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A Hybrid Nash Min-Max Approach for Solving Continuous Static Games

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Abstract: This paper addresses a novel structure of continuous static games (CSGs) called hybrid continuous static games (HCSG). These types of games are based on the fact that the game is presented as a hybrid game between multiple players playing independently using Nash equilibrium solutions (NES) and others playing according to a secure concept using Min-Max solutions (MMS). A Nash Min-Max approach was established to solve such genre games. Also, an algorithm for solving Nash Min-Max hybrid continuous static games (NMMHCSG) has been outlined in clear steps. Moreover, a realistic application of four firms selling replaceable products and seeking to maximize their profits was presented to demonstrate the steps of the algorithm.

Keywords: Continuous static games, Game theory, Hybrid games, Min-max solutions, Nash equilibrium, Nash min-max approach.

1 Introduction

Some decisions extensively affect our life, while others slightly influence it. For instance, we can consider the choice of an item of our clothes and accepting or rejecting a job offer, ...etc. Decision science handles all decision problems and their solutions approaches [1] .

Many decision problems may be considered a parametric system that has one or more cost criteria. The discussion has been limited, however, in the presence of a single decision-maker (controller) with a single cost function and has sole control over the selection of all system parameters constrained by a system of algebraic equality and inequality constraints. The more general case is achieved under the assumption that there exist multiple decision-makers, each with their cost criterion. Now, we have reached the territory of game theory while the game appears when there exists the more general case of multiple decision-makers [2] . This generalization introduces the possibility of competition among the system controllers called players and the optimization problem under consideration has been termed a game. Each player controls a specified subset of the system parameters called his control vector and seeks to minimize his cost criterion subject to specified equality and inequality constraints [3] .

In our real-life, there are several applications of game theory that may be found in economics, biology, political strategies, and many other areas. Competition among firms seeking to maximize their profit and competition for food and territory among biological species are two examples [3] . Also, there are extensive engineering applications that are designed across the game theory. Variety of interesting studies, such as defense applications of wireless networks, cybernetics internet security, smart grid distribution network, electricity marketing to image processing and coding, electromagnetic apparatus design to electrical vehicles, MIMO systems spectrum access, and target tracking cognitive radio. can be obtained by just a quick internet search, see [4–9] .

Although game theory is appropriate for the analysis of multiple controller cooperative systems, it may also be employed effectively in the analysis of uncertain systems via a worst-case analysis [3, 10–12] . Games can be classified into three major classes matrix games [10, 13, 14] , continuous static games [3, 15] and differential games [16] . In this paper, hybrid studies on continuous static games at which the decision possibilities need not be discrete and costs related in a continuous rather than a discrete manner are provided. The game is called static because there exists no time history involved in the relationship between costs and decisions.

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Continuous static games play an extremely important role in various science fields to analyze static optimization issues. In this result, continuous static games offer researchers many opportunities to consider various types of solutions depending on the optimization problem nature. Multiple types of CSGs are presented in the previous works [11, 15, 17–21] . In (1984) [15] , M. Osman introduced different parametric problems in continuous static games. The authors in [17] assumed some general parameters in the objective functions for pareto continuous static games. In (2005), [18] introduced a new problem formulation of special class of large scale continuous static games was given by M. ElShafei. In [19] , the author constructed a Nash cooperative continuous static game solved with an interactive approach. In (2015), [11] presented an interactive approach for solving fuzzy continuous static games. Furthermore, H. A. Kalifa in [20] explored an interactive approach for solving multiobjective nonlinear programming and applied the results to cooperative continuous static games. In (2019), [21] showed that the fractional cooperative continuous static games can be solved using a parametric approach.

In this paper, a hybrid Nash Min-Max approach for solving a novel structure of continuous static games called hybrid continuous static games is established. This approach is based on the fact that the game is presented as a hybrid game between multiple players playing independently using NES [3, 22] and others playing with a secure concept using MMS [3, 10] . Moreover, an algorithm for solving such type of games is stated in clear steps. Furthermore, a competition among firms seeking to maximize their profit as an application to clarify the proposed algorithm steps is presented.

