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Fractional Frobenius Series Solutions of Confluent α-Hypergeometric Differential Equation

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Abstract: In this work, the so-called conformable fractional derivative definition is employed to obtain the fractional Frobenius series solutions around the regular α-singular point $x = 0$ for the confluent α-hypergeometric differential equation. Of course, such solutions for this equation in its classical case are just confluent hypergeometric functions. The proposed method is straightforward to be applied as an algorithm.

Keywords: Confluent hypergeometric function, confluent hypergeometric differential equations, regular α-singular point, conformable fractional derivative definition, fractional Frobenius series.

1 Introduction

The confluent hypergeometric functions have proved their useful and efficiency in several branches of physics. They have been used in many problems involving both diffusion and sedimentation [1]. For instance, such functions can express the solution of the equation for the velocity distribution of electrons in high frequency gas discharges. The high frequency breakdown electric field may then be predicted theoretically for gases by the use of such solutions together with kinetic theory [1]. The confluent hypergeometric function $F(a,b,x)$ means a solution for the confluent hypergeometric equation [1–5]:

$$xy^{''}(x) + (b-x)y^{'}(x) - ay(x) = 0, \tag{1}$$

where a and b are real constants and x is an independent variable.

One may suggest a new form for (1). This form will be called the confluent α-hypergeometric differential equation; $0 < \alpha \leq 1$, which represents (1) itself, but in the fractional case. It may be defined as:

$$x^{\alpha}y^{(2\alpha)}(x) + (\alpha b - x^{\alpha})y^{(\alpha)}(x) - \alpha a y(x) = 0, \tag{2}$$

where a is a real constant, b is a positive real constant, and x is an independent variable.

Throughout this work, the conformable fractional derivative definition, which was introduced in [6] will be utilized, in order to present fractional Frobenius series solutions around the regular α-singular point $x = 0$ for the confluent α-hypergeometric differential equation (i.e. Eq. (2)). Such definition coincides with the classical definitions on polynomials. Further, if $\alpha = 1$, the definition coincides with the classical definition of first derivative [6]. However, the organization of this paper is as follows: In the next section necessary definitions and preliminaries related to the conformable fractional calculus, Frobenius method, and the confluent hypergeometric functions are introduced. Section 3 describes how to use the conformable fractional derivative definition to obtain the general solution of the confluent α-hypergeometric differential equation. Section 4 introduces some simulations of the general solution of the confluent α-hypergeometric differential equation, followed by the final section which presents the conclusion of this work.

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2 Preliminaries

In this section, some essential basic concepts and preliminaries about conformable fractional calculus, Frobenius method, and the confluent hypergeometric functions are presented. These and other related results can be found in [2,4,6,7].

Definition 2.1. Given a function $f : [0,\infty) \to \mathbb{R}$. Then the conformable fractional derivative of f of order α is defined by [6]:

$$f^{(\alpha)}(x) = \lim_{\varepsilon\to\infty} \frac{f\left(x+\varepsilon x^{1-\alpha}\right) - f(x)}{\varepsilon}, \tag{3a}$$

for all $t > 0$, $\alpha \in (0,1]$. If f is an α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{x\to 0^+} f^{(\alpha)}(x)$ exists, then define:

$$f^{(\alpha)}(0) = \lim_{x\to 0^+} f^{(\alpha)}(x). \tag{3b}$$

One can verify that $f^{(\alpha)}$ satisfies the following property ([6]):

$$f^{(\alpha)}\left(x^{\beta}\right) = \beta x^{\beta-\alpha},\ x > 0,\ \alpha \in (0,\ 1],\ \forall\beta \in \mathbb{R}. \tag{4}$$

In order to our proposed method be applicable, the following preliminaries related to Frobenius method and confluent hypergeometric functions are listed for completeness.

Definition 2.2. Let $\alpha \in (0,\ 1]$, $f(x)$ be a real function defined on the interval $[c,\ d]$, and $x_0 \in [c,\ d]$. Then $f(x)$ is said to be α-analytic at x_0 if there exists an interval $N(x_0)$ such that, for all $x \in N(x_0)$, $f(x)$ can be expressed as a series of natural powers of $(x-x_0)^{\alpha}$. That is, $f(x)$ can be expressed as $\sum_{n=0}^{\infty} C_n(x-x_0)^{n\alpha}$, where $C_n \in \mathbb{R}$. This series being absolutely convergent for $|x-x_0| < p$, where $p > 0$. The radius of convergence of the series is p [7].

