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Two Estimation Methods of Burr-XII Distribution based on Step-Stress Model under Type-II Hybrid Censoring

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Abstract: In this article, we think about two estimation methods with only two stress levels based on the Burr Type-XII distribution when the observed failure times are Type-II Hybrid censored. The Bayesian estimation and maximum likelihood estimation of the parameters for Burr XII distribution-based step-stress model are derived. In addition, asymptotic variance and covariance matrix of the estimators are given. They display some numerical illustration through Monte Carlo simulation for the finding. Finally, we present examples to illustrate all the estimation methods discussed.

Keywords: Burr XII distribution; Step-stress model, type-II hybrid censoring data, Bayesian estimation, maximum likelihood estimation, fisher information matrix.

1. Introduction

With step-stress accelerated life testing (ALT) with the emergence of more advanced technology in the manufacturing field it is difficult to produce enough amount of failure samples from ALT with only a constant stress level since failures in ALT are usually dispersed. Therefore time-varying stress ALT is required as an alternative to a constant stress ALT. The step-stress accelerated life testing (SSALT) is one of the most basic time-varying stress ALT. Under SSALT, test units are performed at several stress levels over time (usually in increased order of stress levels); i.e., the test is started at a low testing stress level and then switched to the next lowest stress level at a pre-determined time. Under step-stress, there are some studies that have been made using the usual types of censored data, we review some as follows: Miller and Nelson [1] present optimum simple step-stress plans for accelerated life testing, and Bai et al. [2] presents optimum simple step-stress accelerated life tests with censoring, Abd-Elfattah et al.[3] considers the estimation problem of step-stree model based on Type II censoring data of Burr Type XII distribution via EM-algorithm, Ganguly et al. [4] dispalys the Bayesian analysis of a simple step-stress model under Weibull lifetimes, Firoozeh and Suk [5] consider the reliability estimation from linear degradation and failure time data with competing risks under a step-stress accelerated degradation test, Mohie El-Din et al. [6] estimates the parameter of Lindely distribution, $\hat{\theta}$, and the accelerating factor, $\hat{\beta}$, by using maximum likelihood technique under step stress acceleration with progressive first failure data, Rashad [7] deals with the construct and compute in a Bayesian setting, point and interval predictions based on general progressive Type II censored data from generalized Pareto distribution, Ali [8] considers reliability analysis under constant-stress partially accelerated life tests using hybrid censored data from Weibull distribution, Mohie El-Din et al. [9] evaluates the parametric inference on step-stress accelerated life testing for the extension of exponential distribution under progressive Type-II censoring, Fariba et al. [10] deals with the simple step-stress life test and the tampered failure rate model in the presence of competing risks

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with the possibility of masked causes of failure and interval censoring and assume that the competing risks have independent Weibull distribution with the common shape parameter, Gyan [11] predicts bound lengths for Burr-XII distribution under step-stress PALT. Recently, Mohie El-Din et al. [12] have considered progressive-stress accelerated life testing for power generalized Weibull distribution under progressive Type-II censoring, and Ali [13] have considered the problem of estimation of Weibull distribution parameters and the acceleration factor using three different methods, using Type-I PHCS data under CSPALT in the case of data.

2. Model Description

2.1 Notations

ALT Acceerated Life Testing
ML Maximum likelihood
SST Step Stess Testing
HCS hybrid censored sample
NIP non-informative prior
S_0, S_1 Stress levels
pdf Probability density function
cdf Cumulative distribution function
$G(t)$ Cumulative exposure distribution function
$g(t)$ Probability exposure density function
n Identical units under an initial stress level s_0
$r \leq n$ and $0 < \tau_1 < \tau_2 < \infty$ are fixed in advance.
$t_{\text{i}:n}$ The ordered failure times of the i unit under test
τ_1 A fixed time before which the stress level is changed from S_0 to S_1
τ_2 A fixed time before which the r^{th} failure occurs the experiment is terminated at time τ_2
$t_{r:n}$ The time when the r^{th} failure occurs, the experiment is terminated
T_2 The random time when the life-testing experiment is terminated
N_1 Number of units that fail before time τ_1 at stress level S_0
N_2 Number of units that fail before time τ_2 at stress level S_1
M Number of units that fail before time T_2 at stress level S_1
c_i, k The shape parameters of the Burr Type XII distribution
$\text{L}(\alpha, \beta, k|t_{i:n})$ Likelihood function for the parameters $\theta = (\alpha, \beta, k)$
$\ln L(\alpha, \beta, k|t_{i:n})$ Logarithm of likelihood function for the parameters $\theta = (\alpha, \beta, k)$
$\pi_i(\theta)$ NIP for $\theta = (\alpha, \beta, k)$
$\pi(\alpha, \beta, k)$ The joint NIP
$\pi^*(\alpha, \beta, k|t_{i:n})$ The joint posterior distribution
$\pi(\theta|t_{i:n})$ The marginal posterior density for the parameters $\theta = (\alpha, \beta, k)$
$E[\theta|t_{i:n}]$ The posterior mean for the parameters $\theta = (\alpha, \beta, k)$
$V[\theta|t_{i:n}]$ The posterior variance for the parameters $\theta = (\alpha, \beta, k)$
F Fisher information matrix
MSE Mean square error
ARBias Absolute relative bias
RE Relative error

