Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/110208

Truncated Bivariate Kumaraswamy Exponential Distribution

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Received: 7 Mar. 2021, Revised: 21 May 2021, Accepted: 28 May 2021
Published online: 1 May 2022

Abstract: In this paper we derive the truncated bivariate Kumaraswamy exponential distribution considering the truncated points as parameters. We study various mathematical properties as joint moment generating function, conditional moments, survival function, hazard function, cumulative hazard function, reversed hazard function and stress-strength parameter for the proposed truncated model. Some special states are introduced. ALso, estimation for the unknown parameters is discussed by using maximum likelihood estimation method. Finally a real data set is used to illustrate the superiority of our proposed model for fitting these data set over other compared models.

Keywords: Bivariate Kumaraswamy type; Exponential Distribution; Moments; Stress Strength Parameter; Maximum Liklihood; Fisher Information Matrix.

1 Introduction

In many applications it's important to study two or more random variables determined on the equal sample space. Their models are inflexible meaning that they cannot correspond to different real data situations warranting inspection of each model separately. In modeling situations, when any one of the distributions studied in the literature cannot reach to the adequacy test, all process needs to be defined again by different model. Thus, it's a suitable to begin with adaptable family of distributions that has sufficient members who can cater to various data state. Further, when the information about the data generating mechanism isn't enough, it's appropriate to start with a set of distributions that is more flexible in required properties.

The bivariate exponential distribution(BED) is very important to present the reliance between service times and interarrivals in the failure process in multicomponent systems and in queuing theory. A great number of the BED was studied in literature, such as Arnolda and Straussa[1] studied the bivariate conditional exponential distributions. Abd El-Bar and Ragab[2] introduced the weighted exponential-Gompertz distribution. Afify et al[3] introduced the heavy-tailed exponential distribution. El-Damrawy and Abou-Gabal[4] discussed the random sum of generalized exponential- mixture distribution.Teamah et al[5] studied the generalized weighted exponential-Gompertz distribution. Teamah and Abd El-Bar[6],[7] introduced the compound sum of bivariate distributions. Diawaraa and Carpenter[8] discussed the mixture of BEDs.Teamah et al[9] derived the mixture distribution by using the exponential distribution and Frechet-Weibull distribution along with derived its statistical properties. Mirhosseini, et al[10] introduced the BKED where the two univariate CDFs are F_1 and F_2 and the bivariate distributions are defined by the following relation

$$F(x,y) = 1 - \{1 - F_1(x)F_2(y)\}^{\alpha}\ , x,y \in R. \tag{1}$$

Truncated distributions are defined as the some values of the randomness domain are removed. It has an important general property on real life problems, in many areas as industry of articles, especially in the theory of queues and reliability theory of electronic systems, in medicine as in hospital admonition. Results of studies that are based on the entire population cannot be applied on only a particular class of the society that possesses specific characteristics. The results of such studies will be inaccurate because of the negative influence of certain elements of society on the study group

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or non-importance of some elements of the society that can be neglected. Some conditions can be placed on the population so that the non-important part of the society can be trimmed if that part is outside the interest of the researcher. However, placing such conditions on the community, the data would not follow the same distribution of entire community. Hence, we need to find the distributional characteristics of the truncated data including the PDF and estimation of the parameters for the truncated distributions. Many studies have been made on truncated distributions such as, El-Damrawy and El-shiekh[11] introduced the compound sum for some double truncated distributions. Rosenbaum[12] studied moments for truncated bivariate normal distribution. Shah and Parikh[13] discussed moments for single and double truncated bivariate normal distribution. Arnold,etal[14] discussed the nontruncated marginal for truncated bivariate normal distribution.

Our aim is introducing the bivariate distribution which addresses the problem of the gestation period and Lifespan for various animals by using the truncated bivariate Kumaraswamy exponential distribution(TBKED) considering the truncated points as parameters and studying mathematical properties for TBKED. Some of the special cases are discussed. Finally, we estimate the unknown parameters using the real data. In Section 2 we derived the cumulative distribution function(CDF) and the probability density function(PDF) of the TBKED. Also, we plot the PDF for the TBKED of different values for the parameters. Joint moment generating function(MGF), Pearson's correlation coefficient, conditional moments, survival function(SF), hazard function(HF), cumulative hazard function(CHF), reversed hazard function(RHF) and stress strength parameter are obtained. We study some of the special cases for the TBKED. Also, Maximum Likelihood estimations(MLEs) and fisher information matrix are determined. In Section 3, real data are used to get the value of the unknown parameters for the TBKED.