Definition 2.3. A point $x_0 \in [c,]$ is said to be an α-ordinary point of (2), if the functions $\frac{\alpha b - x^{\alpha}}{x^{\alpha}}$ and $\frac{\alpha ab}{x^{\alpha}}$ are α-analytic in x_0. A point $x_0 \in [c,\ d]$ which is not α-ordinary will be called α-singular [7].

Definition 2.4. Let $x_0 \in [c,\ d]$ be an α-singular point of (2). Then x_0 is said to be a regular α-singular point of this equation if the functions $(x-x_0)^{(n-k)\alpha}.\frac{\alpha b - x^{\alpha}}{x^{\alpha}}$ and $(x-x_0)^{(n-k)\alpha}.\frac{\alpha ab}{x^{\alpha}}$ are α-analytic in x_0. Otherwise, x_0 is said to be an essential α-singular point [7].

Remark 2.5. According to the above three definitions, one might conclude that if $x_0 = 0 \in [c,\ d]$, then $x_0 = 0$ is just an α-singular point. Further, $x_0 = 0$ is also a regular α-singular point of (2). Now, we are ready to define the well-known Frobenius method so as to obtain solutions for (2) around regular α-singular points; $x_0 = 0$.

Definition 2.6. A series is called a fractional Frobenius series solution around regular α-singular points $x_0 = 0$, if it can be written in the form $\sum_{n=0}^{\infty} C_n x^{n\alpha+k}$, for $\alpha \in (0,\ 1]$, where $k,\ C_n \in \mathbb{R}$ [2].

Definition 2.8. The confluent hypergeometric function is defined by [4,7]:

$$F(a;b;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \tag{5}$$

where $x,\ a \in \mathbb{C}$, b is a non-positive integer, and where the Pochhammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\times\cdots\times(a+n-1)$, $n = 1,\ 2,\ 3,\ldots$.

3 Confluent α-hypergeometric differential equation

In this section, the conformable fractional derivative definition will be adopted to obtain the general solution of the confluent α-hypergeometric differential equation defined by (2). This will be shown by stating and proving the following new theorem which outlines the main result of this work, followed by two new results which they could be derived based on such theorem.

Theorem 3.1. If $x_0 = 0$ is a regular α-singular point of (2); $0 < \alpha \leq 1$. Then;

$$y = C_1 y_1(x) + C_2 y_2(x), \tag{6a}$$

is the general solution of this equation, where:

$$y_1(x) = C_0\left[1 + \frac{a}{\alpha b}x^{\alpha} + \frac{a(a+1)}{\alpha^2 b(b+1)}\frac{x^{2\alpha}}{2!} + \frac{a(a+1)(a+2)}{\alpha^3 b(b+1)(b+2)}\frac{x^{3\alpha}}{3!} + \dots\right], \tag{6b}$$

and

$$y_2(x) = b_0 x^{\alpha-\alpha b}\left[1 + \frac{a+1-b}{\alpha(2-b)}x^{\alpha} + \frac{(a+1-b)(a+1-b+1)}{\alpha^2(2-b)(2-b+1)}\frac{x^{2\alpha}}{2!} + \dots\right], \tag{6c}$$

in which $y_1(x)$ and $y_2(x)$ are two linearly independent solutions.
Proof: According to Remark 2.5; we have just noticed that $x_0 = 0$ is a regular α-singular point of (2). Thus; one may assume that $y = \sum_{n=0}^{\infty} C_n x^{n\alpha+k}$ is a fractional Frobenius series solution around $x_0 = 0$, where $k,\ C_n \in \mathbb{R}$. Based on the conformable fractional derivative property in (4), one can obtain the following states:

$$y^{(\alpha)} = \sum_{n=0}^{\infty}(n\alpha+k)C_n x^{n\alpha+k-\alpha}, \tag{7}$$

and

$$y^{(2\alpha)} = \sum_{n=0}^{\infty}(n\alpha+k)(n\alpha+k-\alpha)C_n x^{n\alpha+k-2\alpha}. \tag{8}$$