2.2 Assumptions

Suppose that the data comes from a step-stress model based on Type-II hybrid censored with two stress level S_0 and S_1. The pdf and the cdf under Burr XII distribution are given by

$$f_i(t; c_i) = c_i k t^{c_i - 1}(1 + t^{c_i})^{-(k+1)}, \quad t > 0 \quad k, c_i > 0 \quad i = 1,2 \tag{1}$$

and

$$F_i(t; c_i) = 1 - (1 + t^{c_i})^{-k}, \quad t > 0 \quad k, c_i > 0 \quad i = 1,2. \tag{2}$$

We deduce the cumulative exposure model (see Alhadeed [14]) under Burr Type-XII distribution.
The cdf function of time to failure under a particular step-stress pattern can be expressed mathematically as follows:
The population cumulative fraction of specimen failing by time t in stress level S_0 is

$$G(t) = F_1(t) = 1 - (1 + t^{c_1})^{-k}, \quad 0 \leq t < \tau_1.$$

The population cumulative fraction of specimen failing by time t in stress level S_1 is

$$G(t) = F_2[(t-\tau_1)+u_1] = 1-[1+((t-\tau_1)+u_1)^{c_2}]^{-k}, \quad \tau_1 \le t < \tau_2,$$

where u_1, the equivalent starting time, is the solution of

$$F_2(u_1) = F_1(\tau_1) \text{ and}$$
$$u_1^{c_2} = \tau_1^{c_1}.$$

Then, we get

$$u_1 = \tau_1^{\frac{c_1}{c_2}}.$$

Then, we can rewrite $G(t)$ in stress level S_1 as follows:

$$G(t) = 1-\left[1+\left((t-\tau_1)+\tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-k}, \quad \tau_1 \le t < \tau_2.$$

The simple step stress model is a special case from the cumulative exposure model, so we can say that the cumulative exposure distribution function (cedf) for Burr Type-XII distribution at two stress level is given as

$$G(t) = \begin{cases} G_1(t) = 1-(1+t^{c_1})^{-k}, & 0 \le t < \tau_1, \\ G_2(t) = 1-\left[1+\left(t-\tau_1+\tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-k}, & \tau_1 \le t < \infty. \end{cases} \tag{3}$$

The corresponding probability expsure density function (pedf) becomes

$$g(t) = \begin{cases} g_1(t) = kc_1 t^{c_1-1}(1+t^{c_1})^{-(k+1)}, & 0 \le t < \tau_1, \\ g_2(t) = kc_2\left[t-\tau_1+\tau_1^{\frac{c_1}{c_2}}\right]^{c_2-1} \times \left[1+\left(t-\tau_1+\tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-(k+1)} & \tau_1 \le t < \infty. \end{cases} \tag{4}$$

Based on the Type-II HCS, we have n identical units under an initial stress level, S_0. The stress level is changed to S_1 at time τ_1, and the life-testing experiment is terminated at time τ_2 if the r^{th} failure occurs before τ_2. Otherwise, the experiment is terminated as soon as the r^{th} failure occurs such that, the termination time in this situation is a random variable given by

$$T_2 = \max\{t_{r:n}, \tau_2\}, \tag{5}$$

where

$r \le n$ and $0 < \tau_1 < \tau_2 < \infty$ are fixed in advance, $t_{1:n} < t_{2:n} <\ldots< t_{n:n}$ denote the ordered failure times of the n unit under test. Then it is evident that,

$$M = \begin{cases} r-N_1 & ,\tau_2 < t_{r:n}, \\ N_2 & ,t_{r:n} \le \tau_2, \end{cases} \quad \text{and} \quad T_2 = \begin{cases} t_{r:n} & ,\tau_2 < t_{r:n}, \\ \tau_2 & ,t_{r:n} \le \tau_2. \end{cases}$$

With this notation, we will observe one of the following three cases for the observations:

- *Case 1:* If $t_{r:n} \le \tau_1 < \tau_2$, we will observe

$$t_{1:n} <\ldots< t_{r:n} <\ldots< t_{N_1:n} \le \tau_1 < t_{N_1+1:n} <\ldots< t_{N_1+N_2:n} \le \tau_2.$$

