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Fixed Point Theorem In Polish Spaces With Implicit Relations Satisfying Integral Type Inequality

Manish Kumar Mishra *

Department of Mathematics, R.K.G.I..T., Delhi Meerut Road Ghaziabad, India

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Abstract: In this paper we prove fixed point result in generating Polish space (random space which is more general than the other spaces) with implicit relations stratifying integral type inequality. Fixed-point theory is an important branch of non-linear analysis. A point, which is invariant under any transformation, is termed as “Fixed Point” that is for any transformation T on metric space (X, d), x is fixed point of T if T(x) = x.

Keywords: Fixed point, Random space, Polish spaces, Implicit relation.
Mathematics Subject Classification : 47H10, 54H25

Symbols Used

$\neq$ Not equal to,
$\in$ epsilon,
α alpha
ϕ phi
$\subset$ propersubset
ε varepsilon
$>$ grater than
$<$ less than

1 Introduction

Probabilistic functional analysis has emerged as one of the important mathematical disciplines in view of its role in analyzing probabilistic models in the applied sciences. The study of fixed points of random operators forms a central topic in this area. The place City Prague school of probabilistic initiated its study in the 1950s. However, the research in this area flourished after the publication of the survey article of Bharucha-Reid [3] . Since then, many interesting random fixed point results and several applications have appeared in the literature, see, for example the work of Beg and Shahzad [2], Itoh [5], Lin [7], O’Regan [8], Papageorgiou [9], Dhagat et.l. [4], Shahzad and Latif [10], Tan and Yuan [11], Xu [12], Smriti Mehta [13]. The purpose of this paper is to establish fixed point result in generating Polish space (random space which is more general than the other spaces).

2 Materials and Methods

Let $(\Omega,\sum)$ be a measurable space with $\sum$ a sigma algebra of subsets of $\Omega$and Ma non-empty subset of a metric space $X=(X,d)$. Let 2^Mbe the family of all non-empty subsets of M and $C(M)$ the family of all nonempty closed subsets of M. A mapping $G:\Omega\rightarrow 2^M$ is called measurable if, for each open subset U of M, $G^{-1}(U)\in\sum$ where $G^{-1}(U)=\{w\in\Omega : G(w)\cap U\neq\phi\}$. A mapping $\xi:\Omega\rightarrow M$ is called a measurable selector of a measurable mapping $G:\Omega\rightarrow 2^M$ if ξ is measurable and $\xi(w)\in G(w)$ for each $w\in\Omega$. A mapping $T:\Omega\times M\rightarrow X$ is said to be a random operator if, for each fixed $x\in M, T(.,x):\Omega\rightarrow X$ is measurable. A measurable mapping Ω is a random fixed point of a random operator $T:\Omega\times M\rightarrow X$ if $\xi(w)\in T(w,\xi(w))$ for each $w\in\Omega$.

2.1 *Definition*

Let X be non empty set and $\{d_\alpha : \alpha\in(0,1]\}$ be a family of mappings d_α of $(\Omega\times X)\times(\Omega\times X)$ into $R^+, w\in\Omega$ be

* Corresponding author e-mail: mkm2781@gmail.com

a selector. $\{X, d_\alpha : \alpha \in (0,1]\}$ is called generating Polish space of quasi metric family if it satisfies the following conditions:

1.$d_\alpha((w,x),(w,y)) = 0 \forall \alpha \in (0,1] \Leftrightarrow x = y$
2.$d_\alpha((w,x),(w,y)) = d_\alpha((w,y),(w,x)) \forall x,y \in X, w \in \Omega \text{ and } \alpha \in (0,1]$
3.For any $\alpha \in (0,1]$, there exists a number $\mu \in (0,\alpha]$ such that $d_\alpha((w,x),(w,y)) = d_\mu((w,x),(w,z)) + d_\mu((w,z),(w,y)) \forall x,y \in X, w \in \Omega$ *be a selector.*
4.For any $x,y \in X, w \in \Omega, d_\alpha((w,x),(w,y))$is non - increasing and left continuous in α

