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Bézier type surfaces

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Abstract: In this paper with the help of the fundamental polynomials, from general operators, we construct Bézier-type and GBS Bézier-type surfaces, which correspond to the given control points.

Keywords: Linear positive operators, bivariate operators, GBS operators,Bézier-type and GBS Bézier-type surfaces

1. Introduction

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In this section we recall some notions which we will use in this paper.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the function sets: $B(I) = \{f | f : I \to \mathbb{R}, f \text{ bounded on } I\}$, $C(I) = \{f | f : I \to \mathbb{R}, f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$. For any $x \in I$, let the functions $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$, for any $t \in I$ and $e_0 : I \to \mathbb{R}$, $e_0(x) = 1$ for any $x \in I$.

If $I \subset \mathbb{R}$ is a given interval and $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega(f;\cdot) : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega(f;\delta) = \sup\left\{|f(x') - f(x'')| : x',\, x' \in I,\, |x' - x''| \leq \delta\right\}. \tag{1}$$

Let $I_1, I_2, J_1, J_2 \subset \mathbb{R}$ be intervals, $E(I_1 \times I_2)$, $F(J_1 \times J_2)$ which are subsets of the set of real functions defined on $I_1 \times I_2$, respectively $J_1 \times J_2$ and $L : E(I_1 \times I_2) \to F(J_1 \times J_2)$ be a linear positive operator. The operator $UL : E(I_1 \times I_2) \to F((I_1 \cap J_1) \times (I_2 \cap J_2))$ defined for any function $f \in E(I_1 \times I_2)$, any $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ by

$$(ULf)(x,y) = L\left(f(x,*) + f(\cdot,y) - f(\cdot,*)\right)(x,y) \tag{2}$$

is called GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and second variable (see [2] or [7]).

If $f \in E(I_1 \times I_2)$ and $(x,y) \in I_1 \times I_2$, let the functions $f_x = f(x,*)$, $f^y = f(\cdot,y) : I_1 \times I_2 \to \mathbb{R}$, $f_x(s,t) = f(x,t)$, $f^y(s,t) = f(s,y)$ for any $(s,t) \in I_1 \times I_2$. Then, we can consider that f_x, f^y are functions of real variable, $f_x : I_2 \to \mathbb{R}$, $f_x(t) = f(x,t)$ for any $t \in I_2$ and $f^y : I_1 \to \mathbb{R}$, $f^y(s) = f^y(s,y)$ for any $s \in I_1$.

Let $I_1, I_2 \subset \mathbb{R}$ be given intervals and $f : I_1 \times I_2 \to \mathbb{R}$ be a bounded function. The function $\omega_{total}(f;\cdot,*) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined for any $(\delta_1,\delta_2) \in [0,\infty) \times [0,\infty)$ by

$$\omega_{total}(f;\delta_1,\delta_2) = \sup\left\{|f(x,y) - f(x',y')| : (x,y),(x',y') \in I_1 \times I_2, |x - x'| \leq \delta_1, |y - y'| \leq \delta_2\right\} \tag{3}$$

is called the first order modulus of smoothness of function f or total modulus of continuity of function f (see [2] or [7]).

The first order modulus of smoothness for bivariate functions has properties similar to the properties of the first modulus of smoothness for univariate functions.

If $(L_m)_{m\geq 1}$ is a sequence of operators, $L_m : E(I) \to F(J)$, $m \in \mathbb{N}$, for $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$ define $T_{m,i}$ by

$$(T_{m,i}L_m)(x) = m^i \left(L_m \psi_x^i\right)(x) \tag{4}$$

for any $x \in I \cap J$, where $E(I)$, $F(J)$ are subsets of the set of real functions defined on I, respectively J.

In application, we use the fundamental polynomials from Bernstein and Bleimann-Butzer-Hahn operators.

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For $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$, the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right) \tag{5}$$

where $p_{m,k}(x)$ are Bernstein polynomials defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k} \tag{6}$$

for any $x \in [0,1]$, and any $k \in \{0,1,...,m\}$ (see [3] or [8]).

