Bias | MSE | L.CI | Bias | MSE | L.CI | Bias | MSE | L.CI | Bias | MSE | L.CI |
| 0.75 | 35 | $\lambda_1$ | 0.0289 | 0.0203 | 0.5470 | -0.0158 | 0.0159 | 0.4903 | 0.0195 | 0.0206 | 0.5571 | 0.0094 | 0.0165 | 0.5399 |
| | | $\lambda_2$ | 0.0207 | 0.0090 | 0.3637 | -0.0118 | 0.0071 | 0.3273 | 0.0128 | 0.0087 | 0.3621 | -0.0082 | 0.0071 | 0.3668 |
| | | $\theta$ | -0.0183 | 0.2434 | 1.9338 | -0.0031 | 0.0941 | 1.2029 | 0.2126 | 0.1404 | 1.2103 | -0.1436 | 0.0451 | 1.2682 |
| | | $\tau_k$ | -0.0041 | 0.0120 | 0.4297 | -0.0138 | 0.0104 | 0.3953 | 0.0783 | 0.0152 | 0.3740 | -0.1182 | 0.0244 | 0.8855 |
| | | $\rho_s$ | -0.0061 | 0.0270 | 0.6446 | -0.0230 | 0.0215 | 0.5679 | 0.1130 | 0.0319 | 0.5428 | -0.1576 | 0.0433 | 1.2550 |
| | 70 | $\lambda_1$ | 0.0129 | 0.0088 | 0.3644 | -0.0300 | 0.0083 | 0.3374 | 0.0025 | 0.0086 | 0.3630 | -0.0037 | 0.0079 | 0.3618 |
| | | $\lambda_2$ | 0.0073 | 0.0036 | 0.2349 | -0.0249 | 0.0035 | 0.2108 | 0.0000 | 0.0035 | 0.2314 | -0.0180 | 0.0033 | 0.2412 |
| | | $\theta$ | -0.0135 | 0.1187 | 1.3503 | -0.0418 | 0.0415 | 0.7819 | 0.2350 | 0.0914 | 0.7458 | -0.1622 | 0.0338 | 0.9403 |
| | | $\tau_k$ | -0.0030 | 0.0059 | 0.3001 | -0.0204 | 0.0050 | 0.2652 | 0.0784 | 0.0102 | 0.2490 | -0.1199 | 0.0192 | 0.7600 |
| | | $\rho_s$ | -0.0045 | 0.0132 | 0.4501 | -0.0312 | 0.0105 | 0.3835 | 0.1145 | 0.0216 | 0.3616 | -0.1626 | 0.0349 | 1.0841 |
| | 150 | $\lambda_1$ | 0.0046 | 0.0040 | 0.2472 | -0.0397 | 0.0048 | 0.2228 | -0.0063 | 0.0038 | 0.2414 | -0.0091 | 0.0037 | 0.2406 |
| | | $\lambda_2$ | 0.0030 | 0.0016 | 0.1586 | -0.0285 | 0.0022 | 0.1437 | -0.0046 | 0.0016 | 0.1566 | -0.0199 | 0.0018 | 0.1673 |
| | | $\theta$ | 0.0068 | 0.0614 | 0.9712 | -0.0571 | 0.0214 | 0.5288 | 0.2552 | 0.0822 | 0.5122 | -0.1627 | 0.0298 | 0.7865 |
| | | $\tau_k$ | 0.0015 | 0.0030 | 0.2158 | -0.0230 | 0.0028 | 0.1869 | 0.0824 | 0.0088 | 0.1746 | -0.1148 | 0.0155 | 0.6809 |
| | | $\rho_s$ | 0.0023 | 0.0068 | 0.3237 | -0.0343 | 0.0060 | 0.2721 | 0.1210 | 0.0188 | 0.2542 | -0.1571 | 0.0288 | 0.9771 |