The remainder of this paper is organized , as follows: The next section reviews some basic theorems used as a basis of our proofs in the next sections. A general structure of hybrid continuous static games is formulated in section (3) and its solution concept is developed in section (4). Section (5) presents the algorithm used for solving the introduced Nash Min-Max hybrid continuous static game (NMMHSG). An application of four firms selling substitutable products is introduced in section (6) as a numerical example to clarify the algorithm steps. Conclusion and recommendations are reported in section (7).

2 Basic Theorems

Theorem 2.1. (Taylor's Theorem [3]) If $\hat{y} \in E^m$ and if there exists a ball B centered at $\hat{y}$ such that the function $G(\cdot) : B \to E^1$ is C^{r+1} for every point $\hat{y}+\delta y \in B$, then with $\delta y = \alpha\delta' y$, where $\alpha \geq 0$ and $\delta' y$ is a unit vector in the direction of δy , we have

$$G\left(\hat{y}+\alpha\delta' y\right) = G(\hat{y}) + U\left[G(\hat{y})\right] + \frac{1}{2!}U^2\left[G(\hat{y})\right] + \ldots + \frac{1}{r!}U^r\left[G(\hat{y})\right] + R$$

where,

$$R = \frac{1}{(r+1)!}U^{r+1}\left[G(\hat{y}+\beta\delta' y)\right]$$

For some $\beta \in (0,\alpha)$ and where U is the differential operator defined by:

$$U(\cdot) \triangleq \delta y_1 \frac{\partial(\cdot)}{\partial y_1} + \ldots + \delta y_m \frac{\partial(\cdot)}{\partial y_m}$$

proof. The proof of the above theorem is given in [3] .

Theorem 2.2. (First Order Approximation Theorem [3]) Let $G(\cdot) : E^m \to E^1$ be C^1 for every point $\hat{y}+\delta' y \in B$. Then,

$$G(\hat{y}+\alpha\delta' y) - G(\hat{y}) = \frac{\partial G(\hat{y})}{\partial y}\delta y + R(\alpha)$$

where $\lim_{\alpha\to 0}\left[R(\alpha)/\alpha\right] = 0$ and $\delta y = \alpha\delta' y$.

proof. The proof of **Theorem 2.2** is given in [3] .

Theorem 2.3. (Farkas' Lemma [3]) Let C be the finite convex cone generated by the vectors $y^1, \ldots, y^q$ in E^m and let C^* be the polar cone to C. Then, $y \in C$ if and only if $y^T z \geq 0 \ \forall \ z \in C^*$.

proof. Theorem 2.3 is proved in [3] .

Lemma 2.1. If $\hat{y}$ is a regular point of Y, the tangent cone T to Y at $\hat{y}$ is given by $T = \hat{K}$, where

$$T = \left\{ e \in E^m \,\middle|\, \frac{\partial g_I(\hat{y})}{\partial y}e = 0 \text{ and } \frac{\partial \hat{h}_J(\hat{y})}{\partial y}e \geq 0 \right\},$$

$$K = \left\{ y \in E^m \,\middle|\, \begin{array}{l} y^T = \lambda^T\dfrac{\partial g_I(\hat{y})}{\partial y} + \mu^T\dfrac{\partial h_J(\hat{y})}{\partial y}, \\ \lambda \in E^n,\ \mu \in E^q,\ \mu \geq 0,\ \mu^T h_J(\hat{y}) = 0 \end{array} \right\}$$

and the polar cone to K is given by

$$\hat{K} = \left\{ e \in E^m \,\middle|\, y^T e \geq 0 \ \forall \ y \in K \right\}$$

where $\hat{h}(\cdot) = \left[h_1(\cdot), h_2(\cdot), \ldots, h_{\hat{q}}(\cdot)\right]^T$ denote the active inequality constraints at $\hat{y}$ and $e \in T$ is the tangent vector to Y at $\hat{y}$.

proof. The proof of the above-mentioned lemma is given in [3] .