In 1980, G.Bleiman, P.L.Butzer and L.Hahn introduced in [4] a sequence of linear positive operators $(L_m)_{m\geq 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty)$ by

$$(L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right), \tag{7}$$

for any $x \in [0,\infty)$, and any $m \in \mathbb{N}$, where $C_B([0,\infty)) = \{f | f : [0,\infty) \to \mathbb{R}$, f bounded and continuous on $[0,\infty)\}$. This class of operators has been intensively studied obtaining various generalizations. One of the most recent approaches aimed at q-Calculus (see [1]).

2. Preliminaries

For the following constructions and the results as well, see [7].

In this section let $p_m = m$ for any $m \in \mathbb{N}$ or $p_m = \infty$ for any $m \in \mathbb{N}$ and similarly is defined q_n, $n \in \mathbb{N}$.

Let $I_1, I_2, J_1, J_2 \subset \mathbb{R}$ be intervals with $I_1 \cap J_1 \neq \emptyset$ and $I_2 \cap J_2 \neq \emptyset$. For $m,n \in \mathbb{N}$ and $k \in \{0,1,\dots,p_m\} \cap \mathbb{N}_0$, $j \in \{0,1,\dots,q_n\} \cap \mathbb{N}_0$, we consider $\varphi_{m,k} : J_1 \to \mathbb{R}$, $\varphi_{m,k}(x) \geq 0$ for any $x \in J_1$, $\psi_{n,j} : J_2 \to \mathbb{R}$, $\psi_{n,j}(y) \geq 0$ for any $y \in J_2$ and the linear positive functionals $A_{m,k} : E_1(I_1) \to \mathbb{R}$, $B_{n,j} : E_2(I_2) \to \mathbb{R}$.

For $m,n \in \mathbb{N}$ define the sequences of operators $(L_m)_{m\geq 1}$ and $(K_n)_{n\geq 1}$ by

$$(L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) A_{m,k}(f), \tag{8}$$

$$(K_n g)(y) = \sum_{j=0}^{q_n} \psi_{n,j}(y) B_{n,j}(g) \tag{9}$$

for any $f \in E_1(I_1)$, $g \in E_2(I_2)$, $x \in J_1$ and $y \in J_2$, where $E_1(I_1)$, $E_2(I_2)$ are subsets of the set of real functions defined on I_1, respectively I_2.

In the following let $s \in \mathbb{N}_0$, s even. We suppose that the operators $(L_m)_{m\geq 1}$, $(K_n)_{n\geq 1}$ verify the conditions: there exist the smallest $\alpha_j, \beta_j \in [0,\infty)$, $j \in \{0,2,4,\dots,s+2\}$, such that

$$\lim_{m\to\infty} \frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} = a_j(x) \tag{10}$$

for any $x \in I_1 \cap J_1$,

$$\lim_{n\to\infty} \frac{(T_{n,j} K_n)(y)}{n^{\beta_j}} = b_j(y) \tag{11}$$

for any $y \in I_2 \cap J_2$ and if we note

$$\gamma_s = \max\left\{\alpha_{s-2l+\beta_{2l}} : l \in \left\{0,1,\dots,\frac{s}{2}\right\}\right\}, \tag{12}$$

then

$$\begin{cases} \alpha_{s-2l} + \beta_{2l+2} - \gamma_s - 2 < 0 \\ \alpha_{s-2l+2} + \beta_{2l} - \gamma_s - 2 < 0 \\ \alpha_{s-2l+2} + \beta_{2l+2} - \gamma_s - 4 < 0 \end{cases} \tag{13}$$

where $l \in \left\{0,1,2,\dots\frac{s}{2}\right\}$.

In the following we consider the set $E(I_1 \times I_2) = \{f | f : I_1 \times I_2 \to \mathbb{R}$, $f_x \in E_2(I_2)$ for any $x \in I_1$ and $f^y \in E_1(I_1)$ for any $y \in I_2\}$.

For $m,n \in \mathbb{N}$, let the linear positive functionals $A_{m,n,k,j} : E(I_1 \times I_2) \to \mathbb{R}$ with the property

$$A_{m,n,k,j}\left((\cdot - x)^i (* - y)^l\right) = A_{m,k}\left((\cdot - x)^i\right) B_{n,j}\left((* - y)^l\right) \tag{14}$$

for any $k \in \{0,1,\dots,p_m\} \cap \mathbb{N}_0$, $j \in \{0,1,\dots,q_n\} \cap \mathbb{N}_0$, $i,l \in \{0,1,\dots,s\}$ and $x \in I_1$, $y \in I_2$.

