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Bayesian Analysis of Weighted Boltzmann Maxwell Distribution; a Simulation Study

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Abstract: In this article, we have primarily studied the Bayes' estimator of the parameter of the Weighted Maxwell Boltzmann Distribution under the extended Jeffrey's prior, Gamma & Chi-square prior distributions assuming different loss functions. The extended Jeffrey's prior gives the opportunity of covering wide spectrum of priors to get Bayes' estimates of the parameter - particular cases of which are Jeffrey's prior and Hartigan's prior. A comparative study has been done between the MLE and the estimates of different loss functions (SELF and Al-Bayyati's , Stein &Precautionary new loss function). From the results, we observe that in most cases, Bayesian Estimator under New Loss function (Al-Bayyati's Loss function) has the smallest Mean Squared Error values for both prior's i.e, Jeffrey's and an extension of Jeffrey's prior information. Moreover, when the sample size increases, the MSE decreases quite significantly. The future research may be consider to estimate the parameters using different loss functions especially Linex loss function and Generalized entropy Loss function under different prior distributions like Conjugate priors and double priors etc.

Keywords: Weighted Maxwell Boltzmann Distribution, Prior Distributions, Loss function, R Software, Mean Square Error.

1 Introduction

In Physics and Chemical Sciences, there are a lot of applications of Maxwell-Boltzmann distribution. The Maxwell distribution forms the basis of the kinetic energy of gases, which explains many fundamental properties of gases, including pressure and diffusion. This distribution is sometimes referred to as the distribution of velocities, energy and magnitude of moment of molecules. It was Tyagi and Bhattacharya [1] who considered the Maxwell distribution as a lifetime model for the first time and discussed the Baye's and minimum variance unbiased estimation procedures for its parameter and reliability function. Chaturvedi and Rani [2] obtained classical and Baye's estimators for the Maxwell distribution, after generalizing it by introducing one more parameter. Empirical Baye's estimation for the Maxwell distribution was studied by Bekker and Roux [3]. Kazmi *et al.* [4] carried out the Bayesian estimation for two component mixture of Maxwell distribution, assuming type I censored data. The PDF of a random variable X following weighted Maxwell distribution with rate parameter θ is given by Dar et.al [5].

$$f_\omega(x,\theta,\omega)=\frac{\theta^{\left(\frac{\omega+3}{2}\right)}x^{\omega+2}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\exp\left(-\frac{\theta x^2}{2}\right) \tag{1}$$

Where θ is the rate parameter and ω is the weight parameter (ω>0)

2 Materials and Methods

In comparison to classical approach, Bayesian approach is considered to be fair enough in estimating the parameters of a distribution provided that the prior distribution describes nicely the random behavior of a parameter. Since in Bayesian setup, Parameter of a distribution is considered to be a random variable having the sample space denoted by Θ . Hence, it

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seeks the use of prior information about the parameter. This prior information about the parameter is described in terms of probability distribution usually known as prior distribution. Thus the most important element in Bayesian estimation is the specification of a prior distribution which a parameter as a random variable is supposed to follow. Therefore, an important pre-requisite in Bayesian estimation is the appropriate choice of prior(s) for the estimation of parameter θ. However, Bayesian analysts have pointed out that there isn't any hard and fast rule by which one can conclude that one prior is better than the other. Very often, priors are chosen according to one's subjective knowledge and beliefs that is why Bayesian approach is sometimes called as Subjective approach. If one has adequate information about the parameter(s), one should use informative prior(s); otherwise it is preferable to use non-informative prior(s) (also known as vague prior). Usually, uniform prior is preferred as a non-informative prior. However, Aslam *et al.* [6] have shown an application of prior predictive distribution to elicit the prior density. In this paper, we assume the parameter (θ) to follow extension of Jeffrey's prior (proposed by Al-Kutubi [7] and gamma prior which are given by:

2.1 Extension of the Jeffrey's-prior

Prior proposed by Al-Kutubi, also known as extension of Jeffrey's prior is given by

$$g(\theta) \propto \left[I(\theta)\right]^{c_1} ; c_1 \in R^+$$

Thus the resulting extension of Jeffrey's-prior will be:

$$g(\theta) \propto \left[\frac{1}{\theta^2}\right]^{c_1}$$

2.2 The Gamma (α,β) prior

The density of parameter (θ) on assuming it to follow Gamma(α,β) distribution isgiven by:

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{(\alpha-1)} e^{-\beta\theta}, \theta > 0, \alpha > 0, \beta > 0.$$

2.3 Loss Function

The concept of loss function is as old as Laplace, and was reintroduced in Statistics by Abraham Wald in the middle of 20th century see; A. Wald [8], "Statistical Decision Functions". Sound statistical practice requires selecting an estimator consistent with the actual acceptable variation experienced in the context of a particular applied problem. Thus, in the applied use of loss functions, selecting which statistical method to use to model an applied problem depends on knowing the losses that will be experienced from being wrong under the problem's particular circumstances. One of the consequences of Bayesian inference is that, in addition to experimental data, the loss function does not in itself wholly determine a decision. Moreover, the relationship between the loss function and the posterior probability is important. Thus the choice of loss function is an integral part of Bayesian inference. As there is no specific analytical procedure that allows us to identify the appropriate loss function to be used. Most of the works on point estimation and point prediction assume the underlying loss function to be squared error which is symmetric in nature. However, indiscriminate use of squared error loss function is not appropriate particularly in the cases, where losses are not symmetric. Thus, in order to make the statistical inferences more practical and applicable; we often need to choose an asymmetric loss function.

