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Numerical Solution of the Modified Equal Width Wave Equation Using B-Spline Method

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Abstract: In this paper, the modified equal width wave (MEW) equation is solved numerically using the cubic and septic B-spline. The proposed algorithms are based on Crank-Nicolson formulation and finite difference approximation. The non-linear term for cubic B-spline is computed during executing the algorithm at each time level in terms of the previous level while for septic B-spline we tackle the nonlinear term in the equation using a variant of the linearization technique. The stability analysis using Von-Neumann concept shows the schemes are unconditionally stable. To test accuracy the error norms , are computed.

Keywords: Modified equal width wave (MEW) equation, Cubic, Septic, Stability, Soliton, Solitary waves.

AMS Mathematics Subject Classification: 97N40, 34K28, 74G15, 74G15, 74H15, 32W50.

1 Introduction

In this paper we consider the numerical solution of the modified equal width wave (MEW) equation based upon the equal width wave (EW) equation [1, 2]. The MEW equation as in the form [3, 4, 5]

$$u_t + \varepsilon u^2 u_x - \mu u_{xxt} = 0, \tag{1}$$

where, subscripts x and t denote differentiation and ε , μ are positive parameter with boundary conditions $u \to 0$ as $x \to \pm\infty$. The analytic solution of the (MEW) equation can be expressed in the form

$$u(x,t) = \sqrt{6c/\varepsilon}\ \operatorname{sec}h\ (k(x - ct - x_0)), \tag{2}$$

where, $k = \dfrac{1}{\sqrt{\mu}}$ and c (speed of the wave) are positive constants.

This equation is related with the modified regularized long wave (MRLW) equation [6] and modified Korteweg-de Vries (MKdV) equation [7] and with Burger's equation using septic B-spline [8]. The MEW equation was solved numerically by various forms of finite element method [9-14] such as collocation method using quadratic, cubic, quantic, and septic B-splines. Evan and Raslan [5] studied the generalized EW equation by using collocation method based on quadratic B-spline.

In the present work we solve the MEW equation numerically by a collocation method with cubic and septic B-spline. Moreover, interaction of solitary waves and other properties of the MEW equation are also studied.

2 Solution Methods

2.1 Collocation Method using Cubic B-spline

Boundary conditions and the initial condition are of the form

$$u(a,t) = u(b,t) = 0, \qquad u_x(a,t) = u_x(b,t) = 0, \qquad u(x,0) = f(x), \tag{3}$$

where, $f(x)$ is localized disturbance inside the given closed interval and will be chosen later.

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Let us consider $a = x_0 < x_1 < ... < x_N = b,$ as a partition of $[a,b]$ by knots x_i , and $h = x_m - x_{m-1} = \frac{b-a}{N},$ $m = 1,..., N.$

Partition the interval $[a,b]$ into N finite elements of equal length $h = x_m - x_{m-1}$ where the knots x_m are such that $a = x_0 < x_1 < ... < x_N = b.$ Let $\phi_m(x)$ be those cubic B-splines with knots at x_m defined as [10,16]

$$\phi_m(x) = \frac{1}{h^3}\begin{cases} (x - x_{m-2})^3, & x_{m-2} \le x \le x_{m-1} \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3h(x - x_{m-1})^3, & x_{m-1} \le x \le x_m \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3h(x_{m+1} - x)^3, & x_m \le x \le x_{m+1} \\ (x_{m+1} - x)^3 & x_{m+1} \le x \le x_{m+2} \\ 0, & otherwise \end{cases} \tag{4}$$

The splines $\{\phi_{-1},..., \phi_{N+1}\}$ form a basis for functions defined over $[a, b]$. Our aim is to find an approximate solution $U_N(x,t)$ to the solution $u(x,t)$ which can be expressed in terms of cubic spline trial functions of the form:

$$U_N(x,t) = \sum_{m=-1}^{N+1} \delta_m(t)\,\phi_m(x), \tag{5}$$

where, $\delta_m(t)$ are time dependent parameters for $m = -1,0,..., N+1,$ to be determined from the cubic B-spline collocation form of the MEW Eq. (1) together with the boundary conditions (3) and from conditions based on Eq. (5).

The values of $\phi_m(x)$ and its first and second derivatives at the knots points x_m are given in Table 1 as shown below:

Table 1: values of cubic B-spline and its derivativesat the knots points.

x	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
ϕ_m	0	1	4	1	0
ϕ'_m	0	$\frac{3}{h}$	0	$\frac{-3}{h}$	0
ϕ''_m	0	$\frac{6}{h^2}$	$\frac{-12}{h^2}$	$\frac{6}{h^2}$	0

The nodal values U_m, U'_m, U''_m at the knots x_m are obtained from Eqs. (4), (5) in terms of the element parameters δ_m as:

$$\left.\begin{aligned} U_{\mathrm{m}} &= U(x_m,t) = \delta_{m-1} + 4\delta_m + \delta_{m+1} \\ U'_m &= U'(x_m,t) = \frac{3}{h}(\delta_{m+1} - \delta_{m-1}) \\ U''_m &= U''(x_m,t) = \frac{6}{h^2}(\delta_{m-1} - 2\,\delta_m + \delta_{m+1}) \end{aligned}\right\}, \tag{6}$$

where dashes represent differentiation with respect to space variable.

Substituting from Eq. (6) into Eq. (1) gives the set of coupled first ordinary differential equations

$$\begin{aligned} &(\delta'_{m-1} + 4\delta'_m + \delta'_{m+1}) + \frac{3\varepsilon}{h} z_m(\delta_{m+1} - \delta_{m-1}) \\ &\qquad - \frac{6\mu}{h^2}(\delta'_{m-1} - 2\delta'_m + \delta'_{m+1}) = 0, \end{aligned} \tag{7}$$

where, the nonlinear terms are $z_{\mathrm{m}} = (\delta_{m-1} + 4\delta_m + \delta_{m+1})^2.$

Replacing the time derivative of the parameter δ by usual forward finite difference approximation and parameter δ by the Crank-Nicolson formulation

$$\delta'_m = \frac{1}{\Delta t}(\delta_m^{n+1} - \delta_m^n), \qquad \delta_m = \frac{1}{2}(\delta_m^{n+1} + \delta_m^n) \tag{8}$$

where, Δt time step and the superscripts n and $n+1$ are successive time levels.