2 Main text

2.1 Truncated bivariate exponential distribution

Let the baseline univariate CDFs $F_1(x) = 1 - e^{-\lambda_1(x-a)}$ and $F_2(y) = 1 - e^{-\lambda_2(y-b)}$ are ordinary truncated exponential distributions. We obtain the CDF for the TBKED on the form

$$F(x,y) = 1 - \{1 - (1 - e^{-\lambda_1(x-a)})(1 - e^{-\lambda_2(y-b)})\}^{\alpha},$$
$$x \geq a, y \geq b, \lambda_1, \lambda_2 > 0, 0 < \alpha \leq 1, \tag{2}$$

and the joint PDF is

$$f\ (x,y) = \lambda_1\lambda_2 e^{-(\lambda_1(x-a)+\lambda_2(y-b))} \sum_{j=1}^{\infty} \binom{\alpha}{j}(-1)^{j+1} j^2 \{(1 - e^{-\lambda_1(x-a)})(1 - e^{-\lambda_2(y-b)})\}^{j-1},$$
$$x \geq a,\ y \geq b,\ \lambda_1,\ \lambda_2 > 0,\ 0 < \alpha \leq 1. \tag{3}$$

Figure (1) present PDF of the TBKED for different parameters values $(\alpha, \lambda_1, \lambda_2, a, b)$.

2.2 Properties of truncated bivariate Kumaraswamy exponential distribution

In this part the various properties of the TBKED will be studied.

2.2.1 Moments

Supposing that The random variables (X,Y) ~TBKED$(\lambda_1, \lambda_2, \alpha, a, b)$ then, we will calculate some measures such as the joint MGF, Pearson's correlation coefficient, the conditional moments, the SF, the HF, the CHF and the RHF.

a- The joint MGF of TBKED is

$$M_{X,Y}(t,s) = \sum_{j=1}^{\infty} \binom{\alpha}{j}(-1)^{j+1} j^2 e^{at+bs} B(1 - \frac{t}{\lambda_1}, j) B(1 - \frac{s}{\lambda_2}, j).$$