A number of symmetric and asymmetric loss functions have been shown to be functional, see Dey et al. [9], Norstrom et al. [10] , Podder et al. [11] ,Rasheed et al.[12] ,Spring et al.[13] ,Zellner et al. [14] , Reshi et al.[15] ,Ahmed et al. [16] etc. In the present paper, we have considered squared error, precautionary, Al-Bayyati's and Stein's loss functions for better comparison of Bayes' estimators.

3 Estimation of Parameters of Weighted Maxwell Boltzmann Distribution:

3.1 Maximum Likelihood Estimation:

Let $x = (x_1, x_2, x_3, \ldots, x_n)$ be a random sample of size *n* from Weighted Maxwell Distribution Therefore the maximum likelihood estimator is:

$$\hat{\theta}_{mle} = \frac{n\omega + 3n}{\sum_{i=1}^{n} x_i^{\,2}}$$

Table 3.2.: Bayes' Estimator under Different Combinations of Loss Functions and Prior Distributions:

Prior	Loss Function	Estimator
Extension of Jeffrey's	**Squared-error**	$\hat{\theta}_{sq,EJ)} = \frac{n\omega + 3n - 4c_1 + 2}{\sum_{i=1}^{n} x_i^{\,2}}$
	Precautionary	$\hat{\theta}_{(pr,EJ)} = \frac{\sqrt{(n\omega + 3n - 4c_1 + 4)(n\omega + 3n - 4c_1 + 2)}}{\sum_{i=1}^{n} x_i^{\,2}}$
	Al-Bayyati's	$\hat{\theta}_{(Al,EJ)} = \frac{n\omega + 3n - 4c_1 + 2c_2 + 2}{\sum_{i=1}^{n} x_i^{\,2}}$
	Stein's	$\hat{\theta}_{(St,EJ)} = \frac{n\omega + 3n - 4c_1}{\left(\sum x_i^{\,2}\right)}$
Hartigan's (i.e. c_1=3/2)	**Squared-error**	$\hat{\theta}_{(sq,Hp)} = \frac{n\omega + 3n - 4}{\sum_{i=1}^{n} x_i^{\,2}}$
	Precautionary	$\hat{\theta}_{(pr,Hp)} = \frac{\sqrt{(n\omega + 3n - 2)(n\omega + 3n - 4)}}{\sum_{i=1}^{n} x_i^{\,2}}$
	Al-Bayyati's	$\hat{\theta}_{(Al,Hp)} = \frac{n\omega + 3n + 2c_2 - 4}{\sum_{i=1}^{n} x_i^{\,2}}$
	Stein's	$\hat{\theta}_{(St,Hp)} = \frac{n\omega + 3n - 6}{\sum_{i=1}^{n} x_i^{\,2}}$
Jeffrey's (i.e. c_1=1/2)	**Squared-error**	$\hat{\theta}_{(sq,Jp)} = \frac{n\omega + 3n}{\sum_{i=1}^{n} x_i^{\,2}}$
	Precautionary	$\hat{\theta}_{(pr,Jp)} = \frac{\sqrt{(n\omega + 3n)(n\omega + 3n + 2)}}{\sum_{i=1}^{n} x_i^{\,2}}$
	Al-Bayyati's	$\hat{\theta}_{(Al,Jp)} = \frac{n\omega + 3n + 2c_2 - 4}{\sum_{i=1}^{n} x_i^{\,2}}$