Let $m,n \in \mathbb{N}$. The operator $L^*_{m,n}$ defined for any function $f \in E(I_1 \times I_2)$ and any $(x,y) \in J_1 \times J_2$ by

$$\left(L^*_{m,n} f\right)(x,y) = \sum_{k=0}^{p_m} \sum_{j=0}^{q_n} \varphi_{m,k}(x) \psi_{n,j}(y) A_{m,n,k,j}(f) \tag{15}$$

is named the bivariate operator of LK-type.

In the following we consider that

$$(T_{m,0} L_m)(x) = A_{m,0}(e_0) = 1 \tag{16}$$

for any $x \in I_1 \cap J_1$, $m \in \mathbb{N}$ and

$$(T_{n,0} K_n)(y) = B_{n,0}(e_0) = 1 \tag{17}$$

for any $y \in I_2 \cap J_2$, $n \in \mathbb{N}$.

From (16), (17) it results immediately that

$$\sum_{k=0}^{p_m} \varphi_{m,k}(x) = 1 \tag{18}$$

for any $x \in I_1 \cap J_1$, $m \in \mathbb{N}$,

$$\sum_{j=0}^{q_n} \psi_{n,j}(y) = 1 \tag{19}$$

for any $y \in I_2 \cap J_2$, $n \in \mathbb{N}$ and $\alpha_0 = \beta_0 = 0$.

In the following, in addition we suppose that

$$\alpha_{s+2} < \alpha_s + 2,\ \beta_{s+2} < \beta_s + 2 \tag{20}$$

and for any $f \in E(I_1 \times I_2)$ we have

$$A_{m,n,k,j}(f_x) = B_{n,j}(f_x), \tag{21}$$

$$A_{m,n,k,j}(f^y) = A_{m,k}(f^y), \tag{22}$$

$$A_{m,n,k,j}(f) = A_{m,k}(B_{n,j}(f_x)) = B_{n,j}(A_{m,k}(f^y)) \tag{23}$$

for any $x \in I_1$, $y \in I_2$, $k \in \{0,1,\ldots,p_m\} \cap \mathbb{N}_0$, $j \in \{0,1,\ldots,q_n\} \cap \mathbb{N}_0$; $m,n \in \mathbb{N}$.

Now, let $\left(UL^*_{m,m}\right)_{m,n\geq 1}$ be the GBS operators associated to the $\left(L^*_{m,n}\right)_{m,n\geq 1}$ operators. If $m,n \in \mathbb{N}$, then $UL^*_{m,n}$ have the form

$$\left(UL^*_{m,n}f\right)(x,y) = (K_n f_x)(y) + (L_m f^y)(x) - \left(L^*_{m,n}f\right)(x,y) \tag{24}$$

for any $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$, any $f \in E(I_1 \times I_2)$. Now, we recall two results from [7], which are obtained for $s = 0$ and which we will use in this paper.

Theorem 1.*Let $f : I_1 \times I_2 \to \mathbb{R}$ be a bivariate function.*

If $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ and f is continuous in (x,y), then

$$\lim_{m\to\infty} \left(L^*_{m,m}f\right)(x,y) = f(x,y) \tag{25}$$

and

$$\lim_{m\to\infty} \left(UL^*_{m,m}f\right)(x,y) = f(x,y). \tag{26}$$

Assume that f is continuous on $(I_1 \cap J_1) \times (I_2 \cap J_2)$ and there exist the intervals $K_1 \subset I_1 \cap J_1$, $K_2 \subset I_2 \cap J_2$ such that there exist $m(0) \in \mathbb{N}$ and $a_2, b_2 \in \mathbb{R}$ depending on K_1, respectively K_2 so that for any $m \in \mathbb{N}$, $m \geq m(0)$ and any $x \in K_1$, $y \in K_2$, we have