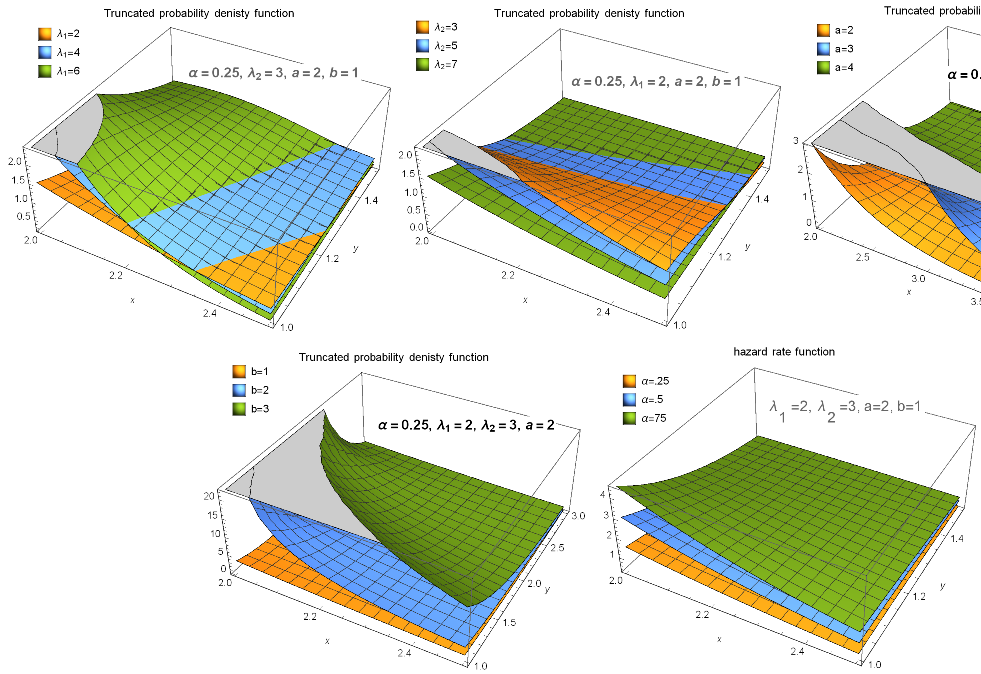

Fig. 1: PDF of TBKED for different parameters values

b- Pearson's correlation coefficient for TBKED is

$$corr(X,Y)=\lambda_1\lambda_2\alpha^2\sum_{j=1}^{\infty}\binom{\alpha}{j}(-1)^{j+1}\sum_{k=-1}^{j}\binom{j}{k+1}(-1)^{k+1}\left[\frac{1}{\lambda_1(k+1)}+a\right]$$
$$\times\sum_{l=-1}^{j}\binom{j}{l+1}(-1)^{l+1}\left[\frac{1}{\lambda_2(l+1)}+b\right]-[\lambda_1\alpha a+1][\lambda_2\alpha b+1].$$

c- The conditional moments of the TBKED are

$$E(Y|X=t)=\frac{1}{\alpha}e^{-\lambda_1(1-\alpha)(t-a)}\sum_{j=1}^{\infty}\binom{\alpha}{j}(-1)^{j+1}j(1-e^{-\lambda_1(t-a)})^{j-1}$$
$$\times\sum_{l=-1}^{j}\binom{j}{l+1}(-1)^{l+1}\left[\frac{1}{\lambda_2(l+1)}+b\right],$$

and,

$$E(Y^2|X=t)=\frac{e^{-\lambda_1(1-\alpha)(t-a)}}{\alpha}\sum_{j=1}^{\infty}\binom{\alpha}{j}(-1)^{j+1}j(1-e^{-\lambda_1(t-a)})^{j-1}\sum_{l=-1}^{j}\binom{j}{l+1}(-1)^{l+1}$$
$$\times\left\{\frac{2}{\lambda_2^2(l+1)^2}+\frac{2b}{\lambda_2(l+1)}+b^2\right\}.$$

d-The SF of the TBKED is

$$S(x,y)=\sum_{j=1}^{\infty}\binom{\alpha}{j}(-1)^{j+1}[1-(1-e^{-\lambda_1(x-a)})^j][1-(1-e^{-\lambda_2(y-b)})^j],$$
$$x\geq a,\ y\geq b,\ \lambda_1,\ \lambda_2>0,\ 0<\alpha\leq 1.$$

e-The HF of the TBKED is

$$h(x,y) = \lambda_1\lambda_2 e^{-(\lambda_1(x-a)+\lambda_2(y-b))} \times \frac{\sum_{j=1}^{\infty}\binom{\alpha}{j}(-1)^{j+1}j^2\{(1-e^{-\lambda_1(x-a)})(1-e^{-\lambda_2(y-b)})\}^{j-1}}{\sum_{j=1}^{\infty}\binom{\alpha}{j}(-1)^{j+1}[1-(1-e^{-\lambda_1(x-a)})^j][1-(1-e^{-\lambda_2(y-b)})^j]}.$$

f- the RHF for TBKED is

$$r(x,y) = \lambda_1\lambda_2 e^{-(\lambda_1(x-a)+\lambda_2(y-b))} \times \frac{\sum_{j=1}^{\infty}\binom{\alpha}{j}(-1)^{j+1}j^2\{(1-e^{-\lambda_1(x-a)})(1-e^{-\lambda_2(y-b)})\}^{j-1}}{1-\{1-(1-e^{-\lambda_1(x-a)})(1-e^{-\lambda_2(y-b)})\}^{\alpha}}.$$

2.2.2 The stress-strength parameter

The stress-strength parameter of the TBKED is helpful for analyzing the data.
If (X,Y)$\sim$ the TBKED$(\lambda_1,\lambda_2,\alpha,a,b)$,then

$$P(X<Y) = \sum_{j=1}^{\infty}\binom{\alpha}{j}(-1)^{j+1}j\sum_{k=0}^{j}\binom{j}{k}(-1)^k B(j,\frac{\lambda_2 k}{\lambda_1}+1)..$$