2.2 Definition

Let $\{X, d_\alpha : \alpha \in (0,1]\}$ be a generating Polish space of quasi metric family and S and T be mappings from $\Omega \times X$ into X. The mapping S and T are said to be quasi compatible if

$$d_\alpha(ST(w,x_n), TS(w,x_n)) \to 0 \text{ as } n \to \infty, \alpha \in (0,1], w \in \Omega$$

whenever $\{w, x_n\}$ be a sequence in $\Omega \times X$ such that

$$\lim_{n\to\infty} S(w,x_n) = \lim_{n\to\infty} T(w,x_n) = p \text{ for some } p \in X.$$

2.3 Definition

Let $\{X, d_\alpha : \alpha \in (0,1]\}$ be a generating Polish space of quasi metric family and S and T be mappings from $\Omega \times X$ into X. The mapping S and T are said to be compatible of type (A) if:

$$d_\alpha(TS(w,x_n), SS(w,x_n)) = 0 \text{ and}$$
$$d_\alpha(ST(w,x_n), TT(w,x_n)) = 0$$

whenever $\{w, x_n\}$ be a sequence in $\Omega \times X$ such that

$$\lim_{n\to\infty} S(w,x_n) = \lim_{n\to\infty} T(w,x_n) = p \text{ for some } p \in X.$$

2.4 Definition

Let $\mathfrak{I}$ be the set of all real functions $\mathfrak{I} : R_+^4 \to R$ such that:
(F_1) : F is continuous in each coordinate variable,
(F_2) : If either $F(u,0,u,v) \le 0$ or $F(u,0,u+v,v) \le 0$ for all $u,v \ge 0$, then there exists a real constant $0 \le h \le 1$such that $v \ge u$.

2.5 Lemma

Let $\{X, d_\alpha : \alpha \in (0,1]\}$ be a generating Polish space of quasi metric family and S and T be mappings from $\Omega \times X$ into X. Suppose that

$$\lim_{n\to\infty} S(w,x_n) = \lim_{n\to\infty} T(w,x_n) = p \text{ for some } p \in X.$$

Then we have the following:
$\lim_{n\to\infty} ST(w,x_n) = Tp$ if T is continuous and
$STp = TSp$ and $Sp = Tp$ if T is continuous

2.6 Lemma

Let $\{X, d_\alpha : \alpha \in (0,1]\}$ be a generating Polish space of quasi metric family and S and T be mappings from $\Omega \times X$ into X. If S and T are compatible of type (A) for $\alpha \in (0,1]$ and for $\mu \in (0,\alpha]$.
Then $STp = TTp = TSp = SSp$.
Let (X,d)be a complete metric space, $\alpha \in [0,1]$, $f : X \to X$ a mapping such that for each

$$x,y \in X,$$

$$\int_0^{d(fx,fy)} \varphi(t)dt \le \alpha \int_0^{\max\{d(x,y),d(x,fx),d(y,fy),\frac{1}{2}[d(x,fy)+d(x,fx)\}} \varphi(t)dt$$

Where $\varphi ; R^+ \to R$ is a lebesgue integrable mapping which is summable, nonnegative and such that, for each $\varepsilon > 0, \int_0^\varepsilon \phi(t)dt > 0$. Then f has a unique common fixed $z \in X$such that for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.
Rhoades(2003), extended this result by replacing the above condition by the following

$$\int_0^{d(fx,fy)} \varphi(t)dt \le \alpha \int_0^{\max\{d(x,y),d(x,fx),d(y,fy),\frac{1}{2}[d(x,fy)+d(x,fx)]} \varphi(t)dt$$

Ojha et al.(2010) Let (X,d) be a metric space and let $f : X \to X$, $F : X \to CB(X)$be a single and a multi-valued map respectively, suppose that f and F are occasionally weakly commutative (OWC) and satisfy the inequality

$$\int_0^{J^P(Fx,Fy)} \phi(t)dt \le \int_0^{\max\left\{\begin{array}{l} ad(fx,fy)d^{P-1}(fx,Fx), ad(fx,fy)d^{P-1}(fy,Fy), \\ ad(fx,Fx)d^{P-1}(fy,Fy), cd^{P-1}(fx,Fy)d(fy,Fx) \end{array}\right\}} \phi(t)dt$$

for all x,y in X,where $p \ge 2$ is an integer $a \ge 0$ and $0 < c < 1$ then f and F have unique common fixed point in X.