$$\frac{(T_{m,2}L_m)(x)}{m^{\alpha_2}} \leq a_2 \tag{27}$$

and

$$\frac{(T_{m,2}K_m)(y)}{m^{\beta_2}} \leq b_2. \tag{28}$$

Then the convergence given in (25) and (26) are uniform on $K_1 \times K_2$ and

$$\left|\left(L^*_{m,m}f\right)(x,y) - f(x,y)\right| \leq (1+a_2)(1+b_2)\omega_{total}\left(f; \frac{1}{\sqrt{m^{\delta_0}}}, \frac{1}{\sqrt{m^{\delta_0}}}\right) \tag{29}$$

and

$$\begin{aligned}&\left|\left(UL^*_{m,m}f\right)(x,y) - f(x,y)\right| \leq \\ &\leq (1+b_2)\omega\left(f_x; \frac{1}{\sqrt{m^{2-\beta_2}}}\right) + (1+a_2)\omega\left(f^y; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \\ &+ (1+a_2)(1+b_2)\omega_{total}\left(f; \frac{1}{\sqrt{m^{\delta_0}}}, \frac{1}{\sqrt{m^{\delta_0}}}\right) \leq \\ &(1+b_2)\omega\left(f_x; \frac{1}{\sqrt{m^{\delta_0}}}\right) + (1+a_2)\omega\left(f^y; \frac{1}{\sqrt{m^{\delta_0}}}\right) + \\ &(1+a_2)(1+b_2)\omega_{total}\left(f; \frac{1}{\sqrt{m^{\delta_0}}}, \frac{1}{\sqrt{m^{\delta_0}}}\right)\end{aligned} \tag{30}$$

for any $(x,y) \in K_1 \times K_2$ and any $m \in \mathbb{N}$, $m \geq m(0)$, where

$$\delta_0 = -\max\left\{\beta_2 - 2, \alpha_2 - 2, \frac{1}{2}(\alpha_2 + \beta_2 - 4)\right\}.$$

3. Bézier type surfaces

Let K_1, K_2 be the intervals from the Theorem 1. For $m,n \in \mathbb{N}$, let the nodes $x_{m,k} \in K_1$, $y_{n,j} \in K_2$, $z_{m,n,k,j} \in \mathbb{R}$ where $k \in \{0,1,...,p_m\} \cap \mathbb{N}_0$ and $j \in \{0,1,2,...,q_n\} \cap \mathbb{N}_0$.

In the following, we consider a continuous function on $K_1 \times K_2$, $f : K_1 \times K_2 \to \mathbb{R}$, so that $f(x_{m,k}; y_{n,j}) = z_{m,n,k,j}$, where $m,n \in \mathbb{N}$, $k \in \{0,1,...,p_m\} \cap \mathbb{N}_0$ and $j \in \{0,1,2,...,q_n\} \cap \mathbb{N}_0$.

Definition 1.*Let $m,n \in \mathbb{N}$. The point $M^{(m,n)}_{k,j} = (x_{m,k}; y_{n,j}; z_{m,n,k,j}) \in K_1 \times K_2 \times \mathbb{R}$, where $k \in \{0,1,...,p_m\} \cap \mathbb{N}_0$ and $j \in \{0,1,2,...,q_n\} \cap \mathbb{N}_0$ is called control point of (m,n) order.*

Definition 2.*Let $m,n \in \mathbb{N}$. The LK-Bézier surface, respectively GBS-Bézier surface of (m,n) order, which correspond to the control points $M^{(m,n)}_{k,j}$, $k \in \{0,1,...,p_m\} \cap \mathbb{N}_0$ and $j \in \{0,1,2,...,q_n\} \cap \mathbb{N}_0$ are defined by*

$$(B_{m,n})(u,v) = \sum_{k=0}^{p_m}\sum_{j=0}^{q_n} \varphi_{m,k}(u)\psi_{n,j}(v)M^{(m,n)}_{k,j}, \tag{31}$$