	Stein's	$\hat{\theta}_{(St,Jp)} = \dfrac{n\omega + 3n - 2}{\sum_{i=1}^{n} x_i^{\,2}}$
uniform (i.e. c_1 =0)	**Squared-error**	$\hat{\theta}_{sq,Up)} = \dfrac{n\omega + 3n + 2}{\sum_{i=1}^{n} x_i^{\,2}}$
	Precautionary	$\hat{\theta}_{(pr,Up)} = \dfrac{\sqrt{(n\omega+3n+4)(n\omega+3n+2)}}{\sum_{i=1}^{n} x_i^{\,2}}$
	Al-Bayyati's	$\hat{\theta}_{(Al,Up)} = \dfrac{n\omega + 3n + 2c_2 + 2}{\sum_{i=1}^{n} x_i^{\,2}}$
	Stein's	$\hat{\theta}_{(St,Up)} = \dfrac{n\omega + 3n}{\sum_{i=1}^{n} x_i^{\,2}}$
Gamma **(α,β)**	**Squared-error**	$\hat{\theta}_{(sq,gp)} = \dfrac{n\omega + 3n + 2\alpha}{\sum_{i=1}^{n} x_i^{\,2} + 2\beta}$
	Precautionary	$\hat{\theta}_{(pr,gp)} = \dfrac{\sqrt{(n\omega+3n+2\alpha+2)(n\omega+3n+2\alpha)}}{\sum_{i=1}^{n} x_i^{\,2} + 2\beta}$
	Al-Bayyati's	$\hat{\theta}_{(Al,gp)} = \dfrac{n\omega + 3n + 2\alpha + 2c_2}{\sum_{i=1}^{n} x_i^{\,2} + 2\beta}$
	Stein's	$\hat{\theta}_{(St,gp)} = \dfrac{n\omega + 3n + 2\alpha - 2}{\left(\sum x_i^{\,2} + 2\beta\right)}$
Chi-squared $(2\alpha\, d.f.\ \&\ \beta = 1/2)$	**Squared-error**	$\hat{\theta}_{(sq,Chi)} = \dfrac{n\omega + 3n + 2\alpha}{\sum_{i=1}^{n} x_i^{\,2} + 1}$
	Precautionary	$\hat{\theta}_{(pr,Chi)} = \dfrac{\sqrt{(n\omega+3n+2\alpha+2)(n\omega+3n+2\alpha)}}{\sum_{i=1}^{n} x_i^{\,2} + 1}$
	Al-Bayyati's	$\hat{\theta}_{(Al,Chi)} = \dfrac{n\omega + 3n + 2\alpha + 2c_2}{\sum_{i=1}^{n} x_i^{\,2} + 1}$
	Stein's	$\hat{\theta}_{(St,Chi)} = \dfrac{n\omega + 3n + 2\alpha - 2}{\left(\sum x_i^{\,2} + 1\right)}$

Chi-squared $(\alpha = 1)$	**Squared-error**	$\hat{\theta}_{(sq,Ep)} = \dfrac{n\omega + 3n + 2}{\sum_{i=1}^{n} x_i^2 + 2\beta}$
	Precautionary	$\hat{\theta}_{(pr,Ep)} = \dfrac{\sqrt{(n\omega + 3n + 4)(n\omega + 3n + 2)}}{\sum_{i=1}^{n} x_i^2 + 2\beta}$
	Al-Bayyati's	$\hat{\theta}_{(Al,Ep)} = \dfrac{n\omega + 3n + 2 + 2c_2}{\sum_{i=1}^{n} x_i^2 + 2\beta}$
	Stein's	$\hat{\theta}_{(St,Ep)} = \dfrac{n\omega + 3n}{\sum x_i^2 + 2\beta}$

4 Simulation Study:

We conduct some simulation studies of weighted Maxwell Boltzmann distribution to examine the performance of the MLEs and Bayesian estimators under different prior's like extension of Jeffreys' prior, Hartigan's prior, Jeffrey's prior, uniform prior, Gamma prior and Chi-square prior under different loss functions in terms of expected estimates, biases, variances and mean squared errors by considering different parameter combinations. Sample size is varied to observe the effect of small, moderate and large samples on the estimators. We have conducted the simulation procedure for different random parameter combinations and the process was repeated 1000 times.

From the results of simulation study, conclusions are drawn regarding the behavior of the estimators in general, which are summarized below.

1. The performances of the Bayesian and MLEs become better when the sample size increases.
2. In terms of MSE, the Bayesian estimators under Gamma prior perform better.

In specific, from Table 4.1, extension of Jeffery's prior under Al-Bayyati's error loss function and stein's loss function gives smaller MSE's as compared to other loss functions.

Table 4.1: Average estimate, Bias, Variance and Mean Squared Error for $(\hat{\theta})$ under extension of Jeffery's prior.

n	θ	ω	c_1	c_2	Criterion	$\hat{\theta}_{ml}$	$\hat{\theta}_{sq}$	$\hat{\theta}_{pre}$	$\hat{\theta}_{alb}$	$\hat{\theta}_{ste}$
10	0.5	0.5	0.5	0.5	$E(\theta)$	0.531764	0.534997	0.520238	0.518804	0.538483
					Bias	0.031764	0.034997	0.020238	0.018804	0.038483
					Variance	0.018123	0.019799	0.018987	0.017248	0.018763
					MSE	0.019132	0.021024	0.019396	0.017602	0.020244
50	0.5	0.5	0.5	0.5	$E(\theta)$	0.503617	0.51494	0.509504	0.507982	0.51186
					Bias	0.003617	0.01494	0.009504	0.007982	0.01186
					Variance	0.002928	0.002992	0.003554	0.002868	0.002902
					MSE	0.002941	0.003215	0.003644	0.002931	0.003043
100	0.5	0.5	0.5	0.5	$E(\theta)$	0.505475	0.499554	0.507244	0.503039	0.500626
					Bias	0.005475	-0.000446	0.007244	0.003039	0.000626
					Variance	0.001510	0.001421	0.001259	0.001205	0.001439
					MSE	0.001540	0.001421	0.001312	0.001214	0.001439
10	1.5	0.5	0.5	0.5	$E(\theta)$	1.599550	1.610747	1.595592	1.577014	1.580433
					Bias	0.099550	0.110747	0.095592	0.077014	0.080433
					Variance	0.130272	0.172294	0.173055	0.118166	0.147757
					MSE	0.140182	0.184559	0.182193	0.124097	0.154226
50	1.5	0.5	0.5	0.5	$E(\theta)$	1.532346	1.518270	1.498868	1.482823	1.501181

