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Some Properties of Commuting Graph of the Ring of All $m_1 \oplus m_2$ Matrices

Manal Al-Labadi

Department of Mathematics, Faculty of Art and Sciences, University Of Petra, Amman, Jordan

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Abstract: The commuting graph of a ring $\mathbf{R}$, denoted by $\Gamma(\mathbf{R})$, is a graph whose vertices are all non-central elements of $\mathbf{R}$ and two distinct vertices u and v are adjacent if and only if $uv = vu$. In this paper let $\mathbf{R}$ be the commutative ring with $1_{\mathbf{R}} \neq 0_{\mathbf{R}}$. In this paper we investigate, some basic properties of $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ we find the $g(\Gamma((M(m_1 \oplus m_2, \mathbf{R}))) = 3$ and we show that $\Gamma((M(m_1 \oplus m_2, \mathbf{R}))$ is not Eulerian, and $\Gamma((M(m_1 \oplus m_2, \mathbf{R}))$ is not planar.

Keywords: Commuting graph, direct sum matrices, planar graph

1 Introduction.

We assume that $\mathbf{R}$ be a commutative ring with unity $1_{\mathbf{R}} \neq 0_{\mathbf{R}}$.
The *distance* between two vertices in a graph G, say m_1 and m_2, is the length of the shortest path between m_1 and m_2 in the graph if such a path exists and ∞ if there is no path. The distance between any two vertices is denoted by $d(m_1, m_2)$. For any graph G, the degree of a vertex m, denoted by $deg(m)$, is the number of edges incident with the vertex m, with loops counted twice if exist. The *diameter* of a graph Γ is the maximum distance between any two vertices in the graph, which is denoted by $diam(\Gamma) = max\{d(m_1, m_2) : m_1, m_2 \in \Gamma\}$, the length of a shortest cycle in G is called the **girth** of G, it is denoted by $g(G)$, if the graph has no cycle then **girth** equal to ∞. We denote the set of all $n \times n$ matrices over $\mathbf{R}$ by $M_{n\times n}(\mathbf{R}) = M(n, \mathbf{R})$. Moreover, for any two matrices $X \in M(m \times n, \mathbf{R})$ and $Y \in M(r \times s, \mathbf{R})$, we define $X \oplus Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in M((m+r) \times (n+s), \mathbf{R})$. We denote the set of all direct sum $X \oplus Y$ where $X \in M(n_1, \mathbf{R})$ and $Y \in M(n_2, \mathbf{R})$ by $M(n_1 \oplus n_2, \mathbf{R})$.
For a ring $\mathbf{R}$, we denote the center of $\mathbf{R}$ by $\mathsf{Z}(\mathbf{R})$ and $\mathbf{Z}(\mathbf{R}) = \{u \in \mathbf{R} : uv = vu, \ \forall v \in \mathbf{R}\}$. If u is an element of $\mathbf{R}$, then $C_{\mathbf{R}}(u)$ denotes the centraliser of u in $\mathbf{R}$ and $C_{\mathbf{R}}(u) = \{v \in \mathbf{R} : uv = vu\}$.

The commuting graphs of groups have been studied deeply, we give some examples in [1,2,3,4,5], and examples of rings in [6,7,8,9].

2 Girth for $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$.

Let $\mathbf{R}$ be a commutative ring with unity $1_{\mathbf{R}} \neq 0_{\mathbf{R}}$. In this section we determine the girth of $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$.
Lemma 1. Suppose that $|\mathbf{R}| \geq 3$. Then $g(\Gamma(M(m_1 \oplus m_2, \mathbf{R}))) = 3$.

Proof. Let $a \in \mathbf{R} \setminus \{0, 1\}$. We have the cycle $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ - $\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ - $\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & 0 & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ - $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$; $a \neq 0$. Hence $g(\Gamma(M(m_1 \oplus m_2, \mathbf{R}))) = 3$.

3 When is $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ Eulerian ?

In this section we determine when $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ is Eulerian.
Definition 1. A graph Γ is called **Eulerian** if there exists a closed trail containing every edge of Γ.

The following well known result characterizes when a graph Γ is Eulerian in [10].

* Corresponding author e-mail: manal.allabadi@uop.edu.jo

Proposition 1. A connected finite graph Γ is Eulerian if and only if the degree of each vertex of Γ is even.