respectively

$$\begin{aligned}(B_{m,n})(u,v) &= \sum_{j=0}^{q_n} \psi_{n,j}(v)M^{(n)}_j(u) + \sum_{k=0}^{p_m} \varphi_{m,k}(u)N^{(m)}_k(v) \\ &- \sum_{k=0}^{p_m}\sum_{j=0}^{q_n} \varphi_{m,k}(u)\psi_{n,j}(v)M^{(m,n)}_{k,j} = \\ &\sum_{k=0}^{p_m}\sum_{j=0}^{q_n} \varphi_{m,k}(u)\psi_{n,j}(v)(M^{(n)}_j(u) + N^{(m)}_k(v) - M^{(m,n)}_{k,j}).\end{aligned} \tag{32}$$

where $(u,v) \in K_1 \times K_2$, $M_j^{(n)}(u) = (u; y_{n,j}; z_{1,j}^{(n)}(u))$, $N_k^{(m)}(v) = (x_{m,k}; v; z_{2,k}^{(m)}(v))$, $z_{1,j}^{(n)}(u) = f(u; y_{n,j})$, $z_{2,k}^{(m)}(v) = f(x_{m,k}; v)$,
$k \in \{0,1,...,p_m\} \cap \mathbb{N}_0$ and $j \in \{0,1,2,...,q_n\} \cap \mathbb{N}_0$.

In the following, we consider that $A_{m,n,k,j}(f_u) = \mathrm{B}_{n,j}(f_u) = f(u, y_{n,j})$, $A_{m,n,k,j}(f^v) = A_{m,k}(f^v) = f(x_{m,k}, v)$, $A_{m,n,k,j}(f) = f(x_{m,k}, y_{n,j})$ for any $(u,v) \in K_1 \times K_2$, $k \in \{0,1,...,p_m\} \cap \mathbb{N}_0$, $j \in \{0,1,2,...,q_n\} \cap \mathbb{N}_0$ and $m,n \in \mathbb{N}$.
Then, from (31), (18) and (19) one obtains

$$(\mathrm{B}_{m,n})(u,v) = ((L_m e_1(u); K_n e_1(v); (L^*_{m,n} f)(u,v)), \quad (33)$$

$$(\mathrm{UB}_{m,n})(u,v) = (u; v; (\mathrm{U}L^*_{m,n} f)(u,v)), \quad (34)$$

for any $(u,v) \in K_1 \times K_2$, $k \in \{0,1,...,p_m\} \cap \mathbb{N}_0$, $j \in \{0,1,2,...,q_n\} \cap \mathbb{N}_0$ and $m,n \in \mathbb{N}$.
In the examples from this paper, we have that $\alpha_2 = \beta_2 = 1$, $\gamma_0 = 1$ and exist the constants a_2, b_2 verifying (27), (28) in every application. Taking Theorem 1 into account for the construction above, the following theorem holds.

Theorem 2. *The following convergence*

$$\lim_{m\to\infty} (B_{m,m})(u,v) = (u; v; f(u,v)) \quad (35)$$

and

$$\lim_{m\to\infty} (UB_{m,m})(u,v) = (u; v; f(u,v)) \quad (36)$$

are uniform in $K_1 \times K_2$

Exists $m(0) \in \mathbb{N}$ so that

$$\left|(L^*_{m,m} f)(u,v)) - f(u,v)\right| \leq (1+a_2)(1+b_2)\omega_{total}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right) \quad (37)$$

and

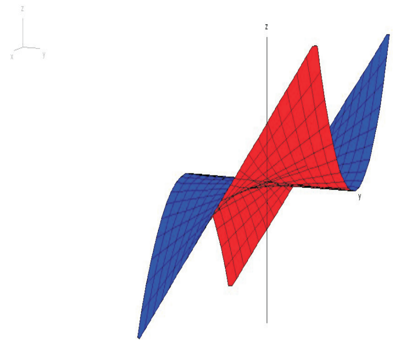
$$\left|(\mathrm{U}L^*_{m,m} f)(u,v)) - f(u,v)\right| \leq (1+b_2)\omega\left(f_u; \frac{1}{\sqrt{m}}\right) + (1+a_2)\omega\left(f^v; \frac{1}{\sqrt{m}}\right) + (1+a_2)(1+b_2)\omega_{total}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right) \quad (38)$$