- *Case 2:* If $\tau_1 < t_{r:n} \le \tau_2$, we will observe

$$t_{1:n} <\ldots< t_{N_1:n} \le \tau_1 < t_{N_1+1:n} <\ldots< t_{r:n} <\ldots< t_{N_1+N_2:n} \le \tau_2.$$

- *Case 3:* If $\tau_2 < t_{r:n}$, we will observe

$$t_{1:n} <\ldots< t_{N_1:n} \le \tau_1 < t_{N_1+1:n} <\ldots< t_{N_1+N_2:n} \le \tau_2 < t_{N_1+N_2+1:n} <\ldots< t_{r:n} < \infty.$$

3. Bayesian Estimation for Hybrid Censoring Under Step-Stress Model

In this section, the Bayesian estimation under step-stress model of Burr-XII distribution will be derived. The bayesian estimators and the posterior variances of the Burr-XII distribution parameters are obtained under Type-II hybrid censoring data. From Eq. (3) and Eq. (4), we obtain the likelihood function of c_1 and c_2 based on the HCS as follows:

$$L = \frac{n!}{(n-R)!}\left[\prod_{i=1}^{N_1} kc_1 t_{i:n}^{c_1-1}(1+t_{i:n}^{c_1})^{-(k+1)}\right] \times \left[1+\left(T_2-\tau_1+\tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-k(n-R)}$$
$$\times \left[\prod_{i=N_1+1}^{R} kc_2\left(t_{i:n}-\tau_1+\tau_1^{\frac{c_1}{c_2}}\right)^{c_2-1}\left[1+\left(t_{i:n}-\tau_1+\tau_1^{\frac{c_1}{c_2}}\right)^{c_2}\right]^{-(k+1)}\right],$$
$$0 < t_{1:n} <\ldots< t_{N_1:n} \le \tau_1 < t_{N_1+1:n} <\ldots< t_{R:n} \le T_2. \tag{6}$$

Assuming that the parameters have independent prior distribution and let the non-informative prior (NIP) for c_1, c_2 and k are respectively given by

$$\pi_1(c_1) \propto \frac{1}{c_1},\ \pi_2(c_2) \propto \frac{1}{c_2},\ \pi_3(k) \propto \frac{1}{k}\ ;\ c_1,\ c_2, k > 0,$$

Consequently, the joint NIP will be as follows:

$$\pi(c_1, c_2, k) = \pi_1(c_1)\pi_2(c_2)\pi_3(k) = \frac{1}{c_1 c_2 k} = (c_1 c_2 k)^{-1}, c_1 > 0, c_2 > 0, k > 0. \tag{7}$$

According to Bayes' theorem, the posterior distribution of the model parameters given data $t_{i:n}$ is

$$\pi^*(c_1, c_2, k|t_{i:n}) = \frac{L(c_1,c_2,k|t_{i:n})\pi(c_1,c_2,k)}{\pi(t_{i:n})}, \tag{8}$$

where $L(c_1, c_2, k|t_{i:n})$ is the likelihood function of the data $t_{i:n}$, and $\pi(t_{i:n})$ is given by

$$\pi(t_{i:n}) = L(c_1, c_2, k|t_{i:n})\pi(c_1, c_2, k)dc_1 dc_2 dk\, , c_1, c_2, k > 0.$$

To simplify, replace the parameters c_1 and c_2 by α and β respectively, and $\pi(t_{i:n})$ is given by

$$\pi(t_{i:n}) = \frac{n!}{(n-R)!}\left\{\alpha^{N_1-1}\beta^{M-1}k^{R-1} \times \left[\prod_{i=1}^{N_1} t_{i:n}^{\alpha-1} B_i^{-(k+1)}\right] \times \left[\prod_{i=N_1+1}^{R} D_i^{\beta-1}\left(1-D_i^{\beta}\right)^{-(k+1)}\right] \times \left[1+E^{\beta}\right]^{-k(n-R)} d\alpha d\beta dk\right\}, \alpha, \beta, k > 0, \tag{9}$$

where

$$B_i = [1 + t_{i:n}^{\alpha}], D_i = \left[t_{i:n} - \tau_1 + \tau_1^{\frac{\alpha}{\beta}}\right], E = \left[T_2 - \tau_1 + \tau_1^{\frac{\alpha}{\beta}}\right],$$

Then, we can obtain

$$\pi(t_{i:n}) = \frac{n!}{(n-R)!}\, \alpha^{N_1-1}\beta^{M-1}k^{R-1}\, H d\alpha d\beta dk, \quad \alpha, \beta, k > 0,$$

where

$$H = \left[\prod_{i=1}^{N_1} t_{i:n}^{\alpha-1} B_i^{-(k+1)}\right] \times \left[\prod_{i=N_1+1}^{R} D_i^{\beta-1}\left(1-D_i^{\beta}\right)^{-(k+1)}\right] \times \left[1+E^{\beta}\right]^{-k(n-R)}.$$