3 General Structure of Hybrid Continuous Static Games

This section addresses a general structure of hybrid continuous static games based on the fact that the game is hybrid between multiple players having different strategies. We can formulate such type of games, as follows:

Let $u = (u_1, u_2, ..., u_m) \in \Re^m$ denote the control vectors for player $i = \{1,2...,m\} \subset \{1,2...,m,m+1,...,r\}$ (the set of all players), $\upsilon = (v_1, v_2, ..., v_r) \in \tau \subset \Re^{s_i - m}$ denote the composite control vectors for player $j = \{m+1,...,r\}$ and $\omega \notin \tau \subset \Re^{s_i - m}$ denote the composite control for other remaining players. Where $(u,\upsilon,\omega) \in \Re^s$, $s = \sum_{e=1}^{r} s_e$ are all composite control.

Each player $l = \{1,2...,m,m+1,...,r\}$ selects his control to minimize a scalar-valued criterion

$$F_l(x,u,\upsilon,\omega) \tag{1}$$

Subject to n equality constraints

$$g_I(x,u,\upsilon,\omega) = 0, \quad I = 1,...,n \tag{2}$$

where $x \in \Re^n$ is the state vector. The composite control is required to be an element of regular control constraint set $\Omega \in \Re^s$, defined by

$$\Omega = \{(u,\upsilon,\omega) \in \Re^s \mid h_J[\zeta(u,\upsilon,\omega),u,\upsilon,\omega] \geq 0\} \tag{3}$$

where $J = 1,...,q$, $x = \zeta(u,\upsilon,\omega)$ is the solution to (2) and gives (u,υ,ω) and the functions $F_l(\cdot) : \Re^n \times \Re^s \to \Re^l$, $g_I(\cdot) : \Re^n \times \Re^s \to \Re^n$, $h_J(\cdot) : \Re^n \times \Re^s \to \Re^q$ are assumed to be C^1 with $|\,\partial g_I(x,u,\upsilon,\omega)/\partial x\,| \neq 0$ in a ball about a solution point (x,u,υ,ω).

At this point, several approaches depending on each team's chosen optimality concept may be used for solving the problem (1) - (3). One of them is discussed in the next section.

4 Nash Min-Max Approach for Solving HCSG

In this section, a hybrid Nash Min-Max approach is established for solving the game (1) - (3). This approach is based on the fact that the game is presented as a hybrid game between multiple players (T_1) using NES and others (T_2) playing on a secure concept using MMS, see NMMHCSG problem (4).

4.1 Problem Formulation

Assume that the game (1) - (3) consists of two teams, the first team decided to play independently using NES with players set $i \in T_1 = \{1,2...,m\} \subset \{1,2...,m,m+1,...,r\}$ (the set of all players). On the other hand, the second team use a MMS between its players set $j \in T_2 = \{m+1,...,r\}$. Then, using the control notations, Nash Min-Max point for any player l can be formulated as the following non-linear programming problem

Find the point $(\zeta(\hat{u},\hat{\upsilon}),\hat{u},\hat{\upsilon})$ that solves

$$\text{NMMHCSG:} \quad \text{Min } F_i(x,u,\hat{\upsilon}), \quad i = 1,2,...,m$$
$$F_j(x,\hat{u},\upsilon), \quad j \in \tau \subset \{m+1,...,r\} \tag{4}$$