					Bias	0.032346	0.018270	-0.001132	-0.017177	0.001181
					Variance	0.024787	0.025524	0.026351	0.019538	0.030989
					MSE	0.025834	0.025858	0.026352	0.019833	0.030991
100	1.5	0.5	0.5	0.5	$E(\theta)$	1.502809	1.526206	1.514425	1.511635	1.512905
					Bias	0.002809	0.026206	0.014425	0.011635	0.012905
					Variance	0.012573	0.012874	0.016993	0.011624	0.014188
					MSE	0.012580	0.013560	0.017201	0.011760	0.014354
10	2.0	1.0	1.5	0.5	$E(\theta)$	1.959142	1.884926	1.903764	1.940722	1.875443
					Bias	-0.040858	-0.115074	-0.096236	-0.059278	-0.124557
					Variance	0.277245	0.184158	0.241101	0.170041	0.217534
					MSE	0.278914	0.197400	0.250362	0.173555	0.233048
50	2.0	1.0	1.5	0.5	$E(\theta)$	2.024636	1.978105	2.003983	1.992229	1.972362
					Bias	0.024636	-0.021895	0.003983	-0.007771	-0.027638
					Variance	0.051391	0.034589	0.043325	0.043106	0.044185
					MSE	0.051998	0.035068	0.043341	0.043166	0.044949
100	2.0	1.0	1.5	0.5	$E(\theta)$	1.978051	2.021427	1.991444	1.984662	1.983395
					Bias	-0.021949	0.021427	-0.008556	-0.015338	-0.016605
					Variance	0.019040	0.019933	0.019731	0.014978	0.020927
					MSE	0.019522	0.020392	0.019805	0.015213	0.021202
10	2.0	1.0	0.5	1.5	$E(\theta)$	2.096885	2.063615	2.105065	2.040458	2.118485
					Bias	0.096885	0.063615	0.105065	0.040458	0.118485
					Variance	0.241319	0.209320	0.328565	0.192860	0.272146
					MSE	0.250706	0.213367	0.339604	0.194497	0.286184
50	2.0	1.0	0.5	1.5	$E(\theta)$	2.003132	1.999828	2.033638	2.020260	2.026221
					Bias	0.003132	-0.000172	0.033638	0.020260	0.026221
					Variance	0.039534	0.038725	0.044800	0.038341	0.044366
					MSE	0.039544	0.038725	0.045932	0.038751	0.045053
100	2.0	1.0	0.5	1.5	$E(\theta)$	2.014654	2.026696	2.031629	1.987300	1.998385
					Bias	0.014654	0.026696	0.031629	-0.012700	-0.001615
					Variance	0.022312	0.020601	0.024237	0.020780	0.021395
					MSE	0.022526	0.021313	0.025237	0.020942	0.021398

ml= Maximum Likelihood, *sq*= Squared error loss function, *pre*= precautionary loss function, *alb*= Al-Bayyati's loss function, *ste*=Stein's Loss function.

Table 4.2: Average estimate, Bias, Variance and Mean Squared Error for $(\hat{\theta})$ under Hartigan's prior.

n	θ	ω	c_1	c_2	Criterion	$\hat{\theta}_{ml}$	$\hat{\theta}_{sq}$	$\hat{\theta}_{pre}$	$\hat{\theta}_{alb}$	$\hat{\theta}_{ste}$
10	0.5	0.5	3/2	0.5	$E(\theta)$	0.428643	0.420860	0.423748	0.425278	0.427150
					Bias	-0.071357	-0.079140	-0.076252	-0.074722	-0.072850
					Variance	0.010714	0.009162	0.008294	0.010153	0.011993
					MSE	0.015806	0.015426	0.014108	0.015736	0.017300
50	0.5	0.5	3/2	0.5	$E(\theta)$	0.443599	0.430645	0.441316	0.429801	0.432513
					Bias	-0.056401	-0.069355	-0.058684	-0.070199	-0.067487
					Variance	0.002294	0.002484	0.002012	0.001681	0.001884
					MSE	0.005475	0.007294	0.005456	0.006609	0.006438
100	0.5	0.5	3/2	0.5	$E(\theta)$	0.438205	0.439445	0.433059	0.434884	0.437340
					Bias	-0.061795	-0.060555	-0.066941	-0.065116	-0.062660
					Variance	0.001070	0.000996	0.001170	0.000743	0.000948
					MSE	0.004889	0.004662	0.005651	0.004983	0.004874