2.3 Special states

In this part we will talk about several special states
Case 1
By taking $\alpha = 1$ in equation (2) we get the PDF of the truncated bivariate independent exponential distribution(TBIED) as:

$$f(x,y) = \lambda_1\lambda_2 e^{-(\lambda_1(x-a)+\lambda_2(y-b))}, x \geq a,\ y \geq b,\ \lambda_1,\lambda_2 > 0.$$

Case 2
When we put $a = b = 0$ in equation (2) we get the PDF of the BKED as:

$$\begin{aligned} f(x,y) &= \alpha\lambda_1\lambda_2 e^{-(\lambda_1 x+\lambda_2 y)}\{1-\alpha(1-e^{-\lambda_1 x})(1-e^{-\lambda_2 y})\} \\ &\quad \times\{1-(1-e^{-\lambda_1 x})(1-e^{-\lambda_2 y})\}^{\alpha-2}, \\ &\quad x,y \geq 0,\ \lambda_1,\lambda_2 > 0,\ 0 \leq \alpha < 1. \end{aligned}$$

Case 3
If we take $\alpha = 1$ and $a = b = 0$ in equation (2) we get the PDF of bivariate independent exponential distribution(BIED) as:

$$f(x,y) = \lambda_1\lambda_2 e^{-(\lambda_1 x+\lambda_2 y)}\ , x,y \geq 0,\ \lambda_1,\lambda_2 > 0.$$

2.4 Maximum likelihood estimation

In this part the MLEs for the the unknown parameters of a random sample of size n from TBKE model are derived. If (X_1,Y_1),...,(X_n,Y_n) be a random sample from (X,Y)~TBKED$(\lambda_1,\lambda_2,\alpha,a,b)$ then,

$$
\begin{aligned}
L(\theta) = {} & n\log\alpha\lambda_1\lambda_2 - \lambda_1\sum_{i=1}^{n}(x_i-a) - \lambda_2\sum_{i=1}^{n}(y_i-b) \\
& + \sum_{i=1}^{n}\log\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\} \\
& + (\alpha-2)\sum_{i=1}^{n}\log\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}.
\end{aligned} \tag{4}
$$

Such that $\theta = (\alpha,\lambda_1,\lambda_2,a,b)$, MLEs are gotten by maximizing equation (4) with respect to the unknown parameters. It cannot be acquired explicitly. We want to solve five nonlinear equations to compute MLEs, then we must use real data for get the values of the unknown parameters.

The first derivatives for equation (5) of parameters $(\alpha,\lambda_1,\lambda_2,a,b)$ are obtained as

$$
\begin{aligned}
\frac{\partial l(\theta)}{\partial\alpha} = {} & \frac{n}{\alpha} + \sum_{i=1}^{n}\log\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\} \\
& - \sum_{i=1}^{n}\frac{(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})}{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})} = 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l(\theta)}{\partial\lambda_1} = {} & \frac{n}{\lambda_1} - \sum_{i=1}^{n}(x_i-a) - (\alpha-2)\sum_{i=1}^{n}\frac{(x_i-a)e^{-\lambda_1(x_i-a)}(1-e^{-\lambda_2(y_i-b)})}{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})} \\
& - \alpha\sum_{i=1}^{n}\frac{(x_i-a)e^{-\lambda_1(x_i-a)}(1-e^{-\lambda_2(y_i-b)})}{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})} = 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l(\theta)}{\partial\lambda_2} = {} & \frac{n}{\lambda_2} - \sum_{i=1}^{n}(y_i-b) - (\alpha-2)\sum_{i=1}^{n}\frac{(y_i-b)e^{-\lambda_2(y_i-b)}(1-e^{-\lambda_1(x_i-a)})}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}} \\
& - \alpha\sum_{i=1}^{n}\frac{(y_i-b)e^{-\lambda_2(y_i-b)}(1-e^{-\lambda_1(x_i-a)})}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}} = 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l(\theta)}{\partial a} = {} & n\lambda_1 + \alpha\sum_{i=1}^{n}\frac{\lambda_1 e^{-\lambda_1(x_i-a)}(1-e^{-\lambda_2(y_i-b)})}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-a)})\}} \\
& + (\alpha-2)\sum_{i=1}^{n}\frac{\lambda_1 e^{-\lambda_1(x_i-a)}(1-e^{-\lambda_2(y_i-b)})}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}} = 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l(\theta)}{\partial b} = {} & n\lambda_2 + \alpha\sum_{i=1}^{n}\frac{\lambda_2 e^{-\lambda_2(y_i-b)}(1-e^{-\lambda_1(x_i-a)})}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-a)})\}} \\
& + (\alpha-2)\sum_{i=1}^{n}\frac{\lambda_2 e^{-\lambda_2(y_i-b)}(1-e^{-\lambda_1(x_i-a)})}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}} = 0.
\end{aligned}
$$