Consequently,

$$\begin{aligned} x^{\alpha}y^{(2\alpha)} &= \sum_{n=0}^{\infty}(n\alpha+k)(n\alpha+k-\alpha)C_n x^{n\alpha+k-\alpha} = k(k-\alpha)C_0 + \sum_{n=1}^{\infty}(n\alpha+k)(n\alpha+k-\alpha)C_n x^{n\alpha+k-\alpha} \\ &= k(k-\alpha)C_0 + \sum_{n=0}^{\infty}(n\alpha+k)(n\alpha+k+\alpha)C_{n+1}x^{n\alpha+k}, \end{aligned} \tag{9}$$

$$\alpha b y^{(\alpha)} = \alpha b\sum_{n=0}^{\infty}(n\alpha+k)C_n x^{n\alpha+k-\alpha} = \alpha bkC_0 + \alpha b\sum_{n=1}^{\infty}(n\alpha+k)C_n x^{n\alpha+k-\alpha} = \alpha bkC_0 + \alpha b\sum_{n=0}^{\infty}(n\alpha+k+\alpha)C_{n+1}x^{n\alpha+k}, \tag{10}$$

$$x^{\alpha}y^{(\alpha)} = \sum_{n=0}^{\infty}(n\alpha+k)C_n x^{n\alpha+k}, \tag{11}$$

and

$$\alpha a y = \sum_{n=0}^{\infty}\alpha a C_n x^{n\alpha+k}. \tag{12}$$

Substituting (9), (10), (11) and (12) in (2) yields;

$$k(k-\alpha)C_0 + \alpha bkC_0 + \sum_{n=0}^{\infty}\{[(n\alpha+k)(n\alpha+k+\alpha) + \alpha b(n\alpha+k+\alpha)]C_{n+1} - [(n\alpha+k)+\alpha a]C_n\}x^{n\alpha+k} = 0. \tag{13}$$

Hence,

$$kC_0[(k-\alpha)+\alpha bk] = 0. \tag{14}$$

Since $C_0 \neq 0$, then:

$$(k-\alpha)+\alpha bk = 0. \tag{15}$$

This means that either $k = 0$ or $k = \alpha - \alpha b = \alpha(1-b)$. Moreover, one might deduce from (13) that:

$$C_{n+1} = \frac{n\alpha+k+\alpha a}{(n\alpha+k+\alpha)(n\alpha+k)+\alpha b(n\alpha+k+\alpha)}C_n; n = 0,\ 1,\ 2,\ \dots \tag{16}$$

Based on the previous discussion, we should concern with the following two cases:

***Case 1*:** If $k=0$ then;

$$C_{n+1}=\frac{n\alpha+\alpha a}{(n\alpha+\alpha)(n\alpha)+\alpha b(n\alpha+\alpha)}C_n=\frac{n+a}{\alpha[(n+1)n+b(n+1)]}C_n;\ n=0,\ 1,\ 2,\ \ldots. \tag{17}$$

Now, one can obtain the following recursive states:

- For $n=0$:
$$C_1=\frac{a}{\alpha b}C_0. \tag{18}$$
- For $n=1$:
$$C_2=\frac{(a+1)}{\alpha[2+2b]}C_1=\frac{(a+1)}{2\alpha(b+1)}C_1=\frac{a(a+1)}{2!\alpha^2 b(b+1)}C_0. \tag{19}$$
- For $n=2$:
$$C_3=\frac{2+a}{\alpha[(3)(2)+3b]}C_2=\frac{3+a}{3\alpha[2+b]}C_2=\frac{a(a+1)(a+2)}{3!\alpha^3 b(b+1)(b+2)}C_0. \tag{20}$$
- For $n=3$:
$$C_4=\frac{3+a}{\alpha[(4)(3)+4b]}C_3=\frac{a+3}{4\alpha(b+3)}C_3=\frac{a(a+1)(a+2)(a+3)}{4!\alpha^4 b(b+1)(b+2)(b+3)}C_0. \tag{21}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

As $y(x)=\sum_{n=0}^{\infty}C_n x^{n\alpha+k}$, then (6b) hold.