Then the marginal posterior distribution of α, β and k are, respectively,

$$\pi(\alpha|t_{i:n}) = \pi^*(\alpha, \beta, k|t_{i:n})d\beta dk \quad \Rightarrow \quad \pi(\alpha|t_{i:n}) = \frac{\int_0^\infty \int_0^\infty \alpha^{N_1-1}\beta^{M-1}k^{R-1} H d\beta dk}{\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{N_1-1}\beta^{M-1}k^{R-1} H d\alpha d\beta dk} \quad , \beta, k > 0,$$

$$\pi(\beta|t_{i:n}) = \pi^*(\alpha, \beta, k|t_{i:n})d\alpha dk \quad \Rightarrow \quad \pi(\beta|t_{i:n}) = \frac{\int_0^\infty \int_0^\infty \alpha^{N_1-1}\beta^{M-1}k^{R-1} H d\alpha dk}{\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{N_1-1}\beta^{M-1}k^{R-1} H d\alpha d\beta dk} \quad , \alpha, k > 0$$

and

$$\pi(k|t_{i:n}) = \pi^*(\alpha, \beta, k|t_{i:n})d\alpha d\beta \quad \Rightarrow \quad \pi(k|t_{i:n}) = \frac{\int_0^\infty \int_0^\infty \alpha^{N_1-1}\beta^{M-1}k^{R-1} H d\alpha d\beta}{\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{N_1-1}\beta^{M-1}k^{R-1} H d\alpha d\beta dk} \quad , \alpha, \beta > 0\,.$$

The posterior mean of α, β and k, given $t_{i:n}$, are respectively,

$$E[\alpha|t_{i:n}] = \tilde{\alpha} = \int_0^\infty \alpha\pi(\alpha|t_{i:n})d\alpha, \tag{10}$$

$$E[\beta|t_{i:n}] = \tilde{\beta} = \int_0^\infty \beta\pi(\beta|t_{i:n})d\beta \text{ and} \tag{11}$$

$$E[k|t_{i:n}] = \tilde{k} = \int_0^\infty k\pi(k|t_{i:n})dk, \tag{12}$$

The posterior variance of α, β and k, given $t_{i:n}$, are respectively,

$$V[\tilde{\alpha}|t_{i:n}] = \int_0^\infty (\tilde{\alpha} - \alpha)^2 \pi(\alpha|t_{i:n})d\alpha, \tag{13}$$

$$V\left[\tilde{\beta}|t_{i:n}\right] = \int_0^\infty \left(\tilde{\beta} - \beta\right)^2 \pi(\beta|t_{i:n})d\beta \text{ and} \tag{14}$$

$$V\left[\tilde{k}|t_{i:n}\right] = \int_0^\infty \left(\tilde{k} - k\right)^2 \pi(k|t_{i:n})dk, \tag{15}$$

Equations (10) to (15) are very hard to obtain, therefore, we use the MATLAB program to obtain the posterior mean and posterior variance of the parameters α, β and k, see table (a-1).

4. Maximum Likelihood Estimation for Hybrid Censoring under Cumulative Exposure Model

From the cumulative exposure distribution function Eq. (3) and the corresponding probability exposure density function Eq. (4), we obtain the likelihood function of α, β and k based on the Type-II hybrid censored sample as follows:

$$L(\alpha, \beta, k) = \frac{n!}{(n-R)!}\left[\prod_{i=1}^{N_1} k\alpha t_{i:n}^{\alpha-1}(1 + t_{i:n}^{\alpha})^{-(k+1)}\right] \times \left[1 + \left(T_2 - \tau_1 + \tau_1^{\frac{\alpha}{\beta}}\right)^{\beta}\right]^{-k(n-R)}$$

$$\times\left[\prod_{i=N_1+1}^{R} k\beta\left(t_{i:n}-\tau_1+\tau_1^{\frac{\alpha}{\beta}}\right)^{\beta-1}\left[1+\left(t_{i:n}-\tau_1+\tau_1^{\frac{\alpha}{\beta}}\right)^{\beta}\right]^{-(k+1)}\right],$$

$$0<t_{1:n}<\ldots<t_{N_1:n}\le\tau_1<t_{N_1+1:n}<\ldots<t_{R:n}\le T_2. \quad (16)$$