for any $(u,v) \in K_1 \times K_2$ and any $m \in \mathbb{N}$, $m \geq m(0)$.
Next, in applications we consider $m = n = 1$ and let be the function
$f : [0,\infty) \times [0,\infty) \to \mathbb{R}, f(u,v) = u^2 v$ for any $(u,v) \in [0,\infty) \times [0,\infty)$.
Also, we take $x_{1,0} = -1$, $x_{1,1} = 1$, $y_{1,0} = 0$, $y_{1,1} = 2$, $z_{1,1,0,0} = -2$, $z_{1,1,0,1} = -6$, $z_{1,1,1,0} = 2$ $z_{1,1,1,1} = 2$,and then the control points of (1,1) order are $M_{0,0}^{(1,1)} = (-1;0;-2)$, $M_{0,1}^{(1,1)} = (-1;2;-6)$ $M_{1,0}^{(1,1)} = (1;0;2)$,$M_{1,1}^{(1,1)} = (1;2;2)$. One obtains $M_0^{(1)}(u) = (u; y_{1,0}; f(u; y_{1,0})) = (u;0;0)$, $M_1^{(1)}(u) = (u; y_{1,1}; f(u; y_{1,1})) = (u;2;2u^2)$,
$N_0^{(1)}(v) = (x_{1,0}; v; f(x_{1,0}; v)) = (-1; v; v)$ and
$N_1^{(1)}(v) = (x_{1,1}; v; f(x_{1,1}; v)) = (1; v; v)$.

In the below figure is the graphical representation of the function f, which have the following parametric equation:

$$\begin{cases} x(u,v) = u \\ y(u,v) = v \\ z(u,v) = u^2 v, \end{cases}$$

where $(u,v) \in [0,\infty) \times [0,\infty)$.



Application 1
Let $K_1 = K_2 = [0,1]$, $\varphi_{m,k}(u) = p_{m,k}(u)$, $\psi_{n,j}(v) = p_{n,j}(v)$, $u,v \in [0,1]$, $m,n \in \mathbb{N}$, $k \in \{0,1,...,m\}$, $j \in \{0,1,...,n\}$ and using the above conditions one obtains:

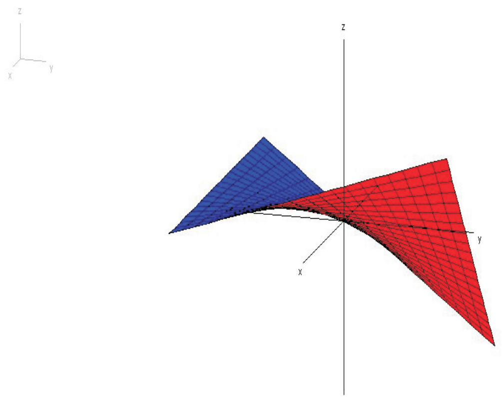
$$(\mathrm{B}_{1,1})(u,v) = p_{1,0}(u)p_{1,0}(v)M_{0,0}^{(1,1)} + p_{1,1}(u)p_{1,0}(v)M_{1,0}^{(1,1)} + p_{1,0}(u)p_{1,1}(v)M_{0,1}^{(1,1)} + p_{1,1}(u)p_{1,1}(v)M_{1,1}^{(1,1)}$$

and using this, one obtains:
$(\mathrm{B}_{1,1})(u,v) = (-1+2u; 2v; -2+4u-4v+4uv)$, $u,v \in [0,1]$

The parametric equations of the above surface are:

$$\begin{cases} x(u,v) = -1+2u \\ y(u,v) = 2v \\ z(u,v) = -2+4u-4v+4uv, \end{cases}$$

where $u,v \in [0,1]$ and the graph of this surface is plotted below:



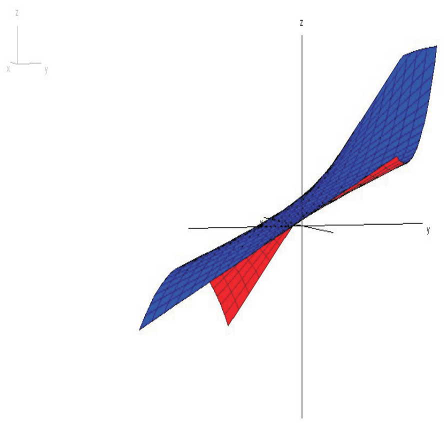
On the other hand, one obtains

$$(UB_{1,1})(u,v) = \left(p_{1,0}(v)M_0^{(1)}(u) + p_{1,1}(v)M_1^{(1)}(u)\right) + (p_{1,0}(u)N_0^{(1)}(v) + p_{1,1}(u)N_1^{(1)}(v)) - (B_{1,1})(u,v) = (u; v; 2-4u+5v-4uv+2u^2v),$$