***Case 2*:** If $k=\alpha(1-b)$. Let us exchange, here, b_n instead of C_n in (16), then we obtain:

$$b_{n+1}=\frac{n+1-b+a}{\alpha[(n+2-b)(n+1-b)+b(n+2-b)]}b_n. \tag{22}$$

Again, based on (22), we can obtain the following recursive states:

- For $n=0$:
$$b_1=\frac{a+1-b}{\alpha[(2-b)(1-b)+b(2-b)]}b_0=\frac{a+1-b}{\alpha(2-b)}b_0. \tag{23}$$
- For $n=1$:
$$b_2=\frac{(a+1-b)(a+1-b+1)}{2!\alpha^2(2-b)(2-b+1)}b_0. \tag{24}$$
- For $n=2$:
$$b_3=\frac{(a+1-b)(a+1-b+1)(a+1-b+2)}{3!\alpha^3(2-b)(2-b+1)(2-b+2)}b_0. \tag{25}$$
- For $n=3$:
$$b_4=\frac{(a+1-b)(a+1-b+1)(a+1-b+2)(a+1-b+3)}{4!\alpha^4(2-b)(2-b+1)(2-b+2)(2-b+3)}b_0. \tag{26}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Now based on our assumption, one can obtain $y_2(x)$ in (6c). In order to complete the prove; we should prove that $y_1(x)$ and $y_2(x)$ are two linearly independent solutions for (2). For this purpose, let k_1 and k_2 be two scalers such that:

$$k_1 y_1(x)+k_2 y_2(x)=0. \tag{27}$$

That is;

$$\begin{aligned}k_1C_0&\left[1+\frac{a}{\alpha b}x^{\alpha}+\frac{a(a+1)}{\alpha^2 b(b+1)}\frac{x^{2\alpha}}{2!}+\frac{a(a+1)(a+2)}{\alpha^3 b(b+1)(b+2)}\frac{x^{3\alpha}}{3!}+\ldots\right]\\&+k_2b_0x^{\alpha-\alpha b}\left[1+\frac{a+1-b}{\alpha(2-b)}x^{\alpha}+\frac{(a+1-b)(a+1-b+1)}{\alpha^2(2-b)(2-b+1)}\frac{x^{2\alpha}}{2!}+\ldots\right]=0.\end{aligned} \tag{28}$$

Obviously, (28) holds iff $k_1 = k_2 = 0$, and this complete the proof. $\square$

Corollary 3.2. If $C_0 = b_0 = 1$, in (6b) and (6c) respectively, then the general solution (6a) of the problem given in (2) can be written in the following form:

$$y(x) = C_1 F\left(a, b, \frac{x^\alpha}{\alpha}\right) + C_2 x^{\alpha-\alpha b} F\left(a+1-b, 2-b, \frac{x^\alpha}{\alpha}\right). \tag{29}$$

where $F(\cdot,\ \cdot,\ \cdot)$ is the confluent α-hypergeometric function.
Proof: In fact, if $C_0 = 1$ and $b_0 = 1$, then $y_1(x)$ and $y_2(x)$ will be, directly, as:

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(a)_n}{\alpha^n (b)_n} \frac{x^{n\alpha}}{n!} = F\left(a, b, \frac{x^\alpha}{\alpha}\right), \tag{30}$$

and

$$y_2(x) = x^{\alpha-\alpha b} \sum_{n=0}^{\infty} \frac{(a+1-b)_n}{\alpha(2-b)_n} \frac{x^{n\alpha}}{n!} = x^{\alpha-\alpha b} F\left(a+1-b, 2-b, \frac{x^\alpha}{\alpha}\right). \tag{31}$$

Hence (29) is also hold. $\square$

Lemma 3.3. Let $F\left(a,\ b, \frac{x^\alpha}{\alpha}\right)$ be a confluent α-hypergeometric function in which $x,\ a \in \mathbb{C}$; b is a non-positive integer and $0 < \alpha \leq 1$, then:

I. $$\frac{d^\alpha}{dx^\alpha} F\left(a, b, \frac{x^\alpha}{\alpha}\right) = \frac{a}{b} F\left(a+1, b+1, \frac{x^\alpha}{\alpha}\right). \tag{32}$$