| 3 | 35 | $\lambda_1$ | 0.0864 | 0.2876 | 2.0757 | -0.2209 | 0.4631 | 2.5244 | 0.0392 | 0.2746 | 2.0494 | 0.0935 | 0.2566 | 1.9465 |
| | | $\lambda_2$ | 0.0129 | 0.0087 | 0.3618 | -0.0272 | 0.0076 | 0.3258 | 0.0045 | 0.0079 | 0.3485 | -0.0204 | 0.0073 | 0.3621 |
| | | $\theta$ | -0.0008 | 0.2209 | 1.8433 | 0.0395 | 0.0997 | 1.2286 | 0.2250 | 0.1326 | 1.1228 | -0.1566 | 0.0388 | 1.1771 |
| | | $\tau_k$ | -0.0002 | 0.0109 | 0.4096 | 0.0010 | 0.0098 | 0.3879 | 0.0814 | 0.0150 | 0.3583 | -0.1239 | 0.0241 | 0.8695 |
| | | $\rho_s$ | -0.0003 | 0.0245 | 0.6144 | -0.0016 | 0.0199 | 0.5539 | 0.1176 | 0.0313 | 0.5186 | -0.1658 | 0.0427 | 1.2306 |
| | 70 | $\lambda_1$ | 0.0499 | 0.1307 | 1.4041 | -0.3231 | 0.3992 | 2.1295 | 0.0066 | 0.1235 | 1.3778 | 0.0578 | 0.1257 | 1.3236 |
| | | $\lambda_2$ | 0.0093 | 0.0042 | 0.2504 | -0.0340 | 0.0048 | 0.2372 | 0.0022 | 0.0040 | 0.2491 | -0.0227 | 0.0040 | 0.2656 |
| | | $\theta$ | -0.0102 | 0.1282 | 1.4038 | -0.0035 | 0.0425 | 0.8087 | 0.2391 | 0.0951 | 0.7638 | -0.1739 | 0.0358 | 0.9412 |
| | | $\tau_k$ | -0.0023 | 0.0063 | 0.3120 | -0.0071 | 0.0048 | 0.2703 | 0.0800 | 0.0106 | 0.2535 | -0.1285 | 0.0206 | 0.7614 |
| | | $\rho_s$ | -0.0034 | 0.0142 | 0.4679 | -0.0119 | 0.0100 | 0.3896 | 0.1169 | 0.0225 | 0.3683 | -0.1742 | 0.0373 | 1.0853 |
| | 150 | $\lambda_1$ | 0.0178 | 0.0595 | 0.9539 | -0.3832 | 0.3181 | 1.6233 | -0.0249 | 0.0587 | 0.9450 | 0.0383 | 0.0563 | 0.8687 |
| | | $\lambda_2$ | 0.0024 | 0.0018 | 0.1660 | -0.0410 | 0.0033 | 0.1600 | -0.0050 | 0.0018 | 0.1632 | -0.0277 | 0.0023 | 0.1798 |
| | | $\theta$ | -0.0062 | 0.0629 | 0.9833 | -0.0143 | 0.0170 | 0.5088 | 0.2508 | 0.0801 | 0.5142 | -0.1761 | 0.0335 | 0.7826 |
| | | $\tau_k$ | -0.0014 | 0.0031 | 0.2185 | -0.0075 | 0.0021 | 0.1763 | 0.0809 | 0.0085 | 0.1742 | -0.1259 | 0.0179 | 0.6841 |
| | | $\rho_s$ | -0.0021 | 0.0070 | 0.3278 | -0.0115 | 0.0044 | 0.2558 | 0.1188 | 0.0183 | 0.2538 | -0.1719 | 0.0331 | 0.9801 |