By using approximations (8) in Eq. (7) and some calculations then we have the following nonlinear recurrence relationship for time parameters between consecutive time n and $n+1$ as:

$$\alpha_m \delta_{m-1}^{n+1} + \beta \delta_m^{n+1} + \gamma_m \delta_{m+1}^{n+1} = \gamma_m \delta_{m-1}^n + \beta \delta_m^n + \alpha_m \delta_{m+1}^n \tag{9}$$

where,

$$\alpha_m = 2h^2 - 3\varepsilon h\ k\ z_m - 12\mu, \quad \beta = 8h^2 + 24\mu, \quad \gamma_m = 2h^2 + 3\varepsilon h\ k\ z_m - 12\mu, \quad m = 0,1,...,N. \tag{10}$$

From the above general scheme as stated in system (9) and using the values of $m = 0,1,..., N,$ the above nonlinear algebraic system (9) is of the $(N+1)$ equations in the $(N+3)$ unknown parameters $\{\delta_{-1}, \delta_0, ...\delta_N, \delta_{N+1}\}$. Thus solving, it to obtain a unique solution we need to two additional constraints which are obtained from the boundary conditions (3) and used to eliminate $\delta_{-1}, \delta_{N+1}$ from (9). Then we have the matrix equation in the simple form as

$$A\delta^{n+1} = B\delta^n, \tag{11}$$

where A and B are tri-diagonal $(N+1)\times(N+1)$ matrices. We can rewrite approximation (5) for the initial condition

$$U_N(x,0) = \sum_{m=-1}^{N+1} \delta_m^0 \phi_m(x), \tag{12}$$

where parameters δ_m^0 will be determined. To determine the parameters $\delta^0 = (\delta_{-1}^0, ..., \delta_{N+1}^0),$ we require the initial numerical approximation $U_N(x,0)$ to satisfy the following conditions:

(a) It must agree with the initial condition $u(x,0)$ at the knots x_m.

(b) The first derivative of the approximate initial condition agrees with those of exact initial conditions at both ends of the range.

These two conditions (a), (b) can be expressed as:

$$\begin{aligned} &(U_x)_N(x_0,0) = u_x(a,0) = 0, \\ &U_N(x_m,0) = u(x_m,0), \qquad m = 0,1,..., N \\ &(U_x)_N(x_N,0) = u_x(b,0) = 0, \end{aligned} \tag{13}$$

Then, we have

$$D\delta^0 = q^0, \tag{14}$$

where, D is the tri-diagonal matrix given by

$$D = \begin{pmatrix} 4 & 2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 2 & 4 \end{pmatrix},$$

and $\delta^0 = [\delta_0^0, \delta_1^0, \delta_2^0, ..., \delta_N^0]^T$, $q^0 = [f(x_0), f(x_1), ..., f(x_N)]^T$. Hence, we can easily determine the initial time parameters δ^0 by solving the above a tri-diagonal system.

2.2 Stability Analysis

The Von-Neumann stability concept will be applied to investigate the stability of the cubic scheme by assuming the nonlinear term as a constant λ where,

$$z_m = (\delta_{m-1} + 4\delta_m + \delta_{m+1})^2 = (6d)^2 = 3d^2 = \lambda. \tag{15}$$

According to the Von-Neumann concept we have

$$\delta_j^n = \xi^n e^{i\varphi jh}, \qquad i = \sqrt{-1}, \tag{16}$$

where φ is the mode number and h is the element size will be determined for a linearization of numerical scheme. At $x = x_j$, system (9) can be written as

$$\alpha_j \delta_{j-1}^{n+1} + \beta \delta_j^{n+1} + \gamma_j \delta_{j+1}^{n+1} = \gamma_j \delta_{j-1}^n + \beta \delta_j^n + \alpha_j \delta_{j+1}^n \tag{17}$$

where

$$\alpha_j = 2h^2 - 3\varepsilon hk\lambda - 12\mu, \qquad \beta = 8h^2 + 24\mu, \qquad \gamma_j = 2h^2 + 3\varepsilon h\ k\ \lambda - 12\mu, \qquad j = 0,1,...,\ N \tag{18}$$

substitute the Fourier mode (16) into the linearized recurrence relationship (17) hence we get

$$g = \frac{A - iB}{A + iB} \tag{19}$$

where, $g = \dfrac{\zeta^{j+1}}{\zeta^j}$ is amplification factor, A and B are as follows

$$\begin{aligned} A &= (4h^2 - 24\mu)\cos\varphi h + 8h^2 + 24\mu, \\ B &= 18h\,\Delta t\,\lambda \sin\varphi\ h. \end{aligned} \tag{20}$$

Therefore, the linearized scheme is marginally stable since $|g| = 1$.

2.3 Numerical Tests and Rresults of MEW Equation

We determine the solution of two and three solitary waves interaction at different time levels. The numerical solutions must preserve the conservation laws during propagation as discussed the in three invariant conditions which correspond to conversation of mass, momentum, and energy [12] respectively

$$\begin{aligned} I_1 &= \int_a^b u(x,t)\,dx \cong h\sum_{m=0}^{N} U_m^n, \\ I_2 &= \int_a^b (u^2(x,t) + \mu u_x^2(x,t))\,dx \cong h\sum_{m=0}^{N} \left\{(U_m^n)^2 + \mu((U_x)_m^n)^2\right\}, \\ I_3 &= \int_a^b u^4(x,t)\,dx \cong h\sum_{m=0}^{N} (U_m^n)^4. \end{aligned} \tag{21}$$

Also, we computed L_2, L_∞ error norms to show how well the numerical schemes models the test problems in terms of accuracy.