For obtaining a confidence intervals for unknown parameters We want to calculate the matrix of Fisher information that can be obtained from The second derivatives of the equation (4) for parameters $(\alpha,\lambda_1,\lambda_2,a,b)$ as seen in the next equations

$$
I_{\alpha^2} = \frac{\partial^2 l(\theta)}{\partial\alpha^2} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n}\left[\frac{(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})}{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})}\right]^2,
$$

$$
\begin{aligned}
I_{\lambda_1^2} = \frac{\partial^2 l(\theta)}{\partial\lambda_1^2} = {} & -\frac{n}{\lambda_1^2} + (\alpha-2)\sum_{i=1}^{n}\frac{(x_i-a)^2 e^{-(\lambda_1(x_i-a)+\lambda_2(y_i-b))}(1-e^{-\lambda_2(y_i-b)})}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2} \\
& + \alpha\sum_{i=1}^{n}\frac{(x_i-a)^2 e^{-\lambda_1(x_i-a)}(1-e^{-\lambda_2(y_i-b)})\{1-\alpha(1-e^{-\lambda_2(y_i-b)})\}}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2},
\end{aligned}
$$

$$I_{\lambda_2^2}=\frac{\partial^2 l(\theta)}{\partial\lambda_2^2}=-\frac{n}{\lambda_2^2}+(\alpha-2)\sum_{i=1}^{n}\frac{(y_i-a)^2e^{-(\lambda_1(x_i-a)+\lambda_2(y_i-b))}(1-e^{-\lambda_1(x_i-a)})}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2}$$
$$+\alpha\sum_{i=1}^{n}\frac{(y_i-b)^2e^{-\lambda_2(y_i-b)}(1-e^{-\lambda_1(x_i-a)})\{1-\alpha(1-e^{-\lambda_1(x_i-a)})\}}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2},$$

$$I_{\alpha a}=\frac{\partial^2 l(\theta)}{\partial\alpha\partial a}=\sum_{i=1}^{n}\frac{\lambda_1e^{-\lambda_1(x_i-a)}(1-e^{-\lambda_2(y_i-b)})}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}}$$
$$+\sum_{i=1}^{n}\frac{\lambda_1e^{-\lambda_1(x_i-a)}(1-e^{-\lambda_2(y_i-b)})}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2},$$

$$I_{\alpha b}=\frac{\partial^2 l(\theta)}{\partial\alpha\partial b}=\sum_{i=1}^{n}\frac{\lambda_2e^{-\lambda_2(y_i-b)}(1-e^{-\lambda_1(x_i-a)})}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}}$$
$$+\sum_{i=1}^{n}\frac{\lambda_2e^{-\lambda_2(y_i-b)}(1-e^{-\lambda_1(x_i-a)})}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2},$$

$$I_{\lambda_1\lambda_2}=\frac{\partial^2 l(\theta)}{\partial\lambda_1\partial\lambda_2}=-\alpha\sum_{i=1}^{n}\frac{(x_i-a)(y_i-b)e^{-\lambda_1(x_i-a)}e^{-\lambda_2(y_i-b)}}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2}$$
$$-(\alpha-2)\sum_{i=1}^{n}\frac{(x_i-a)(y_i-b)e^{-\lambda_1(x_i-a)}e^{-\lambda_2(y_i-b)}}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2},$$

$$I_{\lambda_1 a}=\frac{\partial^2 l(\theta)}{\partial\lambda_1\partial a}=n+(\alpha-2)\sum_{i=1}^{n}\frac{e^{-\lambda_1(x_i-a)}(1-e^{-\lambda_2(y_i-b)})\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2}$$
$$-\alpha\sum_{i=1}^{n}\frac{e^{-\lambda_1(x_i-a)}(1-e^{-\lambda_2(y_i-b)})\{\lambda_1(x_i-a)(1-\alpha)-\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}\}}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2},$$