The logarithm of likelihood function £ is given,

$$£=\ln L(\alpha,\beta,k|t_{i:n})$$
$$=\ln\frac{n!}{(n-R)!}+\sum_{i=1}^{N_1}\ln(\alpha k)+\sum_{i=N_1+1}^{R}\ln(\beta k)+(\alpha-1)\sum_{i=1}^{N_1}\ln(t_{i:n})-(k+1)\sum_{i=1}^{N_1}\ln(1+t_{i:n}^{\alpha})$$
$$+(\beta-1)\sum_{i=N_1+1}^{R}\ln\left[t_{i:n}-\tau_1+\tau_1^{\frac{\alpha}{\beta}}\right]-(k+1)\sum_{i=N_1+1}^{R}\ln\left[1+\left(t_{i:n}-\tau_1+\tau_1^{\frac{\alpha}{\beta}}\right)^{\beta}\right]$$
$$-k(n-R)\ln\left[1+\left(T_2-\tau_1+\tau_1^{\frac{\alpha}{\beta}}\right)^{\beta}\right]. \quad (17)$$

Thus, the estimators of α, β and k are obtained by differentiating Eq. (17) with respect to α,β and k, respectively, and equating to zero as follows:

$$\frac{\partial £}{\partial k}=\frac{R}{k}-(n-R)\ln[1+E^{\beta}]-\sum_{i=1}^{N_1}\ln B_i-\sum_{i=1}^{N_1}\ln\left[1+D_i{}^{\beta}\right]\Big|_{\underline{\theta}=\hat{\underline{\theta}}}=0,$$

$$\frac{\partial £}{\partial \alpha}=\frac{N_1}{\alpha}-\frac{k(n-R)QE^{\beta-1}}{[1+E^{\beta}]}+\sum_{i=1}^{N_1}\ln t_{i:n}-(1+k)\sum_{i=1}^{N_1}\frac{P_i}{B_i}+\left(\frac{\beta-1}{\beta}\right)\sum_{i=N_1+1}^{R}\frac{Q}{D_i}$$
$$-(k+1)\sum_{i=N_1+1}^{R}\frac{QD_i{}^{\beta-1}}{\left[1+D_i{}^{\beta}\right]}\Bigg|_{\underline{\theta}=\hat{\underline{\theta}}}=0 \text{ and}$$

$$\frac{\partial £}{\partial \beta}=\frac{M}{\beta}-\left(\frac{k(n-R)E^{\beta}\left[\ln E-\frac{\alpha Q}{\beta E}\right]}{[1+E^{\beta}]}\right)+\sum_{i=N_1+1}^{R}\ln D_i+\frac{\alpha(1-\beta)}{\beta^2}\sum_{i=N_1+1}^{R}\frac{Q}{D_i}$$
$$-(k+1)\sum_{i=N_1+1}^{R}\left(\frac{D_i{}^{\beta}\left[\ln D_i-\frac{\alpha Q}{\beta D_i}\right]}{\left[1+D_i{}^{\beta}\right]}\right)\Bigg|_{\underline{\theta}=\hat{\underline{\theta}}}=0. \quad (18)$$

Where,

$$R=N_1+M,\quad T_2=max\{t_{r:n},\tau_2\},\quad P_i=t_{i:n}^{\alpha}\log t_{i:n},\quad Q=\tau_1^{\frac{\alpha}{\beta}}\log\tau_1,$$
$$B_i=[1+t_{i:n}^{\alpha}],\quad D_i=\left[t_{i:n}-\tau_1+\tau_1^{\frac{\alpha}{\beta}}\right],\quad E=\left[T_2-\tau_1+\tau_1^{\frac{\alpha}{\beta}}\right].$$

Since the closed form solution to nonlinear equations (18) is very hard to obtain, therefore, we use the MATLAB program to solve the previous nonlinear equations simultaneously to obtain $\hat{\alpha},\hat{\beta}$ and $\hat{k}$, see Table (b-1).

The asymptotic variance and covariance matrix of maximum likelihood estimates are given by the elements of the inverse of Fisher information matrix. The Fisher information matrix is given by

$$I_{ij}(\underline{\theta})\cong\left\{-\frac{\partial^2\ln L}{\partial\theta_i\partial\theta_j}\right\}.$$

Thus, The asymptotic variance covariance matrix F^{-1} for the maximum likelihood estimates can be written as follows:

$$F^{-1}=\begin{pmatrix}-\frac{\partial^2\ln L}{\partial k^2} & -\frac{\partial^2\ln L}{\partial k\partial\alpha} & -\frac{\partial^2\ln L}{\partial k\partial\beta}\\ -\frac{\partial^2\ln L}{\partial\alpha\partial k} & -\frac{\partial^2\ln L}{\partial\alpha^2} & -\frac{\partial^2\ln L}{\partial\alpha\partial\beta}\\ -\frac{\partial^2\ln L}{\partial\beta\partial k} & -\frac{\partial^2\ln L}{\partial\beta\partial\alpha} & -\frac{\partial^2\ln L}{\partial\beta^2}\end{pmatrix}^{-1}$$

Consequently, the maximum likelihood estimators, $\hat{\alpha},\hat{\beta}$ and $\hat{k}$, of α,β and k, respectively, have an asymptotic variance covariance matrix defined by inverting the Fisher information matrix, F, see Table (b-2).