**Table 3:** Point, interval and dependence measure of the bivariate ITL distributions under different copulas: Case 3

| $\lambda_1 = 0.75, \lambda_2 = 0.5$ | | | FGM | | | Clayton | | | AMH | | | Plackett | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta$ | $n$ | | Bias | MSE | L.CI | Bias | MSE | L.CI | Bias | MSE | L.CI | Bias | MSE | L.CI |
| 0.9 | 35 | $\lambda_1$ | 0.0306 | 0.0208 | 0.5526 | -0.0785 | 0.0181 | 0.4291 | -0.0695 | 0.0180 | 0.4495 | 0.0276 | 0.0201 | 0.4002 |
| | | $\lambda_2$ | -0.0084 | 0.0082 | 0.3527 | -0.0545 | 0.0082 | 0.2823 | -0.0487 | 0.0078 | 0.2896 | 0.0200 | 0.0090 | 0.2579 |
| | | $\theta$ | -0.3228 | 0.2799 | 1.6442 | -0.2531 | 0.1881 | 1.3811 | -0.1645 | 0.0549 | 0.6543 | 0.0505 | 0.3162 | 1.2103 |
| | | $\tau_k$ | -0.0717 | 0.0138 | 0.3654 | -0.0784 | 0.0152 | 0.3726 | -0.0640 | 0.0078 | 0.2401 | -0.0207 | 0.0151 | 0.6160 |
| | | $\rho_s$ | -0.1076 | 0.0311 | 0.5481 | -0.1109 | 0.0297 | 0.5170 | -0.0914 | 0.0160 | 0.3427 | -0.0300 | 0.0324 | 0.9000 |
| | 70 | $\lambda_1$ | 0.0137 | 0.0086 | 0.3599 | -0.0924 | 0.0138 | 0.2842 | -0.0823 | 0.0125 | 0.2958 | 0.0113 | 0.0088 | 0.2370 |
| | | $\lambda_2$ | -0.0212 | 0.0036 | 0.2188 | -0.0655 | 0.0064 | 0.1796 | -0.0591 | 0.0058 | 0.1866 | 0.0071 | 0.0037 | 0.1449 |
| | | $\theta$ | -0.2986 | 0.1804 | 1.1846 | -0.3040 | 0.1543 | 0.9755 | -0.1543 | 0.0353 | 0.4203 | -0.0453 | 0.1179 | 0.7686 |
| | | $\tau_k$ | -0.0664 | 0.0089 | 0.2633 | -0.0876 | 0.0129 | 0.2830 | -0.0633 | 0.0057 | 0.1630 | -0.0262 | 0.0072 | 0.5046 |
| | | $\rho_s$ | -0.0995 | 0.0200 | 0.3949 | -0.1223 | 0.0252 | 0.3972 | -0.0896 | 0.0115 | 0.2322 | -0.0386 | 0.0156 | 0.7405 |
| | 150 | $\lambda_1$ | 0.0049 | 0.0039 | 0.2442 | -0.1004 | 0.0125 | 0.1926 | -0.0896 | 0.0106 | 0.1983 | 0.0036 | 0.0040 | 0.1297 |
| | | $\lambda_2$ | -0.0250 | 0.0020 | 0.1476 | -0.0693 | 0.0058 | 0.1249 | -0.0630 | 0.0050 | 0.1281 | 0.0035 | 0.0017 | 0.0773 |
| | | $\theta$ | -0.2755 | 0.1237 | 0.8575 | -0.3260 | 0.1362 | 0.6786 | -0.1477 | 0.0273 | 0.2905 | -0.0517 | 0.0520 | 0.4847 |