2.3.1 The Motion of Single Solitary Wave

For this problem we consider Eq. (1) with the boundary condition $u \to 0$ as $x \to \pm\infty$, and initial condition

$$u(x,0) = A\sec h[k(x - x_0)]. \tag{22}$$

An analytical solution of this problem is given by

$$u(x,t) = A\sec h[k(x - ct - x_0)] \tag{23}$$

which represents the motion of a single solitary wave with amplitude A where the wave velocity $c = A^2/2$ and $k = \sqrt{1/\mu}$. We choose the parameters $c = 1/32$, $k = h = 0.1$ and $x_0 = 30$ to get the conservation quantities and the error norms L_2, L_∞ for cubic B-spline method as shown in Table 2, and the change in I_1, I_2 and I_3 as seen in Table 2, are $2.87\text{x}10^{-4}$, $8\text{x}10^{-5}$ and $6.05\text{x}10^{-6}$ respectively. Also, Fig. 1 shows the computer plot of the interaction of these solitary waves at different time levels.

2.3.2 Interaction of two Solitary Waves

The interaction of two solitary waves having different amplitudes and traveling in the same directions illustrated. We consider the MEW equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

$$u(x,0) = \sum_{i=1}^{2} A_i \operatorname{sech}\,(k(x - x_i)),$$

(24)

where, amplitude $A_i = \sqrt{2c_i}$, $i = 1,2$ and $k = \sqrt{1/\mu}$.

we choose $c_1 = 1/32, c_2 = 1/64$, $x_1 = 15, x_2 = 30$, and $\mu = 1$, $h = k = 0.1$ through the interval $[a,b]$. And the change in I_1, I_2 and I_3 as seen in Table 3, are $3.4\text{x}10^{-4}$, $9.1\text{x}10^{-5}$ and $6.47\text{x}10^{-6}$ respectively. In addition Fig. 2 shows the computer plot of the interaction of these solitary waves at different time levels.

2.3.3 Interaction of Three Solitary Waves

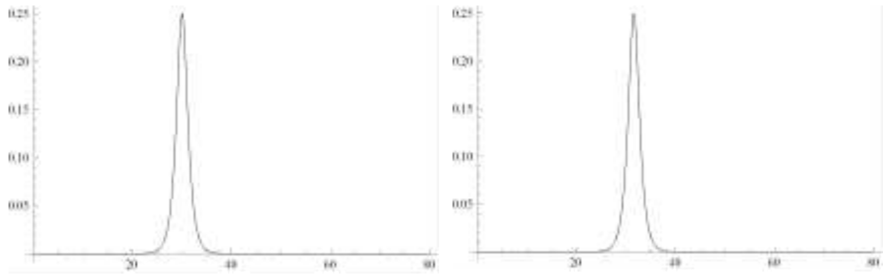
In this subsection interaction of three solitary waves having different amplitudes and traveling in the same directions is studied. We consider the MEW equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes.

$$u(x,0) = \sum_{i=1}^{3} \sqrt{2c_i} \operatorname{sech}\,(k(x - x_i)). \tag{25}$$

we choose $c_1 = 1/32, c_2 = 1/64, c_3 = 1/128$, $h = k = 0.1, \mu = 1$ and $x_1 = 15, x_2 = 30, x_3 = 45$, The change in I_1, I_2 and I_3 as seen in Table 4, are $3.5\text{x}10^{-4}$, $1.11\text{x}10^{-4}$ and $6.64\text{x}10^{-6}$ respectively. Also Fig. 4 shows the computer plot of the interaction of these solitary waves at different time levels.

Table 2: Invariants and error norms for single solitary wave.

t	I_1	I_2	I_3	L_2 -norm	L_∞ -norm
0	**0.785398**	**0.124999**	**0.00520833**	**0.0**	**0.0**
1	**0.785341**	**0.124984**	**0.00520711**	**2.52462×10^{-5}**	**2.13768×10^{-5}**
2	**0.785283**	**0.124968**	**0.0052059**	**5.06155×10^{-5}**	**4.30664×10^{-5}**
3	**0.785226**	**0.124951**	**0.0052046**	**7.61246×10^{-5}**	**6.51776×10^{-5}**
4	**0.785168**	**0.124935**	**0.0052034**	**1.01790×10^{-4}**	**8.74257×10^{-5}**
5	**0.785111**	**0.124919**	**0.00520225**	**1.27629×10^{-4}**	**9.78617×10^{-5}**

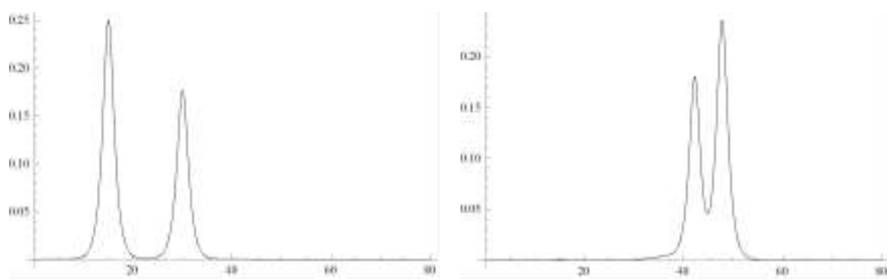


t=0 t=50

Fig. 1: Single solitary wave at different times.