5. Illustrative Examples

In this section, we use illustrative examples by using small and moderately large samples are presented to illustrate all the methods of inference described in this work. In these examples, we illustrate two instances of parameters for Burr XII distribution using ML method and Bayesian methods, they are as follows:
Example 1: In this example, we estimate that parameters $(\alpha, \beta, k) = (1.5, 3, 1.25)$ when $n = 15, 35$ by ML and Bayesian methods, respectively. First: using ML estimation method, we consider the following data when $n = 15$ with $\tau_1 = 3$. Therefore, the failure times of units, see table (E-1-1).

Table E-1-1: The failure times of units for Burr XII distribution when $n = 15$ and $(\alpha, \beta, k) = (1.5, 3, 1.25)$

Time Stress Level	Failure Time of Units under Simple Step-Stress						
$0 < t \leq \tau_1$	0.1503	0.2524	0.2577	0.3023	0.3219	0.3660	0.4271
	0.4316	0.5945	0.6970	0.7841	0.8692	1.0505	
$\tau_1 < t \leq \tau_2$	3.2776	3.5486					

In this case, we fixed time $\tau_2 = 7$ and $r = 60\ \%\ n$, we obtain the estimator of $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{k})$, see table (E-1-2).

Table E-1-2: RE, MSE and ARBais of the parameters using ML estimation method

Estimators	$\hat{\theta}$	RE	MSE	ARbias
$\hat{\alpha}$	1.4939	0.2449	0.1346	0.0098
$\hat{\beta}$	3.0159	0.1516	0.2067	0.0376
$\hat{k}$	1.3502	0.2495	0.0973	0.0747

Note that when $r = 9$, we have $T_2 = \max\{t_{9:15}, \tau_2\} = \max\{0.5945, 7\} = 7$, since $N_1 = 13$ and $M = 2$. In this case, the experiment termination is $T_2 = \tau_2 = 7$.

Second: using Bayesian estimation method, we consider the following data when $n = 35$ with $\tau_1 = 3$. Therefore, the failure times of units, see table (E-1-3).

Table E-1-3: The failure times of units for Burr XII distribution when $n = 35$ and $(\alpha, \beta, k) = (1.5, 3, 1.25)$.

Time Stress Level	Failure Time of Units under Simple Step-Stress						
$0 < t \leq \tau_1$	0.0325	0.1346	0.1430	0.3358	0.3435	0.4195	0.4359
	0.4386	0.4462	0.4519	0.5453	0.5910	0.6219	0.6306
	0.6619	0.6724	0.7586	0.8468	1.0059	1.2181	1.2556
	1.2840	1.3486	1.6691	2.1824	2.4442	2.5312	2.8305
	2.9970						
$\tau_1 < t \leq \tau_2$	3.6996	3.8539	4.2055	4.8852	5.0813		
$t > \tau_2$	9.6135						

In this case, we fixed time $\tau_2 = 7$ and $r = 70\ \%\ n$, we obtain the posterior mean and posterior variance, see table (E-1-4).

Table E-1-4: RE, MSE and ARBais of the parameters using Bayesian estimation method.

Bayesian Estimators	Posterior Mean	Posterior Variance	RE	MSE	ARBias
$\hat{\alpha}$	1.7978	0.0614	0.0685	0.0519	0.1732
$\hat{\beta}$	2.8143	0.1997	0.4456	0.1943	0.2640
$\hat{k}$	0.9330	0.0098	0.0769	0.0142	0.1195

Example 2: In this example, we estimate that parameters $(\alpha, \beta, k) = (1, 2.5, 1)$ when $n = 15, 35$ by ML and Bayesian methods, respectively. First: using ML estimation method, we consider the following data when $n = 15$ with $\tau_1 = 3$. Therefore, the failure times of units, see table (E-2-1).

Table E-2-1: The failure times of units for Burr XII distribution when $n = 15$ and $(\alpha, \beta, k) = (1, 2.5, 1)$

Time Stress Level	Failure Time of Units under Simple Step-Stress
$0 < t \leq \tau_1$	0.0850 0.7096
$\tau_1 < t \leq \tau_2$	3.7552 3.8766 4.4883 4.9020 5.5624 6.0746
$\tau_2 < t$	7.6565 8.0378 12.7858 20.0308 31.8456 35.7937 85.8920

In this case, we fixed time $\tau_2 = 7$ and $r = 80\%$, we obtain the estimator of $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{k})$, see table (E-2-2).