II. $$F\left(a, b, \frac{x^\alpha}{\alpha}\right) = \frac{1}{B(a,\ \ b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1} e^{\frac{x^\alpha t}{\alpha}} .dt. \tag{33}$$

III. $$F\left(a, b, \frac{x^\alpha}{\alpha}\right) = -e^{\frac{x^\alpha}{\alpha}} F\left(b-a,\ b, -\frac{x^\alpha}{\alpha}\right), \tag{34}$$

where $B(\cdot,\ \cdot)$ is the Beta function.
Proof: I. The proof of this part begins from the deduced result in (30). That is;

$$F\left(a, b, \frac{x^\alpha}{\alpha}\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{\alpha^n (b)_n} \frac{x^{n\alpha}}{n!}. \tag{35}$$

By operating $\frac{d^\alpha}{dx^\alpha}(\cdot)$ to the both sides of (40), one obtains:

$$\frac{d^\alpha}{dx^\alpha} F\left(a, b, \frac{x^\alpha}{\alpha}\right) = \sum_{n=1}^{\infty} \frac{(a)_n}{\alpha^n (b)_n} \frac{(n\alpha) x^{n\alpha-\alpha}}{n!} = \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{\alpha^n (b)_{n+1}} \frac{x^{n\alpha}}{n!}. \tag{36}$$

In other words;

$$\frac{d^\alpha}{dx^\alpha} F\left(a, b, \frac{x^\alpha}{\alpha}\right) = \frac{a}{b} \sum_{n=0}^{\infty} \frac{(a+1)_n}{\alpha^n (b+1)_n} \frac{x^{n\alpha}}{n!}, \tag{37}$$

which completes the proof of this part. However, Figure 1 shows the behavior of such derivative of the confluent α-hypergeometric function given in (32) for different values of α. ***II.*** This part begins also by (30), which is equivalent to:

$$\begin{aligned} F\left(a, b, \frac{x^\alpha}{\alpha}\right) &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+n)} \frac{x^{n\alpha}}{\alpha^n n!} = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b-a)}{\Gamma(b-a+a+n)} \frac{x^{n\alpha}}{\alpha^n n!} \\ &= \frac{1}{B(a,\ \ b-a)} \sum_{n=0}^{\infty} \left[\left(\int_0^1 t^{a+n-1}(1-t)^{b-a-1} .dt\right) \frac{x^{n\alpha}}{\alpha^n n!}\right]. \end{aligned} \tag{38}$$

Consequently;

$$F\left(a, b, \frac{x^\alpha}{\alpha}\right) = \frac{1}{B(a,\ \ b-a)} \int_0^1 \left[t^{a-1}(1-t)^{b-a-1} \sum_{n=0}^{\infty} \left(\frac{\left(\frac{tx^\alpha}{\alpha}\right)^n}{n!}\right)\right] .dt, \tag{39}$$

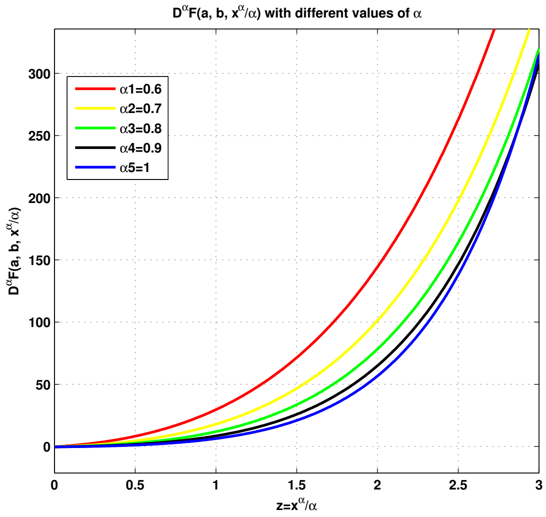

Fig. 1: The behavior of the derivative of the confluent α-hypergeometric function given in (32) for different values of α.

which yields (38).