Table 3: Invariant for the interaction of two solitary waves**.**

t	I_1	I_2	I_3
0	**1.34076**	**0.187502**	**0.00651044**
1	**1.34069**	**0.187483**	**0.00650914**
2	**1.34062**	**0.187465**	**0.00650785**
3	**1.34056**	**0.187447**	**0.00650655**
4	**1.34049**	**0.187429**	**0.00650526**
5	**1.34042**	**0.187411**	**0.00650397**

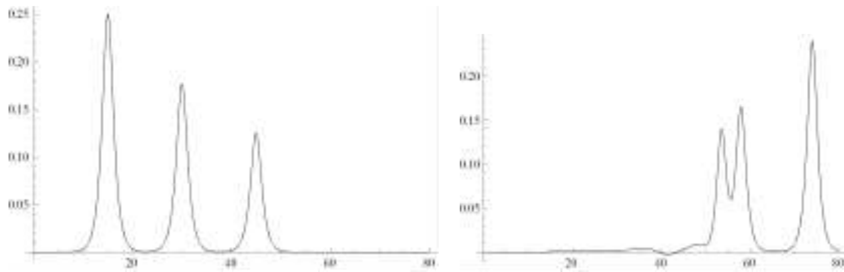


t=0 t=1000

Fig. 2: Two solitary waves at different times.

Table 4: Invariant for the interaction of three solitary waves.

t	I_1	I_2	I_3
0	**1.73346**	**0.218752**	**0.00683596**
1	**1.73339**	**0.218734**	**0.00683466**
2	**1.73332**	**0.218716**	**0.00683336**
3	**1.73325**	**0.218697**	**0.00683207**
4	**1.73318**	**0.218679**	**0.00683077**
5	**1.73311**	**0.218661**	**0.00682947**



t=0 t=1800

Fig.4: Three solitary waves at different times.

3 The Maxwellian Initial Condition

The development of the Maxwellian initial condition

$$u(x,0) = \exp(-(x-7)^2) \tag{26}$$

into a train of solitary waves is examined. We apply it to the problem for different cases:

(I) $\mu = 0.5$, (II) $\mu = 0.1, \mu = 0.05$, (III) $\mu = 0.02$, and (IV) $\mu = 0.005$, when μ is large such as case (I), only single soliton is generated as shown in Fig. 6. However, when μ is reduced more and more as in case (II) two solitons are generated as shown in Fig. 7, and for case (III) three solitons are generated as shown in Fig. 8, for the fourth case (IV), the Maxwellian initial condition has decayed into six stable solitary waves as shown in Fig. 9. The peaks of the well-developed wave lie on a straight line so that their velocities are linearly dependent on their amplitudes and we observe a small oscillating tail appearing behind the wave as shown in the fig. 6, and all states at t =5.

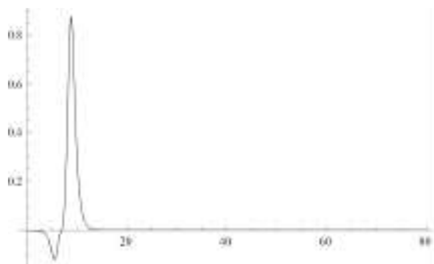
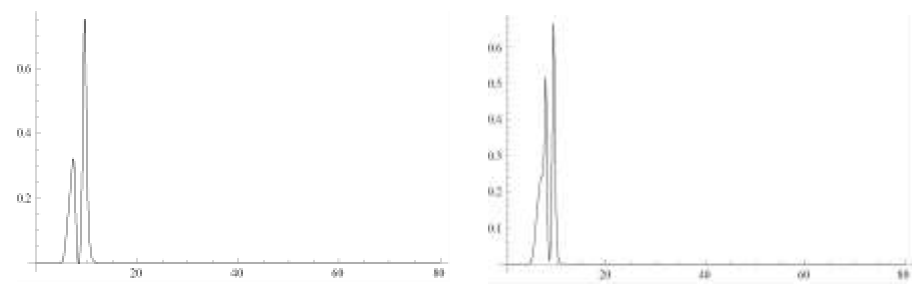


Fig. 6: The Maxwellian initial condition at $\mu = 0.5$ and $t = 5$.



$\mu = 0.1$ $\mu = 0.05$

Fig.7: The Maxwellian initial condition at $t = 5$.

Fig.8: The Maxwellian initial condition at $\mu = 0.02$, and $t = 5$.

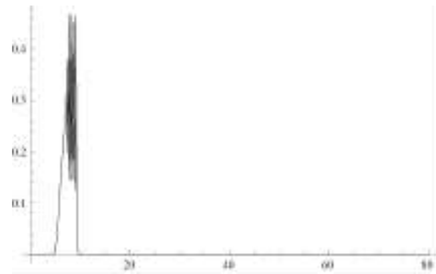


Fig. 9: The Maxwellian initial condition at $\mu = 0.005$, and $t = 5$.

4 Collocation Method using Septic B-spline

The boundary conditions will be chosen from

$$u_x(a,t) = u_x(b,t) = 0,\ u_{xx}(a,t) = u_{xx}(b,t) = 0,\ u_{xxx}(a,t) = u_{xxx}(b,t) = 0, \tag{27}$$

and the initial condition $u(x,0) = f(x)$, where $f(x)$ is localized disturbance inside the given closed interval and will be chosen later. Let $\phi_m(x)$ be those septic B-splines with knots at x_m defined as [11]

$$\phi_m(x) = \frac{1}{h^7}\begin{cases} a_1 = (x - x_{m-4})^7 & , x_{m-4} \le x \le x_{m-3} \\ a_2 = a_1 - 8(x - x_{m-3})^7 & , x_{m-3} \le x \le x_{m-2} \\ a_3 = a_2 + 28(x - x_{m-2})^7 & , x_{m-2} \le x \le x_{m-1} \\ a_4 = a_3 - 56(x - x_{m-1})^7 & , x_{m-1} \le x \le x_m \\ b_4 = b_3 - 56(x_{m+1} - x)^7 & , x_m \le x \le x_{m+1} \\ b_3 = b_2 + 28(x_{m+2} - x)^7 & , x_{m+1} \le x \le x_{m+2} \\ b_2 = b_1 - 8(x_{m+3} - x)^7 & , x_{m+2} \le x \le x_{m+3} \\ b_1 = (x_{m+3} - x)^7 & , x_{m+3} \le x \le x_{m+4} \\ 0 & , otherwise \end{cases} \tag{28}$$