10	1.5	0.5	3/2	0.5	$E(\theta)$	1.248992	1.242909	1.286126	1.286357	1.243080
					Bias	-0.251008	-0.257091	-0.213874	-0.213643	-0.256920
					Variance	0.088653	0.068089	0.129660	0.104914	0.088845
					MSE	0.151658	0.134185	0.175403	0.150558	0.154852
50	1.5	0.5	3/2	0.5	$E(\theta)$	1.302543	1.322182	1.315785	1.290794	1.298256
					Bias	-0.197457	-0.177818	-0.184215	-0.209206	-0.201744
					Variance	0.015040	0.013085	0.016694	0.017968	0.016064
					MSE	0.054029	0.044704	0.050629	0.061735	0.056765
100	1.5	0.5	3/2	0.5	$E(\theta)$	1.294916	1.317093	1.299306	1.307857	1.317998
					Bias	-0.205084	-0.182907	-0.200694	-0.192143	-0.182002
					Variance	0.011964	0.008058	0.008538	0.006709	0.009019
					MSE	0.054024	0.041513	0.048816	0.043628	0.042143
10	2.0	1.0	3/2	1.5	$E(\theta)$	1.896107	1.97495	1.942701	1.939274	2.009747
					Bias	-0.10389	-0.02505	-0.0573	-0.06073	0.009747
					Variance	0.16528	0.235524	0.226484	0.260467	0.2379
					MSE	0.176074	0.236152	0.229767	0.264154	0.237995
50	2.0	1.0	3/2	1.5	$E(\theta)$	1.991933	1.984469	1.955277	2.016252	1.986961
					Bias	-0.00807	-0.01553	-0.04472	0.016252	-0.01304
					Variance	0.042791	0.031745	0.033161	0.046852	0.0362
					MSE	0.042856	0.031986	0.035162	0.047116	0.03637
100	2.0	1.0	3/2	1.5	$E(\theta)$	2.01059	2.004871	1.984062	1.996154	1.984899
					Bias	0.01059	0.004871	-0.01594	-0.00385	-0.0151
					Variance	0.019874	0.017275	0.02029	0.018859	0.01764
					MSE	0.019986	0.017299	0.020544	0.018874	0.017868

ml= Maximum Likelihood, *sq*= Squared error loss function, *pre*= precautionary loss function, *alb*= Al-Bayyati's loss function, *ste*=Stein's Loss function.

From Table 4.2, we can see that the performances of the Bayesian and MLEs become better when the sample size increases. Hartigan's prior under squared error loss function and stein's loss function gives smaller MSE's as compared to other loss functions. Thus, Hartigan's prior under squared error loss function and stein's loss function can be preferred for parameter estimation.

Table 4.3: Average estimate, Bias, Variance and Mean Squared Error for $(\hat{\theta})$ under Jeffery's prior.

n	θ	ω	c_1	c_2	Criterion	$\hat{\theta}_{ml}$	$\hat{\theta}_{sq}$	$\hat{\theta}_{pre}$	$\hat{\theta}_{alb}$	$\hat{\theta}_{ste}$
10	0.5	0.5	1/2	0.5	$E(\theta)$	0.453792	0.471428	0.456535	0.473931	0.455024
					Bias	-0.04621	-0.02857	-0.04347	-0.02607	-0.04498
					Variance	0.012003	0.011383	0.008599	0.016789	0.012807
					MSE	0.014139	0.012199	0.010488	0.017469	0.01483
50	0.5	0.5	1/2	0.5	$E(\theta)$	0.438312	0.443906	0.43689	0.43824	0.444891
					Bias	-0.06169	-0.05609	-0.06311	-0.06176	-0.05511
					Variance	0.001934	0.002788	0.002296	0.002095	0.001799
					MSE	0.005739	0.005935	0.006279	0.005909	0.004836
100	0.5	0.5	1/2	0.5	$E(\theta)$	0.441868	0.438913	0.443871	0.439158	0.444542
					Bias	-0.05813	-0.06109	-0.05613	-0.06084	-0.05546
					Variance	0.000924	0.000959	0.000705	0.000964	0.000978
					MSE	0.004303	0.00469	0.003856	0.004665	0.004053
10	1.5	0.5	1/2	0.5	$E(\theta)$	1.39776	1.434951	1.475452	1.369724	1.410073
					Bias	-0.10224	-0.06505	-0.02455	-0.13028	-0.08993
					Variance	0.093798	0.101177	0.121214	0.09614	0.0914