$u, v \in [0,1]$, and using this one obtains the parametric equations of the GBS-surface, which are

$$\begin{cases} x(u,v) = u \\ y(u,v) = v \\ z(u,v) = 2-4u+5v-4uv+2u^2v, \end{cases}$$

where $u, v \in [0,1]$ and the graph of this surface is plotted below:



Application 2

Let $K_1 = K_2 = [0,\infty)$, $\varphi_{m,k}(u) = \frac{1}{(1+u)^m}\binom{m}{k}u^k$, $\psi_{n,j}(v) = \frac{1}{(1+v)^n}\binom{n}{j}v^j$, $u, v \in [0,\infty)$, $m, n \in \mathbb{N}$, $k \in \{0,1,...,m\}$, $j \in \{0,1,...,n\}$ and using the above conditions one obtains:

$$(\mathrm{B}_{1,1})(u,v) = \varphi_{1,0}(u)\varphi_{1,0}(v)M_{0,0}^{(1,1)} + \varphi_{1,1}(u)\varphi_{1,0}(v)M_{1,0}^{(1,1)} + \varphi_{1,0}(u)\varphi_{1,1}(v)M_{0,1}^{(1,1)} + \varphi_{1,1}(u)\varphi_{1,1}(v)M_{1,1}^{(1,1)}$$

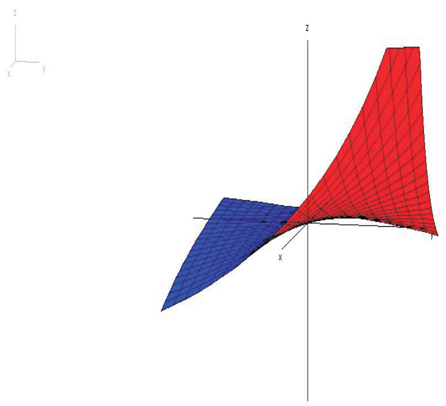
and using this, one obtains:

$(\mathrm{B}_{1,1})(u,v) = \left(\frac{u-1}{1+u}; \frac{2v}{1+v}; \frac{-2+2u-6v+2uv}{(1+u)(1+v)}\right)$.

The parametric equations of the above surface are:

$$\begin{cases} x(u,v) = \frac{u-1}{1+u} \\ y(u,v) = \frac{2v}{1+v} \\ z(u,v) = \frac{-2+2u-6v+2uv}{(1+u)(1+v)}, \end{cases}$$

where $u, v \in [0,\infty)$ and the graph of this surface is plotted below:



The GBS-surface is:

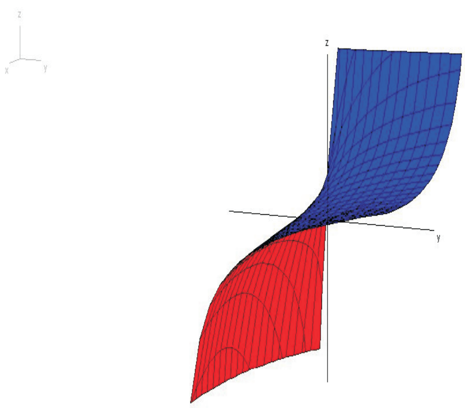
$$(\mathrm{UB}_{1,1})(u,v) = \left(\varphi_{1,0}(v)M_0^{(1)}(u) + \varphi_{1,1}(v)M_1^{(1)}(u)\right) + \left(\varphi_{1,0}(u)N_0^{(1)}(v) + \varphi_{1,1}(u)N_1^{(1)}(v)\right) - (\mathrm{B}_{1,1})(u,v) =$$