The splines $\{\phi_{-3}, \phi_{-2}, ..., \phi_{N+3}\}$ form a basis for functions defined over $[a,b]$. Our aim is to find an approximate solution $U_N(x,t)$ to the solution $u(x,t)$ which can be expressed in terms of septic spline trial functions of the form:

$$U_N(x,t) = \sum_{m=-3}^{N+3} \delta_m(t)\phi_m(x), \tag{29}$$

where $\delta_m(t)$ are time dependent parameters to be determined from the septic B-spline collocation form of the MEW Eq. (1) together with the boundary conditions (27) and from conditions based on Eq. (29). The values of $\phi_m(x)$ and its first and second derivatives at the knots points x_m are given in Table 5 as shown below:

Table 5 The values of septic B-spline and its derivatives at the knots points

x	x_{m-4}	x_{m-3}	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}	x_{m+4}
ϕ_m	0	1	120	1191	2416	1191	120	1	0
ϕ'_m	0	$\frac{-7}{h}$	$\frac{-392}{h}$	$\frac{-1715}{h}$	0	$\frac{1715}{h}$	$\frac{392}{h}$	$\frac{7}{h}$	0
ϕ''_m	0	$\frac{42}{h^2}$	$\frac{1008}{h^2}$	$\frac{630}{h^2}$	$\frac{-3360}{h^2}$	$\frac{630}{h^2}$	$\frac{1008}{h^2}$	$\frac{42}{h^2}$	0

The nodal values U_m, U'_m, U''_m at the knots x_m are obtained from Eqs. (28) and (29) in terms of the element parameters δ_m as:

$$
\begin{aligned}
U_m &= U(x_m,t) = \delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3},\\
U'_m &= U'(x_m,t) = \frac{7}{h}(-\delta_{m-3} - 56\delta_{m-2} - 243\delta_{m-1} + 243\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3}),\\
U''_m &= U''(x_m,t) = \frac{42}{h^2}(\delta_{m-3} + 24\delta_{m-2} + 15\delta_{m-1} - 80\delta_m + 15\delta_{m+1} + 24\delta_{m+2} + \delta_{m+3}),
\end{aligned} \tag{30}
$$

The MEW Eq. (1) can be written as

$$(u - \mu u_{xx})_t + \varepsilon u^2 u_x = 0\,, \tag{31}$$

The time derivative of Eq. (31) is discretized by a first order accurate forward difference formula and by using the θ-weighted $(0 \le \theta \le 1)$ scheme to the space derivative at time levels to get the equation

$$\frac{(U^{n+1} - \mu U^{n+1}_{xx}) - (U^n - \mu U^n_{xx})}{k} + \theta(\varepsilon U^2 U_x)^{n+1} + (1-\theta)(\varepsilon U^2 U_x)^n = 0, \tag{32}$$

where, $k = \Delta t$ time step and the superscripts n and $n+1$ are successive time levels. In this chapter we take $\theta = 0.5$, and we will tackle the nonlinear term $(U^2 U_x)^{n+1}$ by a variant of linearization form introduced by Rubin and Graves [15]

$$(U^2 U_x)^{n+1} + (U^2)^n (U_x)^n = (U^2)^n (U_x)^{n+1} + 2U^n (U_x)^n U^{n+1} - (U^2)^n (U_x)^n. \tag{33}$$

At the n^{th} time step, we denote U_m, U'_m, U''_m at the knots x_m given in (30) by the following expressions

$$
\begin{aligned}
L_{m1} &= (\delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3})^n\\
L_{m2} &= \frac{7}{h}(-\delta_{m-3} - 56\delta_{m-2} - 243\delta_{m-1} + 243\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3})^n,\\
L_{m3} &= \frac{42}{h^2}(\delta_{m-3} + 24\delta_{m-2} + 15\delta_{m-1} - 80\delta_m + 15\delta_{m+1} + 24\delta_{m+2} + \delta_{m+3})^n.
\end{aligned} \tag{34}
$$

Substituting from Eq. (33) in Eq. (32) and some calculation we get the system which can be rewritten in the simple as nonlinear recurrence relationship for time parameters between consecutive time n and $n+1$ as

$$
\begin{aligned}
&z_1\delta^{n+1}_{m-3} + z_2\delta^{n+1}_{m-2} + z_3\delta^{n+1}_{m-1} + z_4\delta^{n+1}_{m} + z_5\delta^{n+1}_{m+1} + z_6\delta^{n+1}_{m+2} + z_7\delta^{n+1}_{m+3}\\
&= z'_1\delta^{n}_{m-3} + z'_2\delta^{n}_{m-2} + z'_3\delta^{n}_{m-1} + z'_4\delta^{n}_{m} + z'_5\delta^{n}_{m+1} + z'_6\delta^{n}_{m+2} + z'_7\delta^{n}_{m+3}
\end{aligned} \tag{35}
$$

where

$$
\begin{aligned}
z'_1 &= 2h^2 - 7\varepsilon k h L^2_{m1} - 84\mu, & z_1 &= z'_1 + 2\varepsilon k h^2 L_{m1}L_{m2}\\
z'_2 &= 240h^2 - 392\varepsilon k h L^2_{m1} - 2016\mu, & z_2 &= z'_2 + 240\varepsilon k h^2 L_{m1}L_{m2}\\
z'_3 &= 2382h^2 - 1715\varepsilon k h L^2_{m1} - 1260\mu, & z_3 &= z'_3 + 2382\varepsilon k h^2 L_{m1}L_{m2}\\
z'_4 &= 4832h^2 + 6720\mu, & z_4 &= z'_4 + 4832\varepsilon k h^2 L_{m1}L_{m2}\\
z'_5 &= 2382h^2 + 1715\varepsilon k h L^2_{m1} - 1260\mu, & z_5 &= z'_5 + 2382\varepsilon k h^2 L_{m1}L_{m2}\\
z'_6 &= 240h^2 + 392\varepsilon k h L^2_{m1} - 2016\mu, & z_6 &= z'_6 + 240\varepsilon k h^2 L_{m1}L_{m2}\\
z'_7 &= 2h^2 + 7\varepsilon k h L^2_{m1} - 84\mu, & z_7 &= z'_7 + 2\varepsilon k h^2 L_{m1}L_{m2}.
\end{aligned}
$$