3 Results and Discussions

3.1 Theorem

Let $\{X, d_\alpha : \alpha \in (0,1]\}$ be a generating Polish space of quasi metric family and $S,T$and $G$are mappings from $\Omega \times X \to X$are continuous random operator w.r.t. d. Suppose there is some $\alpha \in (0,1)$such that for $x,y \in X$ and $w \in \Omega$, we have the following conditions (3.1.1)
$S(X) \subseteq G(X) \text{ and } T(X) \subseteq G(X)$

(3.1.2)$\int_0^{\phi\{d_\alpha(S(w,x),T(w,y)),d_\alpha(S(w,x),G(w,y)),d_\alpha(G(w,x),T(w,y)),d_\alpha(G(w,x),G(w,y))\}} \phi(t)dt \le 0 \ \forall x,y \in X, \text{and } \alpha \in (0,1]$
(3.1.3) G is continuous
(3.1.4) The pairs $\{S,G\}$and $\{T,G\}$ are quasi compatible on X.
Then $S,T$and G have common fixed point.

Proof:
Let $x_0$be any point of X
Since $S(X) \subseteq G(X) \text{ and } T(X) \subseteq G(X)$ and $SG(X) \subseteq GG(X) \text{ and } TG(X) \subseteq GG(X)$
So there exists $x_1$and $x_2$in X such that
$GG(w,x_1) = SG(w,x_0)$and $GG(w,x_2) = TG(w,x_1)$
In general

$GG(w,x_{2n+1}) = SG(w,x_{2n})$ and $GG(w,x_{2n+2}) = TG(w,x_{2n+1})$
for $n = 0,1,2,\ldots\ldots\ldots\ldots$
Let $d_n = d_\alpha(GG(w,x_n) = GG(w,x_{n+1}))$
Also we know
$\int_0^{d_\alpha(GG(w,x_{2n}),GG(w,x_{2n+2}))}\phi(t)dt \leq \int_0^{d_\mu(GG(w,x_{2n}),GG(w,x_{2n+1}))}\phi(t)dt + \int_0^{d_\mu(GG(w,x_{2n+1}),GG(w,x_{2n+2}))}\phi(t)dt \forall\, \alpha \in (0,1]$ and $\mu \in (0,\alpha]$.
Suppose x_{2n},x_{2n+1} satisfy ((3.1.2), then $\forall\, \alpha \in (0,1)$

$$\int_0^{\phi\{d_\alpha(SG(w,x_{2n}),TG(w,x_{2n+1})),d_\alpha(SG(w,x_{2n}),GG(w,x_{2n+1})),d_\alpha(GG(w,x_{2n}),TG(w,x_{2n+1})),d_\alpha(GG(w,x_{2n}),GG(w,x_{2n+1}))\}}\phi(t)dt \leq 0$$
$$\int_0^{\phi\{d_\alpha(GG(w,x_{2n+1}),GG(w,x_{2n+2})),d_\alpha(GG(w,x_{2n+1}),GG(w,x_{2n+1})),d_\alpha(GG(w,x_{2n}),GG(w,x_{2n+2})),d_\alpha(GG(w,x_{2n}),GG(w,x_{2n+1}))\}}\phi(t)dt \leq 0$$
$$\int_0^{\phi\left\{d_\alpha(GG(w,x_{2n+1}),GG(w,x_{2n+2})),0,[d_\mu(GG(w,x_{2n}),GG(w,x_{2n+1})),+d_\mu(GG(w,x_{2n+1}),GG(w,x_{2n+2}))],d_\alpha(GG(w,x_{2n}),GG(w,x_{2n+1}))\right\}}\phi(t)dt \leq 0$$