					MSE	0.104251	0.105408	0.121817	0.113112	0.099487
50	1.5	0.5	1/2	0.5	$E(\theta)$	1.318843	1.307257	1.320483	1.333659	1.307656
					Bias	-0.18116	-0.19274	-0.17952	-0.16634	-0.19234
					Variance	0.013944	0.015033	0.017147	0.021317	0.014775
					MSE	0.046762	0.052183	0.049373	0.048986	0.051772
100	1.5	0.5	1/2	0.5	$E(\theta)$	1.312841	1.300119	1.311205	1.320003	1.310872
					Bias	-0.18716	-0.19988	-0.1888	-0.18	-0.18913
					Variance	0.00702	0.007356	0.006403	0.010931	0.010712
					MSE	0.042049	0.047308	0.042047	0.04333	0.046481
10	2.0	1.0	1/2	1.5	$E(\theta)$	2.094475	2.156969	2.180312	2.176434	2.16855
					Bias	0.094475	0.156969	0.180312	0.176434	0.16855
					Variance	0.194913	0.308231	0.43283	0.32484	0.352547
					MSE	0.203839	0.33287	0.465342	0.355969	0.380956
50	2.0	1.0	1/2	1.5	$E(\theta)$	1.99876	2.008471	1.973695	2.001059	2.000238
					Bias	-0.00124	0.008471	-0.02631	0.001059	0.000238
					Variance	0.036642	0.039834	0.033004	0.040197	0.041898
					MSE	0.036643	0.039906	0.033696	0.040198	0.041898
100	2.0	1.0	1/2	1.5	$E(\theta)$	2.038568	2.043957	2.011401	2.014931	2.002294
					Bias	0.038568	0.043957	0.011401	0.014931	0.002294
					Variance	0.02559	0.019787	0.015701	0.019417	0.01651
					MSE	0.027078	0.021719	0.015831	0.01964	0.016516

ml= Maximum Likelihood, *sq*= Squared error loss function, *pre*= precautionary loss function, *alb*= Al-Bayyati's loss function, *ste*=Stein's Loss function.

From Table 4.3, we can see that the performances of the Bayesian and MLEs become better when the sample size increases. Jeffrey's prior under precautionary loss function gives smaller MSE's as compared to other loss functions.

Table 4.4: Average estimate, Bias, Variance and Mean Squared Error for $(\hat{\theta})$ under Uniform prior.

n	θ	ω	c_1	c_2	Criterion	$\hat{\theta}_{ml}$	$\hat{\theta}_{sq}$	$\hat{\theta}_{pre}$	$\hat{\theta}_{alb}$	$\hat{\theta}_{ste}$
10	0.5	0.5	0	0.5	$E(\theta)$	0.551046	0.575157	0.566045	0.565938	0.552087
					Bias	0.051046	0.075157	0.066045	0.065938	0.052087
					Variance	0.01706	0.026868	0.022625	0.021834	0.019645
					MSE	0.019665	0.032517	0.026987	0.026182	0.022358
50	0.5	0.5	0	0.5	$E(\theta)$	0.504654	0.512814	0.505558	0.502242	0.516267
					Bias	0.004654	0.012814	0.005558	0.002242	0.016267
					Variance	0.003215	0.002832	0.002844	0.002826	0.003289
					MSE	0.003237	0.002996	0.002875	0.002831	0.003554
100	0.5	0.5	0	0.5	$E(\theta)$	0.504082	0.500665	0.504214	0.509798	0.500629
					Bias	0.004082	0.000665	0.004214	0.009798	0.000629
					Variance	0.001255	0.001351	0.001643	0.001322	0.001759
					MSE	0.001271	0.001352	0.001661	0.001418	0.00176
10	1.5	0.5	0	0.5	$E(\theta)$	1.702273	1.704897	1.710616	1.617922	1.660859
					Bias	0.202273	0.204897	0.210616	0.117922	0.160859
					Variance	0.177104	0.218684	0.229024	0.1763	0.177814
					MSE	0.218018	0.260666	0.273383	0.190205	0.20369
50	1.5	0.5	0	0.5	$E(\theta)$	1.526619	1.528865	1.558666	1.550379	1.529697
					Bias	0.026619	0.028865	0.058666	0.050379	0.029697
					Variance	0.034488	0.023227	0.028374	0.024161	0.023936
					MSE	0.035196	0.02406	0.031815	0.026699	0.024818

100	1.5	0.5	0	0.5	$E(\theta)$	1.513331	1.511272	1.512262	1.526888	1.508991
					Bias	0.013331	0.011272	0.012262	0.026888	0.008991
					Variance	0.01263	0.012919	0.013045	0.012629	0.011724
					MSE	0.012808	0.013046	0.013196	0.013352	0.011805
10	2.0	1.0	0	1.5	$E(\theta)$	2.191074	2.204089	2.266874	2.125787	2.252216
					Bias	0.191074	0.204089	0.266874	0.125787	0.252216
					Variance	0.291762	0.21994	0.31334	0.213007	0.269933
					MSE	0.328271	0.261592	0.384562	0.22883	0.333546
50	2.0	1.0	0	1.5	$E(\theta)$	2.036927	2.013713	2.047885	2.027366	2.04506
					Bias	0.036927	0.013713	0.047885	0.027366	0.04506
					Variance	0.053767	0.041889	0.037178	0.036863	0.053688
					MSE	0.055131	0.042077	0.039471	0.037612	0.055718
100	2.0	1.0	0	1.5	$E(\theta)$	2.021633	2.019246	2.019779	2.02441	2.011098
					Bias	0.021633	0.019246	0.019779	0.02441	0.011098
					Variance	0.020048	0.02227	0.026357	0.022009	0.019766
					MSE	0.020516	0.02264	0.026748	0.022605	0.019889

ml= Maximum Likelihood, *sq*= Squared error loss function, *pre*= precautionary loss function, *alb*= Al-Bayyati's loss function, *ste*=Stein's Loss function.