From the above general scheme as stated in system (35) and using the values of $m = 0,1,\ldots, N,$ a septa-diagonal matrix is produced containing $N+1$ equations in $N+7$ unknowns $\{\delta_{-3}, \delta_{-2}, \delta_{-1}, \delta_0, \ldots \delta_N, \delta_{N+1}, \delta_{N+2}, \delta_{N+3}\}$. Thus, solving it, to obtain a unique solution we need six additional constraints which are obtained from the boundary conditions (27) and used to eliminate $\delta_{-3}, \delta_{-2}, \delta_{-1}$ and $\delta_{N+1}, \delta_{N+2}, \delta_{N+3}$ from (35). Hence we have the matrix equation in the simple form as

$$A\,\delta^{n+1} = B\,\delta^{n}, \tag{36}$$

where A and B are septa-diagonal $(N+1)\times(N+1)$ matrices and the vector δ^n is given by $\delta^n = [\delta_0^n, \delta_1^n, ..., \delta_N^n]^T$. We can rewrite approximation (29) for the initial condition

$$U_N(x,0) = \sum_{m=-3}^{N+3} \delta_m^0\, \phi_m(x), \tag{37}$$

where parameters δ_m^0 will be determined. To determine the parameters $\delta^0 = (\delta_{-3}^0, \delta_{-2}^0, \delta_{-1}^0, ..., \delta_{N+1}^0, \delta_{N+2}^0, \delta_{N+3}^0)$, we require the initial numerical approximation $U_N(x,0)$ to satisfy the following conditions:

(a) It must agree with the initial condition $u(x,0)$ at the knots x_m.

(b) The first, second and third derivatives of the approximate initial condition agree with those of the exact initial conditions at both ends of the range.

These two conditions (a), (b) can be expressed as:

$$\begin{aligned}
&(U_x)_N(x_0,0) = u_x(a,0) = 0,\\
&(U_{xx})_N(x_0,0) = u_{xx}(a,0) = 0,\\
&(U_{xxx})_N(x_0,0) = u_{xxx}(a,0) = 0,\\
&U_N(x_m,0) = u(x_m,0), \qquad\qquad m = 0,1,..., N\\
&(U_{xxx})_N(x_N,0) = u_{xxx}(b,0) = 0,\\
&(U_{xx})_N(x_N,0) = u_{xx}(b,0) = 0,\\
&(U_x)_N(x_N,0) = u_x(b,0) = 0,
\end{aligned} \tag{38}$$

then we have

$$D\delta^0 = q^0, \tag{39}$$

where D is the septa-diagonal matrix given by

$$D = \begin{pmatrix}
1536 & 2712 & 768 & 24 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{27.577}{27} & \frac{46.793}{18} & \frac{11.644}{9} & \frac{27.577}{27} & 1 & 0 & 0 & 0 & \cdots & 0\\
\frac{3200}{27} & \frac{10.733}{9} & \frac{21.752}{9} & \frac{32.158}{27} & 120 & 1 & 0 & 0 & \cdots & 0\\
1 & 120 & 1191 & 2416 & 120 & 1 & 0 & 0 & \cdots & 0\\
 & & \cdots\cdots & & & & \cdots\cdots & & & \\
 & & \cdots\cdots & & & & \cdots\cdots & & & \\
0 & \cdots & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1\\
0 & \cdots & 0 & 0 & 1 & 120 & \frac{32.158}{27} & \frac{21.752}{9} & \frac{10.733}{9} & \frac{3200}{27}\\
0 & \cdots & 0 & 0 & 0 & 1 & \frac{27.577}{27} & \frac{46.793}{18} & \frac{46.793}{18} & \frac{3200}{27}\\
0 & \cdots & 0 & 0 & 0 & 0 & 24 & 768 & 2712 & 1536
\end{pmatrix}$$

And $\delta^0 = [\delta_0^0, \delta_1^0, \delta_2^0, ..., \delta_N^0]^T$, $q^0 = [f(x_0), f(x_1), ..., f(x_N)]^T$. Hence, we can easily determine the initial time parameters δ^0 by solving the above a septa-diagonal systems.

5 Stability Analyses

We will be applied the stability of the septic scheme by assuming the nonlinear term as a constant λ. This is equivalent to assuming that all the $\left(\delta_j^n\right)$ are equal to a local constant λ, at $x = x_j$, system (35) can be written as

$$\begin{aligned}&\alpha_1\,\delta_{j-3}^{n+1}+\alpha_2\,\delta_{j-2}^{n+1}+\alpha_3\,\delta_{j-1}^{n+1}+\alpha_4\,\delta_{j}^{n+1}+\alpha_5\,\delta_{j+1}^{n+1}+\alpha_6\,\delta_{j+2}^{n+1}+\alpha_7\,\delta_{j+2}^{n+1}=\\&\alpha_7\,\delta_{j-3}^{n}+\alpha_6\,\delta_{j-2}^{n}+\alpha_5\,\delta_{j-1}^{n}+\alpha_4\,\delta_{j}^{n}+\alpha_3\,\delta_{j+1}^{n}+\alpha_2\,\delta_{j+2}^{n}+\alpha_1\,\delta_{j+2}^{n}\end{aligned}\tag{40}$$