$$I_{\lambda_1 b}=\frac{\partial^2 l(\theta)}{\partial\lambda_1\partial a}=(\alpha-2)\sum_{i=1}^{n}\frac{\lambda_2(x_i-a)e^{-\lambda_1(x_i-a)}e^{-\lambda_2(y_i-a)}}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-a)})\}^2}$$
$$+\alpha\sum_{i=1}^{n}\frac{\lambda_2(x_i-a)e^{-\lambda_1(x_i-a)}e^{-\lambda_2(y_i-a)}}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-a)})\}^2},$$

$$I_{\lambda_2 a}=\frac{\partial^2 l(\theta)}{\partial\lambda_2\partial a}=(\alpha-2)\sum_{i=1}^{n}\frac{\lambda_1(y_i-b)e^{-\lambda_1(x_i-a)}e^{-\lambda_2(y_i-a)}}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-a)})\}^2}$$
$$+\alpha\sum_{i=1}^{n}\frac{\lambda_1(y_i-b)e^{-\lambda_1(x_i-a)}e^{-\lambda_2(y_i-a)}}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-a)})\}^2},$$

$$I_{\lambda_2 b}=\frac{\partial^2 l(\theta)}{\partial\lambda_2\partial b}=n+(\alpha-2)\sum_{i=1}^{n}\frac{e^{-\lambda_2(y_i-b)}(1-e^{-\lambda_1(x_i-a)})\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2}$$
$$-\alpha\sum_{i=1}^{n}\frac{e^{-\lambda_2(y_i-b)}(1-e^{-\lambda_1(x_i-a)})\{\lambda_2(y_i-b)(1-\alpha)-\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}\}}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-b)})\}^2},$$

Table 1: The Gestation and Lifespan of 40 different animals.

No	Animals	X	Y	No	animals	X	Y
1	African giant pouched rat	42	4.5	21	Mouse	19	3.2
2	Baboon	180	27	22	Musk shrew	30	2
3	Big brown bat	35	19	23	N. American opossum	12	5
4	Brazilian tapir	392	30.4	24	Nine-banded armadillo	120	6.5
5	Cat	63	28	25	Owl monkey	140	12
6	Chimpanzee	230	50	26	Patas monkey	170	20.2
7	Chinchilla	112	7	27	Phanlanger	17	13
8	Cow	281	30	28	Pig	115	27
9	Eastern American mole	42	3.5	29	Rabbit	31	18
10	Echidna	28	50	30	Rat	21	4.7
11	European hedgehog	42	6	31	Red fox	52	9.8
12	Galago	120	10.4	32	Rhesus monkey	164	29
13	Goat	148	20	33	Rockhyrax(heterob)	225	7
14	Golden hamster	16	3.9	34	Rockhyrax(procaviahab)	225	6
15	Gray seal	310	41	35	Sheep	151	20
16	Ground squirrel	28	9	36	Tenrec	60	4.5
17	Guinea pig	68	7.6	37	Tree hyrax	200	7.5
18	Horse	336	46	38	Tree shrew	46	2.3
19	Lesser Short-tailed shrew	21.5	2.6	39	Vervet	210	24
20	Little brown bat	50	24	40	Water opossum	14	3

$$
\begin{aligned}
I_{ab} = \frac{\partial^2 l(\theta)}{\partial a \partial b} &= -(\alpha-2)\sum_{i=1}^{n} \frac{\lambda_1\lambda_2 e^{-\lambda_1(x_i-a)} e^{-\lambda_2(y_i-a)}}{\{1-(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-a)})\}^2} \\
&- \alpha\sum_{i=1}^{n} \frac{\lambda_1\lambda_2 e^{-\lambda_1(x_i-a)} e^{-\lambda_2(y_i-a)}}{\{1-\alpha(1-e^{-\lambda_1(x_i-a)})(1-e^{-\lambda_2(y_i-a)})\}^2}.
\end{aligned}
$$