Subject to

$$g(x,u,\hat{\upsilon}) = 0, \ h(x,u,\hat{\upsilon}) \geq 0,$$
$$\varphi(x,\hat{u},\upsilon) = 0, \ \psi(x,\hat{u},\upsilon) \geq 0$$

and

$$\text{Max} \quad F_j(x,\hat{u},\upsilon), \ j \notin \tau$$

Subject to

$$g(x,u,\hat{\upsilon}) = 0, \ h(x,u,\hat{\upsilon}) \geq 0,$$
$$\varphi(x,\hat{u},\upsilon) = 0, \ \psi(x,\hat{u},\upsilon) \geq 0$$

where $x \in \Re^n$ is the state vector, $u = (u_1, u_2, ..., u_m) \in \Re^m$ is the control of the player $i = 1,2,...,m$ (playing independently), $\upsilon = (v_{m+1},...,v_r) \in \Re^{s-m}$ denotes the control vector of the remaining players using MMS, and $s = s_1 + s_2 + ... + s_r$. Also, (u,υ) is required to be an element of a regular control constrain set $\Omega \in \Re^s$, where

$$\Omega = \left\{ (u,\upsilon) \in \Re^s \;\middle|\; \begin{array}{l} h(\zeta(u,\upsilon),u,\upsilon) \geq 0, \\ \psi(\zeta(u,\upsilon),u,\upsilon) \geq 0 \end{array} \right\}$$

where $x = \zeta(u,\upsilon)$ is the solution of $g(x,u,\upsilon) = 0$ and $\varphi(x,u,\upsilon) = 0$ given (u,υ) and we assume that $|\partial g(x,u,v)/\partial x| \neq 0$, $|\partial \varphi(x,u,\upsilon)/\partial x| \neq 0$ in a ball about a solution point $(\hat{x},\hat{u},\hat{\upsilon})$. The functions $F_i(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^m$ and $F_j(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^r$ are of class C^1. The functions $F_j(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^{s-r-m}$, $g(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^{n_1}$, $h(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^{q_1}$, $\varphi(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^{n_2}$ and $\psi(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^{q_2}$ are of class C^2 in x and υ and C^1 in u, $\Re^{n_1} \times \Re^{n_2} \to \Re^n$ and $\Re^{q_1} \times \Re^{q_2} \to \Re^q$.

4.2 Solution Concept

4.2.1 Cost Perturbations and Tangent Cones [3]

Before introducing the formal definitions of the solution concept, it is fundamental to first develop a general relation for the perturbation induced in a cost function $F_e(x,u)$ due to local perturbations about a nominal composite control u. These relations are used to develop the necessary optimal conditions for the Nash Min-Max solutions.
At a point $u \in \Omega = \{(x,u) \in \Re^s | h[\zeta(u),u] \geq 0\}$, let $B \subset E^s$ denote a ball about u and let $x = \zeta(u)$ denote the corresponding solution to $g(x,u)$. Let $T \subseteq E^s$ denote the tangent cone to Ω at u and let $\delta u(\cdot)$ generate a tangent vector $z \in T$, let $u + \alpha\delta u(\alpha) \in \Omega$ for all sufficiently small $\alpha > 0$ and $\delta u(\alpha) \to z$ as $\alpha \to 0$.

For each player $e = 1, 2, \ldots, r$, **Theorem 2.2** yields

$$\begin{aligned}\delta F_e &\triangleq F_e[\zeta[u+\alpha\delta u(\alpha)], u+\alpha\delta u(\alpha)] - F_e[\zeta(u), u] \\ &= \left[\frac{\partial F_e}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial F_e}{\partial u}\right]\alpha\, z + \tilde{R}_i(\alpha)\end{aligned} \quad (5)$$

where $\tilde{R}_i(\alpha)\ /\ \alpha \to 0$ as $\alpha \to 0$ and all quantities are evaluated at (x,u). Applying **Theorem 2.2** to

$$g[\zeta[u+\alpha\delta u(\alpha)],\ u+\alpha\delta u(\alpha)] = g[\zeta(u),\ u] = 0,$$

we have

$$0 = \left[\frac{\partial g}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial g}{\partial u}\right]\alpha\, z + \hat{R}(\alpha)$$

where $\hat{R}(\alpha)/\alpha \to 0$ as $\alpha \to 0$. Thus, in view of

$$\left|\frac{\partial g(x,u)}{\partial x}\right| \neq 0 \quad (6)$$

we have

$$\frac{\partial \zeta}{\partial u}\alpha\, z = -\left[\frac{\partial g}{\partial x}\right]^{-1}\left[\frac{\partial g}{\partial u}\alpha\, z + \hat{R}(\alpha)\right] \quad (7)$$

for all $z \in T$.