Now, we will show that $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ is not Eulerian.

Lemma 2. Let $\mathbf{R}$ be a finite ring such that $|\mathbf{R}|$ is odd. Then for any $X \in \Gamma(M(m_1 \oplus m_2, \mathbf{R}))$, $deg(X)$ is an odd, so $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ can not be Eulerian graph.

*Proof.*Let $X = X_1 \oplus Y_1 \in \Gamma(M(m_1 \oplus m_2, \mathbf{R}))$, then $deg(X) = |C_{\mathbf{R}}(X)|$- $|Z(M(m_1 \oplus m_2, \mathbf{R}))| - 1$, $|C_{\mathbf{R}}(X)|$ and $|Z(M(m_1 \oplus m_2, \mathbf{R}))|$ divide $|\mathbf{R}|$ which is odd and hence $|C_{\mathbf{R}}(X)|$ and $|Z(M(m_1 \oplus m_2, \mathbf{R}))|$ are odd. So, $deg(X)$ =odd-odd-1=odd.

Lemma 3. Let $\mathbf{R}$ be a finite ring such that $|\mathbf{R}|$ is even. Then $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ cannot be Eulerian graph.

*Proof.*Let $X = X_1 \oplus Y_1 \in \Gamma(M(m_1 \oplus m_2, \mathbf{R}))$, then $deg(X) = |C_{\mathbf{R}}(X)| - |Z(M(m_1 \oplus m_2, \mathbf{R}))| - 1$. Let $X_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, $Y_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$. Then

$$C_{\mathbf{R}}(X) = \left\{ \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \oplus \begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \cdots & * \\ 0 & * & \cdots & * \end{pmatrix}, \right.$$
$$\left. Z(M(m_1, \mathbf{R})) \oplus \begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \cdots & * \\ 0 & * & \cdots & * \end{pmatrix}, \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \oplus Z(M(m_2, \mathbf{R})) \right\}$$

where $a,\ b \in \mathbf{R}$ and $\begin{pmatrix} * & \cdots & * \\ \vdots & \cdots & \vdots \\ * & \cdots & * \end{pmatrix} \in M(m_i - 1, \mathbf{R})$ where $i = 1,\ 2$. So $C_{\mathbf{R}}(X) = |\mathbf{R}|^{(m_1-1)(m_1-1)+(m_2-1)(m_2-1)+2} + |\mathbf{R}|^{(m_2-1)(m_2-1)+2} + |\mathbf{R}|^{(m_1-1)(m_1-1)+2}$ which is an even. Then $deg(X) = even - even - 1 = odd$.

Combining the results of Lemma 2 and Lemma 3 we get the following theorem.

Theorem 1. Let $\mathbf{R}$ be a finite ring. Then the commuting graph $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ is not Eulerian.

4 When is $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ Planar ?

Definition 2. A graph Γ is called planar if it can be drawn in a plane with crossing of the edges are only at the vertices of the graph.

We use the following results to show that $\Gamma(M(2 \oplus 2, \mathbf{R}))$ is not Planar, when $|\mathbf{R}| \geq 4$. The following two lemmas were proved in [11] and [2] respectively.

Lemma 4. Let $\mathbf{G}$ be a simple connected planar graph. Then G has at least one vertex of degree less than 6.

Lemma 5. Let $\mathbf{R}$ be an integral domain with order greater than or equal 4. Then the graph $\Gamma(M(2 \oplus 2, \mathbf{R}))$ is a disconnected graph.

Now, we will investigate when $\Gamma(M(2 \oplus 2, \mathbf{R}))$ is planar where $|\mathbf{R}| \geq 4$. Consider the following lemma.

Lemma 6. For any matrix $X \in M(2 \oplus 2, \mathbf{R}) \setminus Z(M(2 \oplus 2, \mathbf{R}))$, the degree of X is the graph $\Gamma(M(2 \oplus 2, \mathbf{R}))$ is greater than or equal to 6.

*Proof.*Let $X = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \oplus Z_1 \in M(2 \oplus 2, \mathbf{R}) \setminus Z(M(2 \oplus 2, \mathbf{R}))$. Suppose $Y = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \oplus Z_2 \in M(2 \oplus 2, \mathbf{R}) \setminus Z(M(2 \oplus 2, \mathbf{R})$ is a matrix that commutes with X. We have several cases to consider.