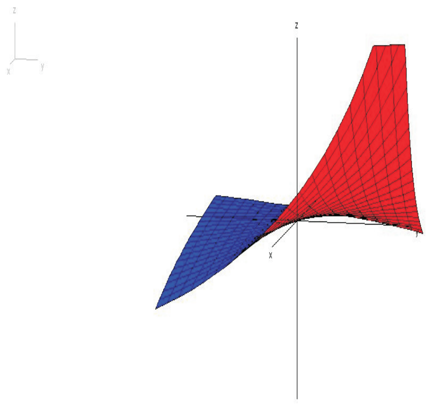
$$= \left(u; v; \frac{2u^3v + 2u^2v + 2uv^2 + v^2 + 7v - uv + 2 - 2u}{(1+u)(1+v)}\right)$$

and

$$\begin{cases} x(u,v) = u \\ y(u,v) = v \\ z(u,v) = \frac{2u^3v+2u^2v+2uv^2+v^2+7v-uv+2-2u}{(1+u)(1+v)}, \end{cases}$$

where $u, v \in [0,\infty)$ and the graph of this surface is plotted below:





Application 3

Let $K_1 = [0,1], K_2 = [0,\infty)$, $\varphi_{m,k}(u) = p_{m,k}(u)$, $\psi_{n,j}(v) = \frac{1}{(1+v)^n}\binom{n}{j} v^j, u \in [0,1], v \in [0,\infty), m,n \in \mathbb{N}, k \in \{0,1,...,m\}$, $j \in \{0,1,...,n\}$ and then:

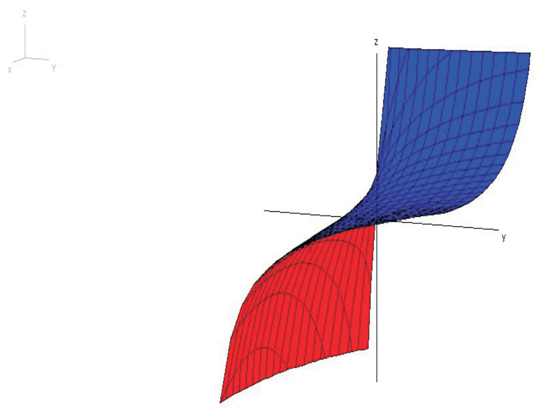
$(\mathtt{B}_{1,1})(u,v) = \varphi_{1,0}(u)\psi_{1,0}(v)M_{0,0}^{(1,1)} + \varphi_{1,1}(u)\psi_{1,0}(v)M_{1,0}^{(1,1)} + \varphi_{1,0}(u)\psi_{1,1}(v)M_{0,1}^{(1,1)} + \varphi_{1,1}(u)\psi_{1,1}(v)M_{1,1}^{(1,1)}$

and using this, one obtains:

$(\mathtt{B}_{1,1})(u,v) = (2u-1; \frac{2v}{1+v}; \frac{-2+4u-6v+8uv}{1+v})$,

$$(\mathtt{UB}_{1,1})(u,v) = \left(\psi_{1,0}(v)M_0^{(1)}(u) + \psi_{1,1}(v)M_1^{(1)}(u)\right) + \left(\varphi_{1,0}(u)N_0^{(1)}(v) + \varphi_{1,1}(u)N_1^{(1)}(v)\right) - (\mathtt{B}_{1,1})(u,v) =$$

$$= \left(u; v; \frac{2u^2v - 8uv - 4u + v^2 + 7v + 2}{1+v}\right).$$



The Bézier surfaces and GBS-Bézier surfaces from this application are given parametrically by

$$\begin{cases} x(u,v) = 2u-1 \\ y(u,v) = \frac{2v}{1+v} \\ z(u,v) = \frac{-2+4u-6v+8uv}{1+v}, \end{cases}$$

respectively

$$\begin{cases} x(u,v) = u \\ y(u,v) = v \\ z(u,v) = \frac{2u^2v-8uv-4u+v^2+7v+2}{1+v}, \end{cases}$$

where $u \in [0,1), v \in [0,\infty)$ and the graphs of these surfaces are plotted below:

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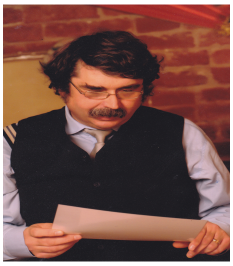

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