Combining (5) and (7) yields

$$\delta F_e = \left[\frac{\partial F_e}{\partial u} - \frac{\partial F_e}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1}\frac{\partial g}{\partial u}\right]\alpha\, z + R_e(\alpha) \quad (8)$$

for all $e = 1, 2, \ldots, r$ and $z \in T$, where $R_e(\alpha)\ /\ \alpha \to 0$ as $\alpha \to 0$. Defining

$$\gamma^T(e) \triangleq \frac{\partial F_e}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1},\quad e = 1, \ldots, r \quad (9)$$

and

$$J_e[x, u, \gamma(e)] \triangleq F_e(x,u) - \gamma^T(e)\, g(x,u),\ e = 1, \ldots, r \quad (10)$$

we can write (8) as

$$\delta F_e = \left[\frac{\partial J_e}{\partial u}\right]\alpha\, z + R_e(\alpha),\ \ e = 1, \ldots, r \quad (11)$$

for all $z \in T$. Note from (9) and (10) that $\gamma(e)$ is the unique solution to

$$\frac{\partial J_e}{\partial x} = 0,\ \ e = 1, \ldots, r \quad (12)$$

for $u + \alpha\delta u(\alpha) \in \Omega$ for all sufficiently small $\alpha > 0$, we have

$$h[\zeta[u+\alpha\delta u(\alpha)],\ \ u+\alpha\delta u(\alpha)] \geq 0 \quad (13)$$

Thus, the first order approximation theorem (**Theorem 2.2**) along with (7) and the regularity assumption yield the following result for T that refers to the tangent cone to Ω at u defined by

$$T = \left\{ z \in \Re^s \ \middle|\ \left[\frac{\partial \hat{h}}{\partial u_e} - \frac{\partial \hat{h}}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1}\frac{\partial g}{\partial u}\right] z \geq 0 \right\} \quad (14)$$

where $\hat{h}(\cdot)$ denotes the active inequality constraints at u.

In discussing a particular player e with control $u_e \in \Re^s$, it is convenient to focus on the control constraints induced on player e by a choice v of the composite control of the remaining players. Thus, for specific v with $u = (u_e, v)$ we have $u_e \in U_e$, where

$$U_e = \{u_e \in \Re^s \mid h[\zeta(u_e,v), u_e, v] \geq 0\} \quad (15)$$

If $U_e \subseteq \Re^{s_e}$ is regular at u_e, $e = 1, 2, \ldots, r$ for a given v, then the tangent cone $T_e \subseteq \Re^{s_e}$ to $U_e \subseteq \Re^{s_e}$ at u_e is given by

$$T_e = \left\{ z^e \in \Re^{s_e} \ \middle|\ \left[\frac{\partial \hat{h}}{\partial u_e} - \frac{\partial \hat{h}}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1}\frac{\partial g}{\partial u}\right] z^e \geq 0 \right\} \quad (16)$$

where $e = 1, 2, \ldots, r$, and $\hat{h}(\cdot)$ denotes the active inequality constraints at $u = (u_e, v)$.