– **Case 1**: Suppose b_1 is a unit. Then $XY = \begin{pmatrix} a_1u_1 + b_1u_3 & a_1u_2 + b_1u_4 \\ c_1u_1 + d_1u_3 & c_1u_2 + d_1u_4 \end{pmatrix} \oplus Z_1Z_2 = \begin{pmatrix} a_1u_1 + c_1u_2 & b_1u_1 + d_1u_2 \\ a_1u_3 + c_1u_4 & b_1u_3 + d_1u_4 \end{pmatrix} \oplus Z_2Z_1 = YX$. So, $b_1u_3 = c_1u_2$, $u_3 = b_1{}^{-1}c_1u_2$, $u_4 = u_1 + b_1{}^{-1}(d_1 - a_1)u_2$. So, X is adjacent to every matrix of the form $\begin{pmatrix} u_1 & u_2 \\ b_1{}^{-1}c_1u_2 & u_1 + b_1{}^{-1}(d_1 - a_1)u_2 \end{pmatrix} \oplus Z$. So, $deg(X) \geq |\mathbf{R}|^2 - |\mathbf{R}| - 1 \geq 6$.

– **Case 2**: Suppose c_1 is a unit. Then $XY = \begin{pmatrix} a_1u_1 + bu_3 & a_1u_2 + b_1u_4 \\ c_1u_1 + du_3 & c_1u_2 + d_1u_4 \end{pmatrix} \oplus Z_1Z_2 = \begin{pmatrix} a_1u_1 + c_1u_2 & b_1u_1 + d_1u_2 \\ a_1u_3 + c_1u_4 & b_1u_3 + d_1u_4 \end{pmatrix} \oplus Z_2Z_1 = YX$. So, $b_1u_3 = c_1u_2$, $u_2 = c_1{}^{-1}b_1u_3$, $u_4 = u_1 + c_1{}^{-1}(d_1 - a_1)u_3$. So, X is adjacent to every matrix of the form $\begin{pmatrix} u_1 & c^{-1}bu_3 \\ u_3 & u_1 + c^{-1}(d - a)u_3 \end{pmatrix} \oplus Z$. Then $deg(X) \geq |\mathbf{R}|^2 - |\mathbf{R}| - 1 \geq 6$.

– **Case 3**: Suppose that neither c_1 nor b_1 is a unit. Then

– **Subcase 3.1**: If $b_1 = c_1 = 0$, then $X = \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix} \oplus Z_1$. Consider $Y = \begin{pmatrix} u_1 & 0 \\ 0 & u_4 \end{pmatrix} \oplus Z_2$, then $XY = \begin{pmatrix} a_1u_1 & 0 \\ 0 & d_1u_4 \end{pmatrix} \oplus Z_1Z_2 = \begin{pmatrix} a_1u_1 & 0 \\ 0 & d_1u_4 \end{pmatrix} \oplus Z_2Z_1 = YX$. Thus X is adjacent to every matrix of the form $\begin{pmatrix} u_1 & 0 \\ 0 & u_4 \end{pmatrix} \oplus Z$. Hence $deg(X) \geq |\mathbf{R}|^2 - |\mathbf{R}| - 1 \geq 6$.

– **Subcase 3.2**: If the matrix X has the form $\begin{pmatrix} a_1 & 0 \\ c_1 & d_1 \end{pmatrix} \oplus Z_1$, $c_1 \neq 0$, $a_1 \neq d_1$. Suppose that $Y = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \oplus Z_2 \in C_{M(2 \oplus 2, \mathbf{R})}(X)$. Then

$XY = \begin{pmatrix} a_1u_1 & a_1u_2 \\ c_1u_1+du_3 & c_1u_2+du_4 \end{pmatrix} \oplus Z_1Z_2 = \begin{pmatrix} a_1u_1+cu_2 & d_1u_2 \\ a_1u_3+cu_4 & d_1u_4 \end{pmatrix} \oplus Z_2Z_1 = YX$. So, $c_1u_2 = 0$, $(a_1-d_1)u_2 = 0$ and $c_1(u_1-u_4) = (a_1-d_1)u_3$.