| | | $\tau_k$ | -0.0612 | 0.0061 | 0.1906 | -0.0908 | 0.0109 | 0.2012 | -0.0622 | 0.0047 | 0.1115 | -0.0204 | 0.0037 | 0.4278 |
| | | $\rho_s$ | -0.0918 | 0.0137 | 0.2858 | -0.1258 | 0.0210 | 0.2830 | -0.0876 | 0.0093 | 0.1598 | -0.0302 | 0.0081 | 0.6304 |
| 0.1 | 35 | $\lambda_1$ | 0.0220 | 0.0189 | 0.5326 | 0.0163 | 0.0156 | 0.4858 | 0.0232 | 0.0185 | 0.5255 | -0.1549 | 0.0326 | 0.6230 |
| | | $\lambda_2$ | 0.0140 | 0.0086 | 0.3592 | 0.0117 | 0.0080 | 0.3468 | 0.0135 | 0.0082 | 0.3511 | -0.1089 | 0.0158 | 0.4215 |
| | | $\theta$ | 0.0242 | 0.2622 | 2.0062 | 0.1487 | 0.0902 | 1.0232 | 0.0565 | 0.2250 | 1.8473 | 0.1121 | 0.0203 | 1.0404 |
| | | $\tau_k$ | 0.0054 | 0.0130 | 0.4458 | 0.0516 | 0.0127 | 0.3921 | 0.0290 | 0.0141 | 0.4520 | 0.1289 | 0.0230 | 0.7829 |
| | | $\rho_s$ | 0.0081 | 0.0291 | 0.6687 | 0.0755 | 0.0270 | 0.5722 | 0.0424 | 0.0309 | 0.6687 | 0.1603 | 0.0363 | 1.1064 |
| | 70 | $\lambda_1$ | 0.0119 | 0.0082 | 0.3512 | 0.0091 | 0.0076 | 0.3395 | 0.0116 | 0.0081 | 0.3499 | -0.1599 | 0.0298 | 0.4737 |
| | | $\lambda_2$ | 0.0098 | 0.0042 | 0.2524 | 0.0089 | 0.0043 | 0.2534 | 0.0096 | 0.0042 | 0.2522 | -0.1114 | 0.0145 | 0.3376 |
| | | $\theta$ | -0.0049 | 0.1436 | 1.4861 | 0.1229 | 0.0421 | 0.6439 | 0.0558 | 0.1251 | 1.3700 | 0.1017 | 0.0135 | 0.7486 |
| | | $\tau_k$ | -0.0011 | 0.0071 | 0.3303 | 0.0478 | 0.0067 | 0.2605 | 0.0213 | 0.0073 | 0.3241 | 0.1263 | 0.0189 | 0.6643 |
| | | $\rho_s$ | -0.0016 | 0.0160 | 0.4954 | 0.0707 | 0.0147 | 0.3852 | 0.0316 | 0.0162 | 0.4833 | 0.1552 | 0.0290 | 0.9437 |
| | 150 | $\lambda_1$ | 0.0043 | 0.0037 | 0.2390 | 0.0027 | 0.0038 | 0.2416 | 0.0038 | 0.0037 | 0.2391 | -0.1640 | 0.0288 | 0.3729 |
| | | $\lambda_2$ | 0.0030 | 0.0018 | 0.1667 | 0.0015 | 0.0019 | 0.1690 | 0.0027 | 0.0018 | 0.1666 | -0.1150 | 0.0141 | 0.2595 |
| | | $\theta$ | -0.0004 | 0.0671 | 1.0158 | 0.1136 | 0.0261 | 0.4504 | 0.0767 | 0.0603 | 0.9146 | 0.1022 | 0.0119 | 0.5051 |
| | | $\tau_k$ | -0.0001 | 0.0033 | 0.2257 | 0.0464 | 0.0044 | 0.1853 | 0.0221 | 0.0037 | 0.2214 | 0.1302 | 0.0183 | 0.5752 |
| | | $\rho_s$ | -0.0001 | 0.0075 | 0.3386 | 0.0691 | 0.0097 | 0.2752 | 0.0329 | 0.0082 | 0.3309 | 0.1594 | 0.0277 | 0.8207 |