From Table 4.4, we can see that the performances of the Bayesian and MLEs become better when the sample size increases. Uniform prior under Stein's loss function gives smaller MSE's as compared to other loss functions and MLEs.

Table 4.5: Average estimate, Bias, Variance and Mean Squared Error for $(\hat{\theta})$ under Gamma prior.

n	θ	ω	α	β	c_2	Criterion	$\hat{\theta}_{ml}$	$\hat{\theta}_{sq}$	$\hat{\theta}_{pre}$	$\hat{\theta}_{alb}$	$\hat{\theta}_{ste}$
10	0.5	0.5	0.5	0.5	0.5	$E(\theta)$	0.535157	0.551611	0.543904	0.524175	0.536501
						Bias	0.035157	0.051611	0.043904	0.024175	0.036501
						Variance	0.014338	0.016222	0.013776	0.020955	0.013319
						MSE	0.015574	0.018886	0.015703	0.02154	0.014652
50	0.5	0.5	0.5	0.5	0.5	$E(\theta)$	0.517194	0.502874	0.505094	0.509343	0.506384
						Bias	0.017194	0.002874	0.005094	0.009343	0.006384
						Variance	0.003036	0.003514	0.003103	0.003185	0.003136
						MSE	0.003332	0.003522	0.003129	0.003272	0.003177
100	0.5	0.5	0.5	0.5	0.5	$E(\theta)$	0.500249	0.505043	0.505794	0.503998	0.507433
						Bias	0.000249	0.005043	0.005794	0.003998	0.007433
						Variance	0.001711	0.001355	0.001591	0.001459	0.001666
						MSE	0.001711	0.001381	0.001624	0.001475	0.001722
10	1.5	0.5	0.3	0.5	0.5	$E(\theta)$	1.531032	1.576751	1.492511	1.584728	1.576001
						Bias	0.031032	0.076751	-0.00749	0.084728	0.076001
						Variance	0.165335	0.126774	0.128802	0.131472	0.118435
						MSE	0.166298	0.132665	0.128858	0.138651	0.124211
50	1.5	0.5	0.3	0.5	0.5	$E(\theta)$	1.527785	1.546209	1.487002	1.513218	1.502551
						Bias	0.027785	0.046209	-0.013	0.013218	0.002551
						Variance	0.022702	0.031425	0.021552	0.026998	0.021121
						MSE	0.023474	0.03356	0.021721	0.027173	0.021128
100	1.5	0.5	0.3	0.5	0.5	$E(\theta)$	1.499727	1.511451	1.507952	1.488217	1.479905
						Bias	-0.00027	0.011451	0.007952	-0.01178	-0.0201
						Variance	0.015073	0.011254	0.015098	0.014241	0.014446
						MSE	0.015073	0.011386	0.015161	0.014379	0.01485
10	2.0	1.0	0.3	1.5	1.5	$E(\theta)$	1.837421	1.847674	1.916624	1.989052	1.962612

						Bias	-0.16258	-0.15233	-0.08338	-0.01095	-0.03739
						Variance	0.137147	0.1562	0.144334	0.174077	0.202594
						MSE	0.163579	0.179403	0.151286	0.174197	0.203992
50	2.0	1.0	0.3	1.5	1.5	$E(\theta)$	1.956766	1.948355	2.01772	1.949	1.993626
						Bias	-0.04323	-0.05165	0.01772	-0.051	-0.00637
						Variance	0.035192	0.044494	0.043551	0.033277	0.036726
						MSE	0.037061	0.047161	0.043865	0.035878	0.036767
100	2.0	1.0	0.3	1.5	1.5	$E(\theta)$	1.987497	2.002142	1.989095	2.011745	1.986053
						Bias	-0.0125	0.002142	-0.01091	0.011745	-0.01395
						Variance	0.019337	0.016733	0.018671	0.015889	0.024234
						MSE	0.019494	0.016737	0.01879	0.016027	0.024428

ml= Maximum Likelihood, *sq*= Squared error loss function, *pre*= precautionary loss function, *alb*= Al-Bayyati's loss function, *ste*=Stein's Loss function.

From Table 4.5, we can see that the performances of the Bayesian and MLEs become better when the sample size increases. For large samples, Gamma prior under squared error loss function and Al-Bayyati's loss function gives smaller MSE's as compared to other loss functions and MLEs.