Then, the Fisher information matrix $I_n(\theta) = I_n(\alpha, \lambda_1, \lambda_2, a, b)$ can be written as shown in the following matrix

$$
I_n(\theta) = \begin{bmatrix} I_{\alpha^2} \dots I_{\alpha\lambda_1} \dots I_{\alpha\lambda_2} \dots I_{\alpha a} \dots I_{\alpha b} \\ I_{\lambda_1\alpha} \quad I_{\lambda_1^2} \quad I_{\lambda_1\lambda_2} \dots I_{\lambda_1 a} \dots I_{\lambda_1 b} \\ I_{\lambda_2\alpha} \quad I_{\lambda_2\lambda_1} \quad I_{\lambda_2^2} \dots I_{\lambda_2 a} \dots I_{\lambda_2 b} \\ I_{a\alpha} \quad I_{a\lambda_1} \quad I_{a\lambda_2} \dots I_{a^2} \dots I_{ab} \\ I_{b\alpha} \quad I_{b\lambda_1} \quad I_{b\lambda_2} \dots I_{ba} \dots I_{b^2} \end{bmatrix}
$$

3 Real data application

In this part we want to matching the TBKED to a real data. Consider the following data which presents Gestation period (X variable) and Lifespan(Y variable) of 40 different animals. We cannot obtain analytical solution for the five equations hence; we can get the values of the unknown parameters as seen in table 2.

In Table 2. the MLEs for the parameters of BKED$(\alpha, \lambda_1, \lambda_2)$, TBKED$(\alpha, \lambda_1, \lambda_2, a, b)$ and TBIED$(\lambda_1, \lambda_2, a, b)$ are obtained. The value of the negative log likelihood function (-LOG), Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC), Hanan-Quinn Information Criterion (HQIC) and Second Order of AKaike Information Criterion(AICc)are obtained in Table 3.

It has shown from Table (2) and Table (3) that, the application of TBKED for the real data appears the superiority of this new distribution by comparing the compatibility with BKED and TBIED.

Table 2: MLEs for the data set.

Distribution	α	λ_1	λ_2	a	b
BKED	0.47	0.133	0.47	-	-
TBKED	0.046	0.0897	0.399	14.369	6.125
TBIED	-	0.0567	0.64881	25.665	4.372

Table 3: Statistics -LOG, AIC, BIC, AICc and HQC of the data set

Distribution	-LOG	AIC	BIC	AICc	HQIC
BKED	690.657	1387.31	1392.38	1387.98	1389.15
TBKED	529.067	1068.13	1076.58	1069.9	1071.19
TBIED	663.85	1335.7	1342.46	1336.84	1338.14

4 Conclusions

Now, we offered the TBKED distribution. The statistical properties of TBKED are calculated. MLEs parameters of the TBKED distribution are given.Finally, data analysis is discussed to illustrate the behavior of TBKED distribution in practice and proofed that TBKED is better than BKED and TBIED.