Thus from definition of implicit relation 2.4, we have
$\int_0^{\phi\{d_\alpha(GG(w,x_{2n+1}),GG(w,x_{2n+2}))\}}\phi(t)dt \leq \int_0^{h\left\{d_\mu(GG(w,x_{2n}),GG(w,x_{2n+1}))\right\}}\phi(t)dt$
$\int_0^{d_{2n+1}}\phi(t)dt \leq \int_0^{h\,d_{2n}}\phi(t)dt$
$\int_0^{d_{2n+1}}\phi(t)dt \leq \int_0^{d_{2n}}\phi(t)dt$
$Similarly\ \int_0^{d_{2n}}\phi(t)dt \leq \int_0^{h\,d_{2n-1}}\phi(t)dt$
Thus $\{d_{2n}\}$be monotone decreasing and hence converge to zero.
Therefore $\{GG(w,x_{2n})\}$is a Cauchy sequence and converge to G_p and hence to point X.
Since $\{SG(w,x_{2n})\}$and $\{TG(w,x_{2n})\}$are subsequence of $\{GG(w,x_{2n})\}$and so converge to same point p. Now by lemma 2.5 we obtain
$SGp = GSp$ and $Sp = Gp$
Similarly $TGp = GTp$ and $Gp = Tp$
Hence $Sp = Gp = Tp$.
Also $Sp = p = Gp = Tp$as $Gp = p$.
Hence p is common fixed point of $S,T$and G .
This completes the proof.

3.2 Theorem

Let $\{X,d_\alpha : \alpha \in (0,1]\}$ be a generating Polish space of quasi metric family and $S,T$and G be mappings from $\Omega \times X \to X$satisfying
we have the following conditions

$$4.2.1) S(X) \subseteq G(X)\ and\, T(X) \subseteq G(X)$$

3.2.2) $\int_0^{\phi\{d_\alpha(S(w,x),T(w,y)),d_\alpha(S(w,x),G(w,y)),d_\alpha(G(w,x),T(w,y)),d_\alpha(G(w,x),G(w,y))\}}\phi(t)dt \leq 0\ \forall x,y \in X, and\ \alpha \in (0,1]$
3.2.3) G is continuous
3.2.4) The pairs $\{S,G\}$and $\{T,G\}$ are quasi compatible of type(A).
Then $S,T$and G have common fixed point.

Proof: Similar to the proof of the theorem 4.1 by using lemma 2.6.

3.3 Corollary

Let $\{X,d_\alpha : \alpha \in (0,1]\}$ be a generating Polish space of quasi metric family and $S,T$and $G$are mappings from $\Omega \times X \to X$are continuous random operator w.r.t. d. Suppose there is some $\alpha \in (0,1)$such that for $x,y \in X$ and $w \in \Omega$,
we have the following conditions

$$3.3.1) S(X) \subseteq G(X)\ and\, T(X) \subseteq G(X)$$

3.3.2) $\int_0^{\phi\{d_\alpha(S(w,x),T(w,y)),d_\alpha(S(w,x),G(w,y)),d_\alpha(G(w,y),T(w,y)),d_\alpha(G(w,x),G(w,y))\}}\phi(t)dt \leq 0\ \forall x,y \in X, and\ \alpha \in (0,1]$

3.3.3) G is continuous
3.3.4) The pairs $\{S,G\}$and $\{T,G\}$ are quasi compatible on X.
Then $S,T$and G have common fixed point.
Proof :
Let $x_0$be any point of X
Since $S(X) \subseteq G(X)\ and\, T(X) \subseteq G(X)$ and $SG(X) \subseteq GG(X)\ and\, TG(X) \subseteq GG(X)$
So there exists $x_1$and $x_2$in X such that
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In general
$GG(w,x_{2n+1}) = SG(w,x_{2n})$and $GG(w,x_{2n+2}) = TG(w,x_{2n+1})$
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Let $d_n = d_\alpha(GG(w,x_n) = GG(w,x_{n+1}))$
Also we know
$\int_0^{d_\alpha(GG(w,x_{2n}),GG(w,x_{2n+2}))}\phi(t)dt \leq \int_0^{d_\mu(GG(w,x_{2n}),GG(w,x_{2n+1}))}\phi(t)dt + \int_0^{d_\mu(GG(w,x_{2n+1}),GG(w,x_{2n+2}))}\phi(t)dt \forall\, \alpha \in (0,1]$ and $\mu \in (0,\alpha]$.
Suppose x_{2n},x_{2n+1} satisfy (3.1.2), then $\forall\, \alpha \in (0,1]$