If (a_1-d_1) is a unit, then we can take $u_3 = (a_1-d_1)^{-1}c_1(u_1-u_4)$ and $u_2 = 0$. So, X is adjacent to every matrix of the form $\begin{pmatrix} u_1 & 0 \\ (a_1-d_1)^{-1}c_1(u_1-u_4) & u_4 \end{pmatrix} \oplus Z$. Hence $deg(X) \geq |\mathbf{R}|^2 - |\mathbf{R}| - 1 \geq 6$.
If (a_1-d_1) is a zero divisor, then there exists nonzero element, say $(a_1-d_1)^*$, with $(a_1-d_1)(a_1-d_1)^* = 0$. Also c_1 is a zero divisor, so there exists nonzero element say $c_1{}^*$ with $c_1c_1{}^* = 0$. One can easily check that X is adjacent to every matrix of the form $\begin{pmatrix} u_4+k_jc_1{}^* & 0 \\ n_j(a_1-d_1)^* & u_4 \end{pmatrix} \oplus Z$ where $k_j,\ n_j \in \{0,1\}$. Hence $deg(X) \geq 2.2.|\mathbf{R}| - |\mathbf{R}| - 1 \geq 6$.

–Subcase 3.3: If the matrix X has the form $X = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \oplus Z_1$, $b_1 \neq 0$, then one can check that X is adjacent to every matrix of the form $\begin{pmatrix} u_1 & u_2 \\ 0 & u_4 \end{pmatrix} \oplus Z$, for all $u_1,\ u_2 \in \mathbf{R}$. Hence $deg(X) \geq |\mathbf{R}|^2 - |\mathbf{R}| - 1 \geq 6$.

–Subcase 3.4: If the matrix X has the form $\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \oplus Z_1$, $b_1 \neq 0$, $a_1 \neq d_1$. Suppose that $Y = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \oplus Z_2 \in C_{M(2\oplus 2\mathbf{R})}(X)$. Then $XY = \begin{pmatrix} a_1u_1+b_1u_3 & a_1u_2+b_1u_4 \\ d_1u_3 & d_1u_4 \end{pmatrix} \oplus Z_1Z_2 = \begin{pmatrix} a_1u_1 & b_1u_1+d_1u_2 \\ a_1u_3 & b_1u_3+d_1u_4 \end{pmatrix} \oplus Z_2Z_1 = YX$. So, $b_1u_3 = 0$, $(d_1-a_1)u_2 = b_1(u_4-u_1)$.
If (d_1-a_1) is a unit, then we can take $u_2 = (d_1-a_1)^{-1}b_1(u_4-u_1)$ and $u_3 = m_jb^*$. So, X is adjacent to every matrix of the form $\begin{pmatrix} u_1 & (d_1-a_1)^{-1}b_1(u_4-u_1) \\ 0 & u_4 \end{pmatrix} \oplus Z$. Then $deg(X) \geq |\mathbf{R}|^2 - |\mathbf{R}| - 1 \geq 6$.
If (d_1-a_1) is a zero divisor, then there exists nonzero element, say $(d_1-a_1)^*$, with $(d_1-a_1)(d_1-a_1)^* = 0$. Also b_1 is a zero divisor, so there exists nonzero element say b^* with $b_1b^* = 0$. One can easily check that X is adjacent to every matrix of the form $\begin{pmatrix} u_1 & n_j(d_1-a_1)^* \\ 0 & u_1+k_jb^* \end{pmatrix} \oplus Z$, $k_j,\ n_j \in \{0,1\}$. Then $deg(X) \geq 2.2.|\mathbf{R}| - |\mathbf{R}| - 1 \geq 6$.

–Subcase 3.5: If the matrix X has the form $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \oplus Z_1$ where $b_1,\ c_1 \neq 0$, $b_1,\ c_1$ are zero divisors. If $Y \in C_{M(2\oplus 2,\mathbf{R})}(X)$, then $XY = \begin{pmatrix} a_1u_1+b_1u_3 & b_1u_4+a_1u_2 \\ a_1u_3+c_1u_1 & c_1u_2+a_1u_4 \end{pmatrix} \oplus Z_1Z_2 = \begin{pmatrix} a_1u_1+c_1u_2 & a_1u_2+b_1u_1 \\ a_1u_3+c_1u_4 & a_1u_4+b_1u_3 \end{pmatrix} \oplus Z_2Z_1 = YX$. Since b_1 and c_1 are zero divisors there exists $b^*,\ c^* \neq 0$ with $b_1b^* = 0$ and $c_1c^* = 0$. So, the matrix X is adjacent to all matrices of the form $\begin{pmatrix} u_1 & m_jc^* \\ n_jb^* & u_1 \end{pmatrix} \oplus Z$ where $m_j,\ n_j \in \{0,1\}$, $u_1 \in \mathbf{R}$. Thus $deg(X) \geq 2.2.|\mathbf{R}| - |\mathbf{R}| - 1 \geq 6$.