III. For this part; the result in (38) will be considered. Replacing $(b-a)$ and $(-x^{\alpha})$ instead of (a) and (x^{α}) in (38), respectively, yields:

$$F\left(b-a,b,\frac{-x^{\alpha}}{\alpha}\right)=\frac{1}{B(b-a,\ a)}\int_0^1 t^{b-a-1}(1-t)^{a-1}e^{\frac{-x^{\alpha}t}{\alpha}}.dt. \tag{40}$$

Letting $u=1-t$ and considering one of the most well-known properties of Beta function, $B(b-a,\ a)=B(a,\ b-a)$, in (40) give:

$$F\left(b-a,b,\frac{-x^{\alpha}}{\alpha}\right)=\frac{-e^{\frac{-x^{\alpha}}{\alpha}}}{B(a,\ \ b-a)}\int_0^1 u^{a-1}(1-u)^{b-a-1}e^{\frac{x^{\alpha}u}{\alpha}}.du=-e^{\frac{-x^{\alpha}}{\alpha}}F\left(a,\ b,\ \frac{x^{\alpha}}{\alpha}\right). \tag{41}$$

Consequently (39) is identified and this completes the proof of the theorem. □

4 Simulation results

In this section, the general solution of the confluent α-hypergeometric differential equation given in (29) is graphically simulated. The behaviors of such solution for different values of a and also for different values of α, are shown in Figures (2) and (3), respectively. The two constants C_1 and C_2 are assumed in such equation to be equal 1. The parameter b is fixed to be equal -1.5 in both simulations. Firstly, let us see, in Figure 1, how the solution, given in (29), looks like for different values of a and for a fixed value of α to be 0.5. These values are chosen to be $a=0.2$, $a=0.4$, $a=0.6$, $a=0.8$ and $a=1$. On the other hand, Figure 3 shows the behavior of (29) for different values of α; $\alpha=0.8$, $\alpha=0.85$, $\alpha=0.9$, $\alpha=0.95$ and $\alpha=1$, with considering $a=0.5$.

5 Conclusion

This work successfully obtains the fractional Frobenius series solutions around the regular α-singular point $x=0$ for the confluent α-hypergeometric differential equation through using the conformable fractional-order operator. It has been shown that these solutions in its classical case are just confluent hypergeometric functions.

Conflicts of Interests

The authors declare that they have no conflicts of interests.

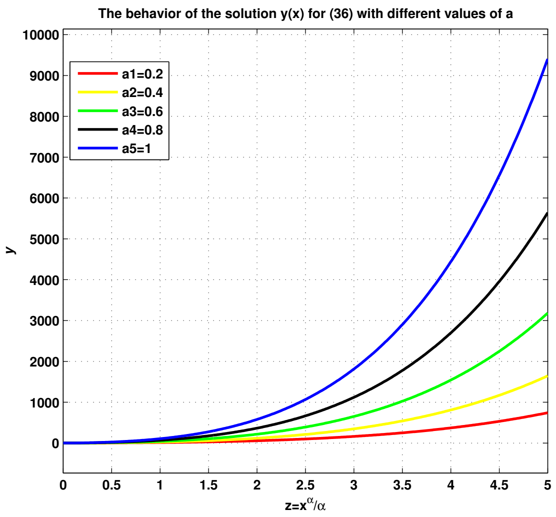

Fig. 2: The behavior of the solution given in (29) for different values of a.

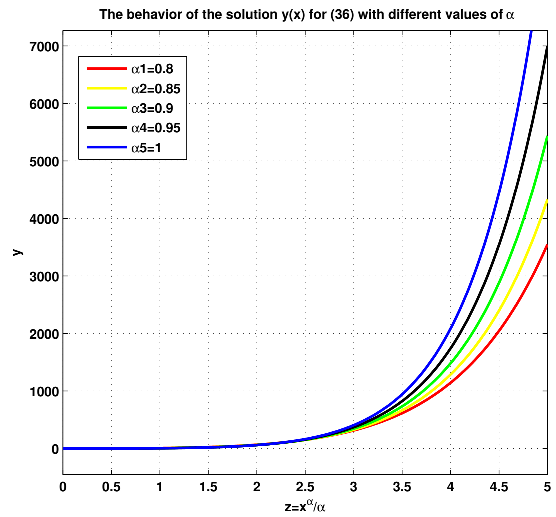

Fig. 3: The behavior of the solution given in (29) for different values of α.

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