4.2.2 Formal Definitions of the Solution Concept

Definition 4.1. (Regular Point [23, 24]) A point $u = (u_i, v) \in \Omega = \{(x,u) \in \Re\, s | h_J[\zeta(u), u] \geq 0\}$ is a completely regular point if and only if for each $i = 1, \ldots, r$

1. u is a regular point of Ω.
2. u_i is a regular point of U_i.

where $x = \zeta(u)$ denotes the corresponding solution to $g_I(x,u) = 0, I = 1, \ldots, n, J = 1, \ldots, q$ and

$$U_i = \{u_i \in \Re^s \mid h_J[\zeta(u_i,v), u_i, v] \geq 0,\ i = 1, \ldots, r\}$$

Definition 4.2. A point $\hat{u} = (\hat{u}_i, \hat{v}_i, \hat{\omega}) \in \Omega$ is a completely regular Nash Min-Max point if and only if for each $i = 1, 2, \ldots, m$ and $j = m+1, \ldots, r$

$$F_i[\zeta(\hat{u}), \hat{u}] \ \leq\ F_i[\zeta(u_i, \hat{v}_j, \hat{\omega}), u_i, \hat{v}_j, \hat{\omega}] \quad (17)$$

and

$$\begin{cases} F_j[\zeta(\hat{u}_i, \hat{v}_j, \omega), u_i, \hat{v}_j, \omega] \leq \\ \qquad F_j[\zeta(\hat{u}), \hat{u}] \leq \\ \qquad\qquad F_j[\zeta(\hat{u}_i, v_j, \hat{\omega}), \hat{u}_i, v_j, \hat{\omega}] \end{cases} \quad (18)$$

for all (u_i, v_j, ω) such that $(u_i, \hat{v}_j, \hat{\omega}) \in \Omega$, $(\hat{u}_i, v_j, \hat{\omega}) \in \Omega$ and $(\hat{u}_i, \hat{v}_j, \omega) \in \Omega$ defined by

$$\Omega = \left\{(u_i, v_j, \omega) \in \Re^s \ \middle|\ h_J[\zeta(u_i, v_j, \omega), u_i, v_j, \omega] \geq 0\right\}$$

where $u \in \Re^m$, $\upsilon \in \tau \subset \Re^{s_i - m}$ denote the control vectors for player i and j, respectively. Also, $\omega \notin \tau \subset \Re^{s_i - m}$

denotes the composite control vector for other remaining players. $(u,\upsilon,\omega) \in \Re^s$, $s = \sum_{e=1}^{r} s_e$ are all composite control and $x = \zeta(u,\upsilon,\omega)$ is the solution to $g_I(x,u,\upsilon,\omega)$ and gives (u,υ,ω). The functions $F_i(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^m$ and $F_j(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^r$ are of class C^1. The functions $F_j(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^{s-r-m}$, $g_I(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^n$, $I = 1,\dots,n$ and $h_J(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^r \to \Re^q$, $J = 1,\dots,q$ are of class C^2 in x and υ and C^1 in u. For a local Nash Min-Max point, replace Ω by $B \cap \Omega$ for some ball $B \subset \Re^s$ centered at $\hat{u}$.

Lemma 4.1. If $(\hat{u},\hat{\upsilon}) = (\hat{u}_i,\hat{v}_j) \in \Omega$ is a local Nash Min-Max point for NMMHCSG, problem (4), and if $\hat{x} = \zeta(\hat{u},\hat{\upsilon})$ is solution to $g(x,\hat{u},\hat{\upsilon}) = 0$ and $\varphi(x,\hat{u},\hat{\upsilon}) = 0$, then for each $i = 1,2,\dots,m$ and $j = m+1,\dots,r$ there exist vectors $\gamma(i) \in \Re^n$ and $\gamma(j) \in \Re^m$ defined by

$$\frac{\partial J_i\left[\hat{x},\hat{u},\hat{\upsilon},\gamma(i),\gamma(j)\right]}{\partial x} = 0, \quad i = 1,2,\dots,m \tag{19}$$

$$\frac{\partial J_j\left[\hat{x},\hat{u},\hat{\upsilon},\gamma(i),\gamma(j)\right]}{\partial x} = 0, \quad j = m+1,\dots,r \tag{20}$$