Table E-2-2: RE, MSE and ARBais of the parameters using ML estimation method

Estimators	$\hat{\theta}$	RE	MSE	ARbias
$\hat{\alpha}$	1.1048	0.2036	0.0467	0.0893
$\hat{\beta}$	2.3601	0.2578	0.4153	0.1381
$\hat{k}$	0.9904	0.2637	0.1611	0.0651

Note that when $r = 12$, we have $T_2 = \max\{t_{12:15}, \tau_2\} = \max\{20.031, 7\} = 20.031$ since $N_1 = 2$ and $M = 10$. In this example, the experiment termination is $T_2 = t_{12:15} = 20.031$.

Second: using Bayesian estimation method, we consider the following data when $n = 15$ with $\tau_1 = 3$. Therefore, the failure times of units, see table (E-2-3).

Table E-2-3: The failure times of units for Burr XII distribution when $n = 15$ and $(\alpha, \beta, k) = (1, 2.5, 1)$

Time Stress Level	Failure Time of Units under Simple Step-Stress
$0 < t \leq \tau_1$	0.0418 0.2453 0.3517 0.6763 0.9112 0.9938 1.0741 1.2657 2.1347
$\tau_1 < t \leq \tau_2$	4.1987 4.9171 4.5042 6.9595
$t > \tau_2$	11.6936

In this case, we fixed time $\tau_2 = 7$ and $r = 90\ \%\ n$, we obtain the posterior mean and posterior variance, see table E-2-4.

Table E-2-4: RE, MSE and ARBais of the parameters using Bayesian estimation method

Bayesian Estimators	Posterior Mean	Posterior Variance	RE	MSE	ARBias
$\hat{\alpha}$	1.1095	0.0187	0.1847	0.0315	0.1065
$\hat{\beta}$	2.3836	0.2495	0.2061	0.2834	0.0491
$\hat{k}$	0.9170	0.0394	0.1950	0.0371	0.0919

From the above examples, we note that the number of units used in the experiment is increasing; the results are better.

6. Conclusion

In this article, we have considered a cumulative exposure model with two stress levels from Burr-XII distribution. The estimation of the unknown parameters are derived by using two different methods, which are the Bayesian and ML methods when the data are Type II-hybrid. We derive the posterior distribution of the model parameters and the marginal posterior distribution when the observed failure times are Type-II Hybrid censored. Moreover, the posterior mean and posterior variance of the unknown parameter are given. Finally, we present the result of a Monte Carlo simulation study carried out in order to compare the performance of the methods of inference described in section 6. Also note that whatever number of units increases, the MSE is reduced when we estimate the parameters by using ML method.

Conflict of Interest

The authors declare that they have no conflict of interest.

Table a-1: The posterior mean and posterior variance of the parameter $\theta = (\alpha, \beta, k)$ based on hybrid Type II censoring from Burr XII distribution

Values of Parameters		Bayesian estimators	N = 15	25	35	40
(1, 2.5, 1)	Posterior Means	$\hat{\alpha}$	1.163936	1.149332	1.091959	1.089737
		$\hat{\beta}$	2.042249	2.371250	2.398104	2.430229
		$\hat{k}$	1.138893	1.064839	1.041665	1.034454
	Posterior Variances	$\hat{\alpha}$	0.047981	0.045086	0.056537	0.007529
		$\hat{\beta}$	0.581397	0.200872	0.294845	0.219069
		$\hat{k}$	0.150879	0.064995	0.051848	0.035588
(1.5, 3, 1.25)	Posterior Means	$\hat{\alpha}$	1.909896	1.7-27820	1.688896	1.564794
		$\hat{\beta}$	1.725439	2.289788	2.545392	2.651807
		$\hat{k}$	1.417912	1.380676	1.357457	1.282051
	Posterior Variances	$\hat{\alpha}$	0.215151	0.042412	0.057664	0.087274
		$\hat{\beta}$	4.482149	0.096258	0.250936	0.278375
		$\hat{k}$	0.127126	0.107727	0.057826	0.014678

Table b-1: ARBais, MSE and RE of the parameter $\theta = (\alpha, \beta, k)$ based on hybrid Type II censoring from Burr XII distribution