where

$$\left.\begin{array}{l}\alpha_1=2h^2-7\phi h\varepsilon\lambda-84\mu,\;\;\alpha_2=240h^2-392\phi h\varepsilon\lambda-2016\mu,\;\;\;\alpha_3=2382h^2-1715\phi h\varepsilon\lambda-1260\mu,\\\alpha_4=4832h^2+6720\mu,\;\;\;\alpha_5=2382h^2+1715\phi h\varepsilon\lambda-1260\mu,\;\;\alpha_6=240h^2+392\phi h\varepsilon\lambda-2016\mu,\\\alpha_7=2h^2+7\phi h\varepsilon\lambda-84\mu,\end{array}\right\},$$

substitute the Fourier mode (16) into the linearized recurrence relationship (40) hence we get

$$g=\frac{A-i\,B}{A+i\,B},\tag{41}$$

where,

$A=(4\,h^2-168\mu)\cos(3\theta)+(480h^2-4032\mu)\cos(2\theta)+(4764h^2-5220\mu)\cos\theta+\alpha,$

$B=14hk\lambda\sin(3\theta)+784hk\lambda\sin(2\theta)+3430\,hk\lambda\sin\theta,\quad \alpha=(4832h^2+6720\mu),\;\;\theta=k\,h,$

Therefore the linearized scheme is marginally stable since $|g|=\sqrt{g\,\bar{g}}=1$

6 Test Problems

The purpose of this section is to examine our numerical method using different test problems concerned with the development, migration and interaction of two and three solitary waves.

6.1 Motion of Single Solitary Wave

For the computational work we put $c=1/32$, $\varepsilon=3$, $\mu=1$, $x_0=30$, $\Delta x=0.1$ and $\Delta t=0.05$ with the interval [0,80]. Initial conditions enable the conservations to be determined analytically as

$$I_1=\pi\sqrt{\frac{6c}{\varepsilon}},\;I_2=\frac{16c}{\varepsilon},\;I_3=\frac{48c^2}{\varepsilon^2}\;.\tag{42}$$

For our treatment, we find the values of conservation laws from the analytical form are $I_1=0.785398$, $I_2=0.166667$, $I_3=0.00520833$. The simulations are done up to $t=5$. The invariants I_1, I_2 and I_3 approach to zero in the computer program for the scheme. Errors, also, at $t=5$ are satisfactorily small L_2-error = 2.5178×10^{-8} and L_∞ -error $=1.46041\times10^{-8}$ for the scheme. Our results are recorded in Table 6 and the motion of solitary wave is plotted at $t=20$ in Fig. 10.

6.2 Interaction of two Solitary Waves

We study the interaction of two positive solitary waves with an amplitude ratio $2:1$. For computation, we have chosen the amplitudes $A_1=1, A_2=0.5,$ and $h=0.1, \Delta t=0.2, x_1=15, x_2=30$. The simulations are done up to $t=55$, and the change in I_1 approach to zero, the change in I_2 and I_3 as seen in Table 7, are $2.09\text{x}10^{-3}$, $2.08\text{x}10^{-3}$ respectively, also Fig. 11 shows the computer plot of the interaction of these solitary waves at different time levels.

6.3 Interaction of Three Solitary Waves

We study the interaction of three positive solitary waves with an amplitude ratio $4:2:1$. We have chosen the amplitudes $A_1=1$, $A_2=0.5$, $A_3=0.25$, and $x_1=15, x_2=30, x_3=45$, $h=0.1$, $\Delta t=0.2$ and the interval $[0,80]$, the three invariants in this case are shown in Table 8. Fig. 11 shows the interaction of these three solitary waves at different times.

Table 6:Invariants and error norms for single solitary waves.

t	I_1	I_2	I_3	L_2 -norm	L_∞ -norm
1	**0.785398**	**0.166667**	**0.00520833**	**5.03592 E-9**	**2.97232 E-9**
2	**0.785398**	**0.166667**	**0.00520833**	**1.00718 E-8**	**5.92458 E-9**
3	**0.785398**	**0.166667**	**0.00520833**	**1.51075 E-8**	**8.84653 E-9**
4	**0.785398**	**0.166667**	**0.00520833**	**2.01433 E-8**	**1.17278 E-8**
5	**0.785398**	**0.166667**	**0.00520833**	**2.51780 E-8**	**1.46041 E-8**
$5[11]_a$	**0.785396**	**0.1666666**	**0.0052083**	**0.00979 E-5**	**0.00622 E-5**
$5[11]_b$	**0.7853966**	**0.1666664**	**0.0052083**	**0.00972 E-5**	**0.00627 E-5**
$5[11]_c$	**0.7854325**	**0.1666908**	**0.0052098**	**2.37333 E-5**	**2.28190 E-5**

where a, b and c refer to different three linearization techniques implemented in [11].

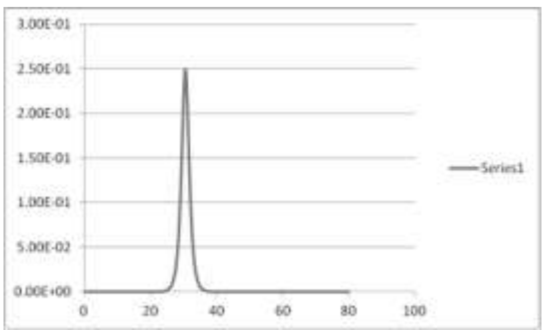
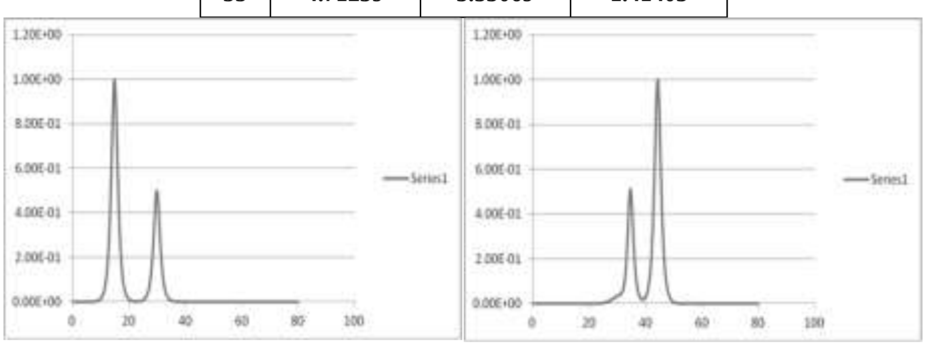


Fig.10: Single solitary wave.