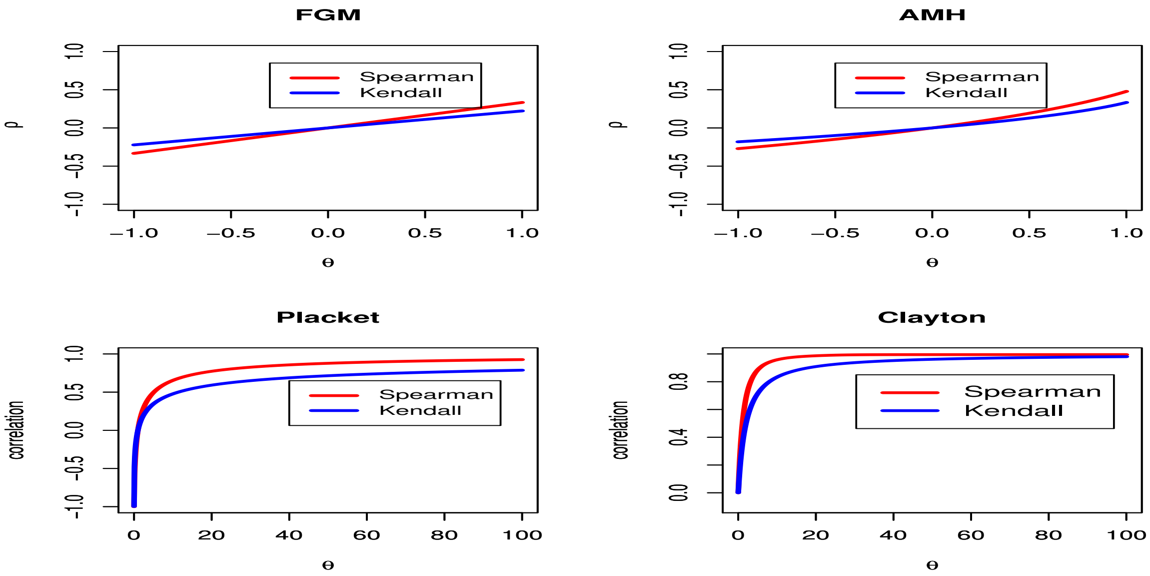


**Fig. 5:** Kendall's Tau, and Spearman's Rho coefficients limiting for BITL

**Table 4:** MLE and dependance measure of FGMBITL, AMHBITL, CBITL and PBITLF distributions for medical data

| | $\lambda_1$ | $\lambda_2$ | $\theta$ | $\tau_k$ | $\rho_s$ |
|---|---|---|---|---|---|
| FGMBITL | 0.5121 | 0.3981 | 0.4670 | 0.1038 | 0.1557 |
| AMHBITL | 0.4083 | 0.2407 | 0.4975 | 0.1280 | 0.1913 |
| CBITL | 0.3605 | 0.3409 | 3.0826 | 0.1317 | 0.1964 |
| PBITL | 0.3359 | 0.3290 | 1.6633 | 0.1138 | 0.1682 |

**Table 5:** The model criterion selection and goodness of fit for FGMBITL, AMHBITL, CBITL and PBITLF distributions for medical data

| | AIC | CAIC | BIC | HQIC | AD | p-value | CVM | p-value |
|---|---|---|---|---|---|---|---|---|
| FGMBITL | 754.0737 | 754.9968 | 758.2773 | 755.4185 | 0.2903 | 0.4000 | 0.02503 | 0.2532 |
| AMHBITL | 749.8751 | 750.7982 | 754.0787 | 751.2199 | 0.2114 | 0.5509 | 0.020513 | 0.4363 |
| CBITL | 731.5039 | 732.4270 | 735.7075 | 732.8487 | 0.1442 | 0.7494 | 0.020806 | 0.4744 |
| PBITL | 734.7395 | 735.6626 | 738.9431 | 736.0843 | 0.2623 | 0.4434 | 0.023196 | 0.2972 |