$$\int_0^{\phi\{d_\alpha(SG(w,x_{2n}),TG(w,x_{2n+1})),d_\alpha(SG(w,x_{2n}),GG(w,x_{2n+1})),d_\alpha(GG(w,x_{2n}),TG(w,x_{2n+1})),d_\alpha(GG(w,x_{2n}),GG(w,x_{2n+1}))\}}\phi(t)dt \leq 0$$

$$\int_0^{\phi\{d_\alpha(GG(w,x_{2n+1}),GG(w,x_{2n+2})),d_\alpha(GG(w,x_{2n+1}),GG(w,x_{2n+1})),d_\alpha(GG(w,x_{2n}),GG(w,x_{2n+2})),d_\alpha(GG(w,x_{2n}),GG(w,x_{2n+1}))\}}\phi(t)dt \leq 0$$

$$\int_0^{\phi\left\{d_\alpha(GG(w,x_{2n+1}),GG(w,x_{2n+2})),0,[d_\mu(GG(w,x_{2n}),GG(w,x_{2n+1})),+d_\mu(GG(w,x_{2n+1}),GG(w,x_{2n+2}))],d_\alpha(GG(w,x_{2n}),GG(w,x_{2n+1}))\right\}}\phi(t)dt \leq 0$$

Thus from definition of implicit relation 2.4, we have

$\int_0^{\phi\{d_\alpha(GG(w,x_{2n+1}),GG(w,x_{2n+2}))\}}\phi(t)dt \leq \int_0^{h\left\{d_\mu(GG(w,x_{2n}),GG(w,x_{2n+1}))\right\}}\phi(t)dt$
$\int_0^{d_{2n+1}}\phi(t)dt \leq \int_0^{h\,d_{2n}}\phi(t)dt$
$\int_0^{d_{2n+1}}\phi(t)dt \leq \int_0^{d_{2n}}\phi(t)dt$
$Similarly\ \int_0^{d_{2n}}\phi(t)dt \leq \int_0^{h\,d_{2n-1}}\phi(t)dt$

Thus $\{d_{2n}\}$be monotone decreasing and hence converge to zero.
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Hence p is common fixed point of $S,T$and G .
This completes the proof.

3.4 Corollary

Let $\{X, d_\alpha : \alpha \in (0,1]\}$ be a generating Polish space of quasi metric family and $S,T$and $G$are mappings from $\Omega \times X \to X$are continuous random operator w.r.t. d. Suppose there is some $\alpha \in (0,1)$such that for $x, y \in X$ and $w \in \Omega$,
we have the following conditions

$$3.3.1) S(X) \subseteq G(X) \ and\, T(X) \subseteq G(X)$$

3.3.2) $\int_0^{\phi\{d_\alpha(S(w,x),T(w,y)),d_\alpha(S(w,x),G(w,y)),d_\alpha(G(w,y),T(w,y)),d_\alpha(G(w,x),G(w,y))\}} \phi(t)dt \leq 0 \, \forall x,y \in X, and\ \alpha \in (0,1]$

3.3.3) G is continuous
3.3.4) The pairs $\{S,G\}$and $\{T,G\}$ are quasi compatible of type(A).
Then $S,T$and G have common fixed point.

Proof : Similar to the proof of the corollary 4.3 by using lemma 2.6.

4 Conclusion

We establish fixed point result in generating Polish space (random space which is more general than the other spaces) with Implicit Relations Satisfying Integral Type Inequality.

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Manish Kumar Mishra is an Assistant Professor of Department of Applied Science (Mathematics) at Raj Kumar Goel institute of Technology. He published more than 18 international papers. He is referee, Editorial Board member and member of several international journals in the frame of pure and applied mathematics. His main research interests are: fixed point theorem, computer.