(n, r)	θ	Case 1 (1, 2.5 ,1) $\hat{\theta}$	RE	MSE	RABias	Case 2 (1.5, 3, 1.25) $\hat{\theta}$	RE	MSE	RABias
(15,7)	α	1.0336	0.01501	0.0225	0.0336	1.9968	0.5019	0.5668	0.3312
	β	2.3340	0.1804	0.2035	0.0640	2.7601	0.2514	0.5690	0.0799
	k	1.0153	0.2161	0.0467	0.0156	1.1769	0.2034	0.0646	0.0585
(15,14)	α	1.0032	0.2177	0.0474	0.0032	1.6274	0.2566	0.1482	0.0849
	β	2.1070	0.2802	0.4906	0.1572	2.7095	0.2529	0.5758	0.0968
	k	1.1105	0.2637	0.0696	0.1105	1.0158	0.2442	0.0932	0.1874
(25,10)	α	0.9899	0.1014	0.0103	0.0102	1.6361	0.1916	0.0826	0.0907
	β	2.4362	0.1401	0.1227	0.0255	2.3756	0.2504	0.5644	0.2081
	k	0.9998	0.2422	0.0587	0.0002	2.3756	0.1535	0.0368	0.0356
(25,18)	α	1.0445	0.1593	0.0254	0.0445	1.4902	0.2036	0.0933	0.0066
	β	2.0805	0.2305	0.3321	0.1678	2.7284	0.1606	0.2322	0.0906
	k	1.0308	0.1585	0.0251	0.0308	1.2288	0.1094	0.0187	0.0169
(35,14)	α	1.0760	0.1499	0.0225	0.0760	1.6313	0.2078	0.0972	0.0875
	β	2.2839	0.2122	0.2813	0.0864	2.6634	0.2149	0.4158	0.1122
	k	1.0007	0.1649	0.0272	0.0007	1.1757	0.2221	0.0771	0.0594
(35,31)	α	1.0929	0.2319	0.0538	0.0929	1.5665	0.1672	0.0629	0.0444
	β	2.3910	0.1875	0.2198	0.0436	2.5461	0.2469	0.5484	0.1513
	k	1.0469	0.2223	0.0494	0.0468	1.2579	0.1263	0.0249	0.0063
(40,16)	α	1.1058	0.2151	0.0463	0.1058	1.5225	0.0904	0.0184	0.0149
	β	2.2514	0.1816	0.206	0.0994	2.1616	0.2991	0.8053	0.2795
	k	0.9244	0.1758	0.0309	0.0756	1.3433	0.1462	0.0334	0.0747
(40,36)	α	1.0898	0.2285	0.0522	0.0898	1.4712	0.1392	0.0436	0.0192
	β	2.3996	0.1951	0.2378	0.0402	2.3592	0.2574	0.5962	0.2136
	k	1.0191	0.2386	0.0569	0.0191	1.1914	0.1953	0.0596	0.0469

Table b-2: Asymptotic variance and covariance of estimates based on hybrid Type II censoring data

(n, r)	$\hat{\theta}$	Case 1(1, 2.5,1)			Case 2(1.5,3,1.25)		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$
(15,7)	$\hat{\alpha}$	0.0937	-0.0445	-0.1695	0.1145	-0.0510	-0.1900
	$\hat{\beta}$	-	0.1743	0.1399	-	0.2265	0.1705
	$\hat{k}$	-	-	1.3328	-	-	2.2743
(15,14)	$\hat{\alpha}$	0.1227	-0.0180	-0.1799	0.0952	-0.0428	-0.1908
	$\hat{\beta}$	-	0.0653	0.0457	-	0.1647	0.1417
	$\hat{k}$	-	-	0.8339	-	-	1.6762
(25,10)	$\hat{\alpha}$	0.0621	-0.0118	-0.1179	0.0796	-0.0181	-0.0987
	$\hat{\beta}$	-	0.0396	0.0334	-	0.0835	0.0529
	$\hat{k}$	-	-	0.6711	-	-	0.9131
(25,18)	$\hat{\alpha}$	0.0598	-0.0114	-0.0856	0.0736	-0.0167	-0.1151
	$\hat{\beta}$	-	0.0419	0.0289	-	0.0734	0.0516
	$\hat{k}$	-	-	0.4645	-	-	1.1650
(35,14)	$\hat{\alpha}$	0.0406	-0.0090	-0.0669	0.0495	-0.0169	-0.0830
	$\hat{\beta}$	-	0.0328	0.0246	-	0.0634	0.0506
	$\hat{k}$	-	-	0.4042	-	-	0.9267
(35,31)	$\hat{\alpha}$	0.0438	-0.0090	-0.0720	0.0552	-0.0139	-0.0862
	$\hat{\beta}$	-	0.0320	0.0242	-	0.0554	0.0418
	$\hat{k}$	-	-	0.4436	-	-	0.9163
(40,16)	$\hat{\alpha}$	0.0326	-0.0091	-0.0585	0.0518	-0.0097	-0.0566
	$\hat{\beta}$	-	0.0321	0.0257	-	0.0450	0.0267
	$\hat{k}$	-	-	0.3357	-	-	0.5769
(40,36)	$\hat{\alpha}$	0.0376	-0.0085	-0.0649	0.0445	-0.0121	-0.0640
	$\hat{\beta}$	-	0.0289	0.0229	-	0.0446	0.0325
	$\hat{k}$	-	-	0.3877	-	-	0.5546

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