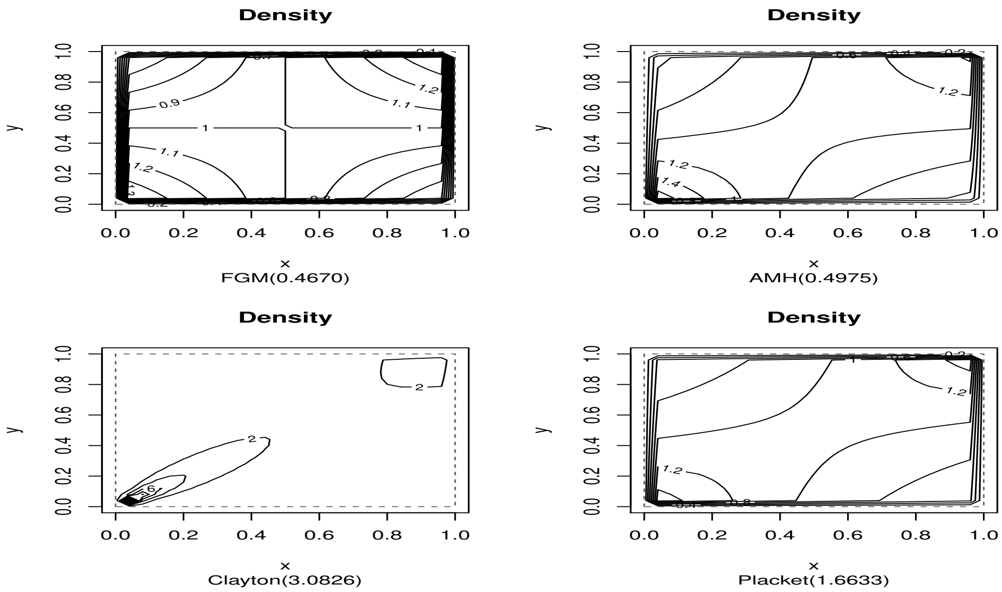


**Fig. 6:** Density contour plot for BITL distribution

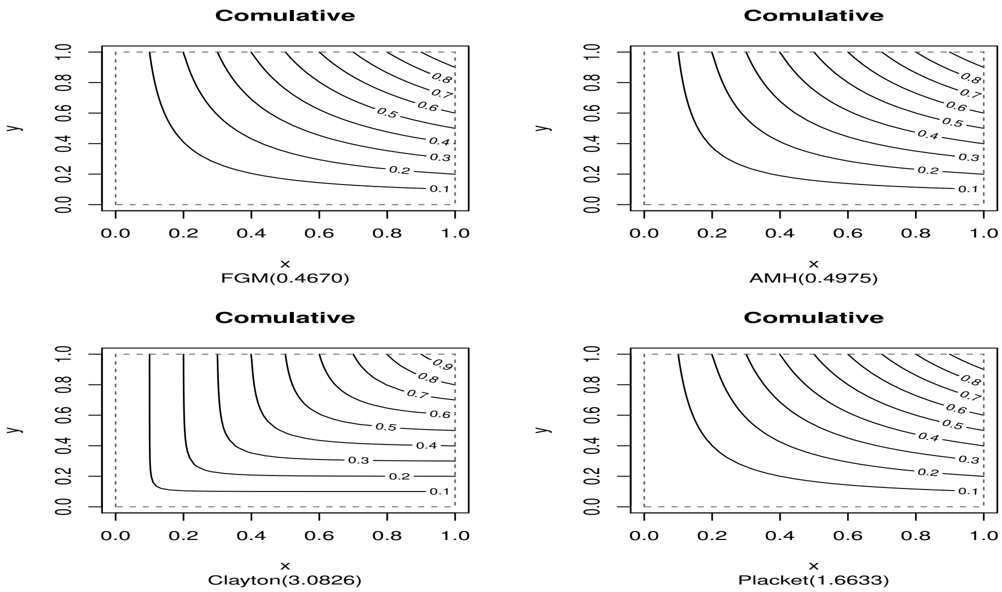


**Fig. 7:** Comulative contour plot for BITL distribution

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## References

[1] T. Okimoto, New evidence of asymmetric dependence structures in international equity markets. Journal of financial and quantitative analysis, **43**(3), 787-815, (2008).
[2] Y. R. Wang, H. Huang, Review on statistical methods for gene network reconstruction using expression data. Journal of theoretical biology, **362**, 53-61, (2014).
[3] M. M. Al-Sadoon, Testing subspace Granger causality. Econometrics and statistics, **9**, 42-61, (2019).
[4] A. Bárdossy, S. H orning, Process-driven direction-dependent asymmetry: identification and quantification of directional dependence in spatial fields. Mathematical Geosciences, **49**(7), 871-891, (2017).
[5] A. R. Coenen, J. S. Weitz, Limitations of correlation-based inference in complex virus-microbe communities. Msystems, **3**(4), 1-14, (2018).
[6] R. R. Junker, F. Griessenberger, & W. Trutschnig, Estimating scale-invariant directed dependence of bivariate distributions. Computational Statistics & Data Analysis, **153**, 107058, (2021).
[7] P. Embrechts, F. Lindskog, & A. McNeil, Modelling dependence with copulas. Rapport technique, Département de mathématiques, Institut Fédéral de Technologie de Zurich, Zurich, **14**, (2001).
[8] M. S. Smith, Q. Gan, & R. J. Kohn, Modelling dependence using skew t copulas: Bayesian inference and applications. Journal of Applied Econometrics, **27**(3), 500-522, (2012).
[9] G. J. Székely, M. L. Rizzo, & N. K. Bakirov, Measuring and testing dependence by correlation of distances. The annals of statistics, **35**(6), 2769-2794, (2007).
[10] Y. A. Reshef, D. N. Reshef, H. K. Finucane, P. C. Sabeti, & M. Mitzenmacher, Measuring dependence powerfully and equitably. The Journal of Machine Learning Research, **17**(1), 7406-7468, (2016).
[11] D. N. Reshef, Y. A. Reshef, P. C. Sabeti, & M. Mitzenmacher, An empirical study of the maximal and total information coefficients and leading measures of dependence. The Annals of Applied Statistics, **12**(1), 123-155, (2018).
[12] Y. Chang, Y. Li, A. Ding, & J. Dy, A robust-equitable copula dependence measure for feature selection. In Artificial Intelligence and Statistics pp. 84-92. PMLR, (2016).
[13] R. B. Nelsen. *An Introduction to Copulas*. Springer, New York (2006).
[14] M. K. Elaal, R. S. Jarwan. Inference of Bivariate Generalized Exponential Distribution Based on Copula Functions. Applied Mathematical Sciences, 11(24), 1155-1186 (2017).
[15] E. M. Almetwally, H. Z. Muhammed and E. S. El-Sherpieny. Bivariate Wei bull Distribution: Properties and Different Methods of Estimation. Annals of Data Science, 7(1), 163-193, (2020).
[16] E. M. Almetwally, and H. Z. Muhammed, On a bivariate Fréchet distribution. J Stat Appl Probab, **9**(1), 1-21, (2020).