Table 4.6: Average estimate, Bias, Variance and Mean Squared Error for $(\hat{\theta})$ under Chi-square prior.

n	θ	ω	α	β	c_2	Criterion	$\hat{\theta}_{ml}$	$\hat{\theta}_{sq}$	$\hat{\theta}_{pre}$	$\hat{\theta}_{alb}$	$\hat{\theta}_{ste}$
10	0.5	0.5	1	0.5	0.5	$E(\theta)$	0.537762	0.534916	0.555513	0.544	0.537518
						Bias	0.037762	0.034916	0.055513	0.044	0.037518
						Variance	0.017204	0.015407	0.018719	0.017708	0.022259
						MSE	0.01863	0.016626	0.021801	0.019644	0.023666
50	0.5	0.5	1	0.5	0.5	$E(\theta)$	0.510991	0.520883	0.51213	0.514233	0.49978
						Bias	0.010991	0.020883	0.01213	0.014233	-0.00022
						Variance	0.003362	0.003338	0.002619	0.002826	0.003596
						MSE	0.003483	0.003774	0.002766	0.003029	0.003596
100	0.5	0.5	1	0.5	0.5	$E(\theta)$	0.502419	0.513712	0.504947	0.504172	0.510825
						Bias	0.002419	0.013712	0.004947	0.004172	0.010825
						Variance	0.001495	0.001044	0.001439	0.001489	0.001421
						MSE	0.001501	0.001232	0.001463	0.001507	0.001538
10	1.5	0.5	1	0.5	0.5	$E(\theta)$	1.607841	1.554018	1.67936	1.630788	1.655879
						Bias	0.107841	0.054018	0.17936	0.130788	0.155879
						Variance	0.190556	0.150285	0.149174	0.137779	0.146063
						MSE	0.202186	0.153203	0.181344	0.154884	0.170361
50	1.5	0.5	1	0.5	0.5	$E(\theta)$	1.49362	1.525872	1.540543	1.533499	1.507352
						Bias	-0.00638	0.025872	0.040543	0.033499	0.007352
						Variance	0.022291	0.02407	0.03457	0.024438	0.02618
						MSE	0.022332	0.024739	0.036214	0.02556	0.026234
100	1.5	0.5	1	0.5	0.5	$E(\theta)$	1.495624	1.502956	1.513457	1.515378	1.521578
						Bias	-0.00438	0.002956	0.013457	0.015378	0.021578
						Variance	0.013522	0.014112	0.015997	0.012318	0.011654
						MSE	0.013541	0.014121	0.016178	0.012555	0.01212
10	2.0	1.0	2	1.5	1.5	$E(\theta)$	1.987128	1.999155	2.056052	2.038252	2.049806
						Bias	-0.01287	-0.00085	0.056052	0.038252	0.049806
						Variance	0.216818	0.224497	0.197312	0.191381	0.174968
						MSE	0.216984	0.224498	0.200454	0.192844	0.177449
50	2.0	1.0	2	1.5	1.5	$E(\theta)$	2.0147	2.009206	2.002258	1.998735	2.019777

						Bias	0.0147	0.009206	0.002258	-0.00127	0.019777
						Variance	0.031948	0.022386	0.041348	0.036296	0.036437
						MSE	0.032164	0.022471	0.041353	0.036297	0.036828
100	2.0	1.0	2	1.5	1.5	$E(\theta)$	2.02209	2.007718	2.000391	1.992023	2.005752
						Bias	0.02209	0.007718	0.000391	-0.00798	0.005752
						Variance	0.019201	0.016932	0.0199	0.019235	0.020151
						MSE	0.019689	0.016991	0.0199	0.019299	0.020184

ml= Maximum Likelihood, *sq*= Squared error loss function, *pre*= precautionary loss function, *alb*= Al-Bayyati's loss function, *ste*=Stein's Loss function.

From Table 4.6, we can see that the performances of the Bayesian and MLEs become better when the sample size increases. Chi-square prior under squared error loss function gives smaller MSE's as compared to other loss functions and MLEs. Hence we expect better estimates, under squared error loss function using Chi-square prior as compared to other loss functions.

5 Conclusions

It has been observed that the performances of the Bayesian and MLEs become better when the sample size increases. Hartigan's prior under squared error loss function and stein's loss function gives smaller MSE's as compared to other loss functions. Thus, Hartigan's prior under squared error loss function and stein's loss function can be preferred for parameter estimation. It has evident that the performances of the Bayesian and MLEs become better when the sample size increases. Jeffrey's prior under precautionary loss function gives smaller MSE's as compared to other loss functions. It has been observed that the performances of the Bayesian and MLEs become better when the sample size increases. Uniform prior under Stein's loss function gives smaller MSE's as compared to other loss functions and MLEs. From Table 4.5, we can see that the performances of the Bayesian and MLEs become better when the sample size increases. For large samples, Gamma prior under squared error loss function and Al-Bayyati's loss function gives smaller MSE's as compared to other loss functions and MLEs. From Table 4.6, we can see that the performances of the Bayesian and MLEs become better when the sample size increases. Chi-square prior under squared error loss function gives smaller MSE's as compared to other loss functions and MLEs. Hence we expect better estimates, under squared error loss function using Chi-square prior as compared to other loss functions. The future research may be considered to estimate the parameters using different loss functions especially Linex loss function and generalized entropy Loss function under different prior distributions like Conjugate priors and double priors etc.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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