such that

$$\frac{\partial J_i\left[\hat{x},\hat{u},\hat{\upsilon},\gamma(i),\gamma(j)\right]}{\partial u_i} z^i \geq 0, \quad i = 1,2,\dots,m \tag{21}$$

for all $z^i \in T_i$,

$$\frac{\partial J_j\left[\hat{x},\hat{u},\hat{\upsilon},\gamma(i),\gamma(j)\right]}{\partial v_j} z^j \geq 0, \quad j \in \tau \subset \{m+1,\dots,r\} \tag{22}$$

for all $z^j \in T_j$, $j \in \tau \subset \{m+1,\dots,r\}$ and

$$\frac{\partial J_j\left[\hat{x},\hat{u},\hat{\upsilon},\gamma(i),\gamma(j)\right]}{\partial v_j} z^j \geq 0, \quad j \notin \tau \tag{23}$$

for all $z^j \in T_j$, $j \notin \tau$, where T_i and T_j denote the tangent cones to Ω at $(\hat{u},\hat{\upsilon})$ defined by

$$T_i = \left\{ z^i \in \Re^{s_i} \middle| \left[\frac{\partial \hat{h}}{\partial u_i} - \frac{\partial \hat{h}}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1}\frac{\partial g}{\partial u} \right] z^i \geq 0 \right\} \tag{24}$$

$$T_j = \left\{ z^j \in \Re^{s_j} \middle| \left[\frac{\partial \hat{\psi}}{\partial \upsilon_j} - \frac{\partial \hat{\psi}}{\partial x}\left[\frac{\partial \varphi}{\partial x}\right]^{-1}\frac{\partial \varphi}{\partial \upsilon} \right] z^j \geq 0 \right\} \tag{25}$$

where

$$J_i\left[x,u,\upsilon,\gamma(i),\gamma(j)\right] \triangleq F_i(x,u,\upsilon) - \gamma^T(i)g(x,u,\upsilon) - \gamma^T(j)\varphi(x,u,\upsilon)$$

and

$$J_j\left[x,u,\upsilon,\gamma(i),\gamma(j)\right] \triangleq F_j(x,u,\upsilon) - \gamma^T(i)g(x,u,\upsilon) - \gamma^T(j)\varphi(x,u,\upsilon)$$

for all $i \in T_1 = \{1,2,\dots,m\}$ and $j \in T_2 = \{m+1,\dots,r\}$.

Proof. From (11) and **Definition 4.2** for each player $i = 1,2,\dots,m$ we have

$$\delta F_i = \left[\frac{\partial J_i\left[\hat{x},\hat{u},\hat{\upsilon},\gamma(i),\gamma(j)\right]}{\partial u_i}\right] \alpha\, z + R_i(\alpha) \tag{26}$$

for all $i = 1,2,\dots,m$ and $z \in T$, with

$$z^i = 0 \;\forall\; k \in \{1,2,\dots,m\},\; k \neq i,\; i = 1,2,\dots,m \tag{27}$$

Combining (26) and (27) yields

$$\left[\frac{\partial J_i\left[\hat{x},\hat{u},\hat{\upsilon},\gamma(i),\gamma(j)\right]}{\partial u_i}\right] \alpha\, z^i + R_i(\alpha) \geq 0 \tag{28}$$

for all $z^i \in T_i$, $i = 1,2,\dots,m$. Dividing (28) by $\alpha > 0$ and taking the limit as $\alpha \to 0$ yields (21).
Similarly, from (11) and the **Definition 4.2** for each player $j = m+1,\dots,r$, we have

$$\left[\frac{\partial J_j\left[\hat{x},\hat{u},\hat{\upsilon},\gamma(i),\gamma(j)\right]}{\partial v_j}\right] \alpha\, z^j + R_j(\alpha) \geq 0 \tag{29}$$