Table 7: Invariant for the interaction of two solitary waves.

t	I_1	I_2	I_3
10	**4.71239**	**3.33278**	**1.41611**
20	**4.71239**	**3.33219**	**1.41552**
30	**4.71239**	**3.33077**	**1.41386**
40	**4.71239**	**3.33094**	**1.41404**
50	**4.71239**	**3.33097**	**1.41431**
55	**4.71239**	**3.33069**	**1.41403**

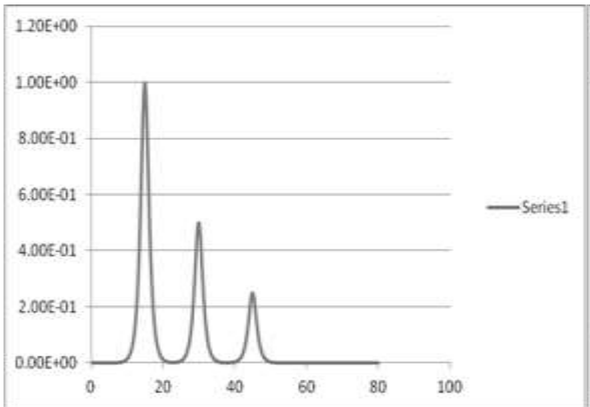


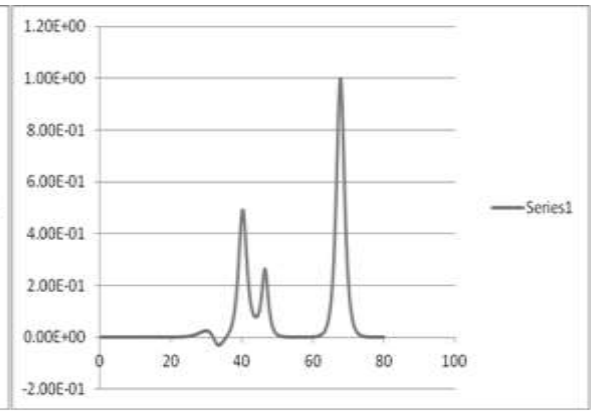
t=0 t=55.2

Fig.11: Interaction of two solitary waves at different times.

Table 8: Invariant for the interaction of three solitary waves.

t	I_1	I_3	I_3
20	**5.49779**	**3.49886**	**1.42073**
40	**5.49779**	**3.49761**	**1.41925**
60	**5.49779**	**3.49633**	**1.41821**
80	**5.49779**	**3.49624**	**1.41811**
100	**5.49777**	**3.49511**	**1.41698**





t=0 t=100

Fig. 11: Interaction of three solitary waves at different times.

7 Comparison between Septic B-spline and Cubic B-spline

Now, a comparison between the collocation method using septic B-spline and the cubic B-spline method is carried out. We find that the numerical solution for the collocation method using septic B-spline provides better accuracy than the cubic B-spline collocation method. These obtained numerical results are illustrated in the Tables 9, 10 given as follows:

Table 9: Comparison for L_2, L_∞-norm with $\Delta t = 0.05, A = 0.25, h = 0.1$ and $x_0 = 30,\ 0 \le x \le 80$.

t	cubic B-spline		Septic B-spline	
	L_2 -norm	L_∞ -norm	L_2 -norm	L_∞ -norm
1	**1.48379E-5**	**1.47435E-5**	**5.03592 E-9**	**2.97232 E-9**
2	**2.97761E-5**	**2.94297 E-5**	**1.00718 E-8**	**5.92458 E-9**
3	**4.4823E-5**	**4.44288 E-5**	**1.51075 E-8**	**8.84653 E-9**
4	**5.9975E-5**	**5.94433 E-5**	**2.01433 E-8**	**1.17278 E-8**
5	**7.52477E-5**	**7.43001 E-5**	**2.51780 E-8**	**1.46041 E-8**

Table 10: Comparison for conservation quantities for motion of single solitary waves with $\Delta t = 0.05, A = 0.25, h = 0.1$ and $x_0 = 30,\ 0 \le x \le 80$.

t	Cubic B-spline			Septic B-spline		
	I_1	I_2	I_3	I_1	I_2	I_3
1	**0.785369**	**0.124992**	**0.00520772**	**0.785398**	**0.166667**	**0.00520833**
2	**0.785341**	**0.124984**	**0.00520711**	**0.785398**	**0.166667**	**0.00520833**
3	**0.785312**	**0.124976**	**0.00520651**	**0.785398**	**0.166667**	**0.00520833**
4	**0.785283**	**0.124968**	**0.0052059**	**0.785398**	**0.166667**	**0.00520833**
5	**0.785255**	**0.124959**	**0.00520529**	**0.785398**	**0.166667**	**0.00520833**

8 Conclusions

In this paper, a numerical solution of the MEW equation based on the cubic and septic B-spline finite element are presented using a variant of the linearization technique. Three test problems are worked out to examine the performance of the algorithm. The performance and accuracy of the method are demonstrated by calculating the error norms L_2 and L_∞ on the motion of a single solitary wave. We obtained small errors for the solitary wave solution and conservation constants have been keeping satisfactorily constant during the computer run.

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