–Subcase 3.6: If the matrix X has the form $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \oplus Z_1$ where $b_1,\ c_1$ are nonzero zero divisors. Since b_1 and c_1 are nonzero zero divisors then there exists $b^*,\ c^* \neq 0$ such that $b_1b^* = 0$, $c_1c^* = 0$. So, the matrix X is adjacent to every matrix of the form $\begin{pmatrix} a_1+c_2 & b_1 \\ c_1 & d_1+c_2 \end{pmatrix} \oplus Z$ where $c_2 \in \mathbf{R}$. Also X is adjacent to every matrix of the form $\begin{pmatrix} b^*a+c_2 & 0 \\ b^*c_1 & b^*d+c_2 \end{pmatrix}$.
If $b^*c_1 \neq 0$, then $deg(X) \geq (|\mathbf{R}|-1) + |\mathbf{R}| \geq 6$.
If $b^*c_1 = 0$, then X is adjacent to all matrices of the form $\begin{pmatrix} a+c_2 & b \\ c_1 & d+c_2 \end{pmatrix}$ and all matrices of the form $\begin{pmatrix} m_jb^*+c_3 & 0 \\ 0 & l_jb^*+c_3 \end{pmatrix}$ where $c_2,\ c_3 \in \mathbf{R}$. So, $deg(X) \geq (|\mathbf{R}|-1) + (2.2.|\mathbf{R}| - |\mathbf{R}|) \geq 6$.

Now, we give the final result that shows that $\Gamma(M(2\oplus 2,\mathbf{R}))$ is not Planar, when $|\mathbf{R}| \geq 4$.
Theorem 2. Suppose that $\mathbf{R}$ is a finite ring with $|\mathbf{R}| \geq 4$. Then $\Gamma(M(2\oplus 2,\mathbf{R}))$ is not Planar.

*Proof.*Using the previous lemma, every vertex of $\Gamma(M(2\oplus 2,\mathbf{R}))$ has degree greater than 6. Hence by lemma 4, is not Planar.

Theorem 3. Suppose that $\mathbf{R}$ is a finite ring. Then $\Gamma(M(m_1\oplus m_2,\mathbf{R}))$ is not Planar.

*Proof.*Let X be any matrix $X \in M(m_1\oplus m_2,\mathbf{R}) \setminus Z(M(m_1\oplus m_2,\mathbf{R}))$. Then $X = A_1 \oplus B_1 \in M(m_1\oplus m_2,\mathbf{R}) \setminus Z(M(m_1\oplus m_2,\mathbf{R}))$ is adjacent to every matrix of the form $\{A_1 + c_1 \oplus B_1 + c_2, Z(M(m_1,\mathbf{R})) \oplus B_1 + c_2, A_1 + c_1 \oplus Z(M(m_2,\mathbf{R}))\}$ where $c_1,\ c_2 \in \mathbf{R}$. So, $deg(X) \geq 3|\mathbf{R}|^2 - |\mathbf{R}|^2 - 1 \geq 6$. Hence by lemma 4, $\Gamma(M(m_1\oplus m_2,\mathbf{R}))$ is not Planar.

5 Perspective.

In this article, We give, some basic properties of $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ we find the $g(\Gamma((M(m_1 \oplus m_2, \mathbf{R}))) = 3$, $\Gamma((M(m_1 \oplus m_2, \mathbf{R}))$ is not Eulerian, and $\Gamma((M(m_1 \oplus m_2, \mathbf{R}))$ is not planar.

One can ask the following questions:

– **(1)** When the complement of commuting graph $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ is Planar graph?

– **(2)** When the complement of commuting graph $\Gamma(M(m_1 \oplus m_2, \mathbf{R}))$ is Eulerian graph ?

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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