Conflicts of Interests

The authors declare that they have no conflicts of interests

Abbreviations

CDF Cumulative Distribution Function

PDF Probability Density Function

MGF Moment Generating Function

HF Hazard Function

SF Survival Function

CHF Cumulative Hazard Function

RHF Reversed Hazard Function

BED Bivariate Exponential Distribution

BKED Bivariate Kumarswamy Exponential Distribution

TBKED Truncated Bivariate Kumaryswamy Exponential Distribution

TBIED Truncated Bivariate Independent Exponential Distribution

BIED Bivariate Independent Exponential Distribution

MLEs Maximum Likelihood Estimations

-LOG negative log likelihood function

AIC Akaike Information Criterion

BIC Bayesian Information Criterion

AICc Second Order of AKaike Information Criterion

HQIC Hanan-Quinn Information Criterion

Acknowledgement

The authors would like to thank the referees for providing very helpful comments and suggestions that helped in improving the quality of the paper.
Conflicts of Interest The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] *B. C. Arnolda and D. Straussa, Bivariate distributions with exponential conditionals. Journal of the American Statistical Association,83, 522–527(1988).*
[2] *A. M. Abd El-Bar, A. M. Teamah and I. E.Ragab, On weighted exponential-Gomperetz distributions: properties and application. Journal of Taibah University for science. 13(1): 616-627(2019).*
[3] *A. Z. Afify, A. M. Gemeay and N. A. Ibrahim, The heavy-tailed exponential distribution: Risk measures, estimation, and application to actuarial data. Mathematics, 8(8), 1276 (2020).*
[4] *H. H. El-Damrawy, and H. M. Abou-Gabal, Sum of generalized exponential mixture distribution with random parameter, Afrika Matematika, 26: 239–246 (2015).*
[5] *A.M.Teamah, H. H. El-Damrawy and Sh. M. Swan, Generalized weighted exponential-gompertez distribution. Journal of Applied Mathematics. 11: 97-118(2020).*
[6] *A.M.Teamah, and A. M. Abd El-Bar, Random sum of new bivariate negative binomial distributions. Journal of the Egyptian Mathematical Society. 18(1): 107-114(2010).*
[7] *A.M.Teamah, and A. M. Abd El-Bar, Random sum of sum of bivariate exponential distributions. Journal of Mathematics and Statistics. 5(4):270-275(2009).*
[8] *N. Diawaraa and M. Carpenter, Mixture of bivariate exponential distributions. Commun. Statist. Theory and Methods,39: 2711–2720(2010).*
[9] *A.M.Teamah, A. A. Elbanna and A. M. Gemeay, Frechet-Weibull Mixture Distribution, Properties and Applications. Applied Mathematical Sciences, 14(2), 75-86(2020).*
[10] *S. M. Mirhosseinia,A. Dolati and M. Amini, On a bivariate Kumaraswamy type exponential distribution. Communication in statistics-Theory and Methods. 45(18): 5461-5477(2016).*
[11] *H. H. El-Damrawy, and B. M. El-shiekh, Random sum for some doubly truncated distributions, Wulfenia Journal, 21(4): 45-54(2014).*
[12] *S. Rosenbaum, Moments of a truncated bivariate normal distribution, Journal of the Royal Statistical Society", Series B, 23: 405-408(1961).*
[13] *S. M. Shah, and N. T.Parikh, Moments of singly and doubly truncated standard bivariate normal distribution, Vidya, 7: 82-9(1964).*
[14] *Arnold, c. Barry . and Others, The nontruncated marginal of a truncated bivariate normal distribution Psychometrika, 58(3):471-488 (1993).*

Apendix A

These codes were performed by using Wolfram Mathematica software version 10.4.
i = {42, 180, 35, 392, 63, 230, 112, 281, 42, 28, 42, 120, 148, 16,
310, 28, 68, 336, 21.5, 50, 19, 30, 12, 120, 140, 170, 17, 115, 31,
21, 52, 164, 225, 225, 151, 60, 200, 46, 210, 14};
j = {4.5, 27, 19, 30.4, 28, 50, 7, 30, 3.5, 50, 6, 10.4, 20, 3.9, 41,
9, 7.6, 46, 2.6, 24, 3.2, 2, 5, 6.5, 12, 20.2, 13, 27, 18, 4.7, 9.8,
29, 7, 6, 20, 45, 7.5, 2.3, 24, 3};
n = 40;
$I1 = Total[log[1-\alpha(1-e^{-\lambda_1(i-a)})(1-e^{-\lambda_2(j-b)})]];$
$I2 = (\alpha-2)Total[log[1-(1-e^{-\lambda_1(i-a)})(1-e^{-\lambda_2(j-b)})]];$
$Maxlik = nlog[\alpha\lambda_1\lambda_2]-\lambda_1 Sum[(i-a),\{i,$ {42, 180, 35, 392, 63, 230, 112, 281, 42, 28, 42, 120, 148, 16,
310, 28, 68, 336, 21.5, 50, 19, 30, 12, 120, 140, 170, 17, 115, 31,
21, 52, 164, 225, 225, 151, 60, 200, 46, 210, 14}}] $-\lambda_2 Sum[(j-b),\{j,$ {4.5, 27, 19, 30.4, 28, 50, 7, 30, 3.5, 50, 6, 10.4, 20, 3.9, 41,
9, 7.6, 46, 2.6, 24, 3.2, 2, 5, 6.5, 12, 20.2, 13, 27, 18, 4.7, 9.8,
29, 7, 6, 20, 45, 7.5, 2.3, 24, 3}}] $+I1+I2$;
D $_{alfa}$= D[Maxlik, α];
D $_{lambda1}$= D[Maxlik, λ_1];
D $_{lambda2}$= D[Maxlik, λ_2];
D $_a$= D[Maxlik, a];
D $_b$= D[Maxlik, b];
FindRoot[{D $_{alfa}$ == 0, D $_{lambda1}$== 0, D $_{lambda2}$ == 0, D $_a$== 0, D $_b$ == 0}, {{α, .047}, {λ_1, .1}, {λ_2, .3}, {a, 12}, {b, 10}}]