[17] E. S. El-Sherpieny, E. M. Almetwally. Bivariate Generalized Rayleigh Distribution Based on Clayton Copula. In Proceedings of the Annual Conference on Statistics **54**, Computer Science and Operation research , Faculty of Graduate Studies for Statistical Research, Cairo University, 1-19 (2019).
[18] E. S. A. El-Sherpieny, H. Z. Muhammed, and E. M. Almetwally. Bivariate Weibull-G Family Based on Copula Function: Properties, Bayesian and non-Bayesian Estimation and Applications. Statistics, Optimization & Information Computing, 9(4), (2021). https://doi.org/10.19139/soic-2310-5070-1129.
[19] E. S. A. El-Sherpieny, E. M. Almetwally, & H. Z. Muhammed, Bayesian and Non-Bayesian Estimation for the Parameter of Bivariate Generalized Rayleigh Distribution Based on Clayton Copula under Progressive Type-II Censoring with Random Removal. Sankhya A, 1-38, (2021). https://doi.org/10.1007/s13171-021-00254-3
[20] A. S. Hassan, M. Elgarhy, and R. Ragab, Statistical Properties and Estimation of Inverted Topp-Leone Distribution. Journal of Statistics Applications Probability. **9**(2), 319-331 (2020).
[21] E. M. Almetwally, R. Alharbi, D. Alnagar, & E. H. Hafez, A New Inverted Topp-Leone Distribution: Applications to the COVID-19 Mortality Rate in Two Different Countries. Axioms, **10**(1), 25, (2021).
[22] H. Z. Muhammed, On the inverted Topp Leone distribution. International Journal of Reliability and Applications, **20**(1), 17-28, (2019).
[23] E. M. Almetwally, The Odd Weibull Inverse Topp-Leone Distribution with Applications to COVID-19 Data. Annals of Data Science, 1-20, (2021). https://doi.org/10.1007/s40745-021-00329-w
[24] T. A. Abushal, A. S. Hassan, A. R. El-Saeed, & S. G. Nassr, Power Inverted Topp-Leone Distribution in Acceptance Sampling Plans. CMC-COMPUTERS MATERIALS & CONTINUA, **67**(1), 991-1011, (2021).
[25] A. S. Hassan, E. M. Almetwally, & G. M. Ibrahim, Kumaraswamy Inverted Topp-Leone Distribution with Applications to COVID-19 Data. CMC-COMPUTERS MATERIALS & CONTINUA, **68**(1), 337-358, (2021).
[26] G. M. Ibrahim, A. S. Hassan, E. M. Almetwally, & H. M. Almongy, Parameter Estimation of Alpha Power Inverted Topp-Leone Distribution with Applications. Intelligent Automation & Soft Computing, **29**(2), 353-371, (2021).
[27] C. Genest, & L. P. Rivest, Statistical inference procedures for bivariate Archimedean copulas. Journal of the American statistical Association, **88**(423), 1034-1043, (1993).
[28] A. Charpentier, & J. Segers, Tails of multivariate Archimedean copulas. Journal of Multivariate Analysis, **100**(7), 1521-1537, (2009).
[29] R. L. Plackett, A class of bivariate distributions. Journal of the American Statistical Association, **60**(310), 516-522, (1965).
[30] H. Bekrizadeh, G. A. Parham, & M. R. Zadkarmi, The new generalization of Farlie–Gumbel–Morgenstern copulas. Applied Mathematical Sciences, **6**(71), 3527-3533, (2012).
[31] G. Grover, A. Sabharwal, & J. Mittal, Application of Multivariate and Bivariate Normal Distributions to Estimate Duration of Diabetes. International Journal of Statistics and Applications, **4**(1), 46-57, (2014).

[32] C. Genest, W. Huang, & J. M. Dufour, A regularized goodness-of-fit test for copulas. Journal de la Société française de statistique, **154**(1), 64-77, (2013).