for all $z^j \in T_j$, $j \in \tau \subset m+1,\dots,r$, and

$$\left[\frac{\partial J_j\left[\hat{x},\hat{u},\hat{\upsilon},\gamma(i),\gamma(j)\right]}{\partial v_j}\right] \alpha\, z^j + R_j(\alpha) \geq 0, \quad j \notin \tau \tag{30}$$

for all $z^j \in T_j$, where $R_j(\alpha)/\alpha \to 0$ as $\alpha \to 0$. Dividing (29) and (30) by $\alpha > 0$ and taking the limit as $\alpha \to 0$ yield (22) and (23) which establish the lemma. ■

Necessary optimal conditions used to define Nash Min-Max points for problem (4) may now be stated in the following theorem.

Theorem 4.1. If $\hat{w} = (\hat{u},\hat{\upsilon}) = (\hat{u}_i,\hat{v}_j) \in \Omega$ is a completely regular local Nash Min-Max point and if $\hat{x} = \zeta(\hat{w})$ is solution to $g(x,\hat{w}) = 0$ and $\varphi(x,\hat{w}) = 0$, then for each $i = 1,2,\dots,m$ and $j = m+1,\dots,r$ there exist vectors $\lambda(i) \in \Re^{n_1}$, $\mu(i) \in \Re^{q_1}$, $\rho(j) \in \Re^{n_2}$, $\beta(j) \in \Re^{q_2}$, $\bar{\rho}(j) \in \Re^{n_2}$, and $\bar{\beta}(j) \in \Re^{q_2}$ such that

$$\frac{\partial L_i\left[\hat{x},\hat{w},\lambda,\mu,\rho,\beta\right]}{\partial x} = 0, \quad i = 1,2,\dots,m \tag{31}$$

$$\frac{\partial L_i\left[\hat{x},\hat{w},\lambda,\mu,\rho,\beta\right]}{\partial u_i} = 0, \quad i = 1,2,\dots,m \tag{32}$$

$$\frac{\partial L_j\left[\hat{x},\hat{w},\lambda,\mu,\rho,\beta\right]}{\partial x} = 0, \quad j \in \tau \subset \{m+1,\dots,r\} \tag{33}$$

$$\frac{\partial L_j\left[\hat{x},\hat{w},\lambda,\mu,\rho,\beta\right]}{\partial x} = 0, \quad j \notin \tau \tag{34}$$

$$\frac{\partial L_j\left[\hat{x},\hat{w},\lambda,\mu,\rho,\beta\right]}{\partial v_j} = 0, \quad j \in \tau \subset \{m+1,\dots,r\} \tag{35}$$

$$\frac{\partial L_j\left[\hat{x},\hat{w},\lambda,\mu,\rho,\beta\right]}{\partial v_j} = 0, \quad j \notin \tau \tag{36}$$

$$g(\hat{x},\hat{w}) = 0 \tag{37}$$
$$\varphi(\hat{x},\hat{w}) = 0 \tag{38}$$
$$h(\hat{x},\hat{w}) \geq 0 \tag{39}$$
$$\psi(\hat{x},\hat{w}) \geq 0 \tag{40}$$
$$\mu^t(i)\, h(\hat{x},\hat{w}) \geq 0,\ i = 1,2,\ldots,m \tag{41}$$
$$\beta^t(j)\, \psi(\hat{x},\hat{w}) \geq 0,\ j \in \tau \subset \{m+1,\ldots,r\} \tag{42}$$
$$\bar{\beta}^t(j)\, \psi(\hat{x},\hat{w}) \geq 0,\ j \notin \tau \tag{43}$$
$$\mu(i) \geq 0,\ i = 1,2,\ldots,m \tag{44}$$
$$\beta(j) \geq 0,\ j \in \tau \subset \{m+1,\ldots,r\} \tag{45}$$
$$\bar{\beta}(j) \geq 0,\ j \notin \tau \tag{46}$$

where,

$$L_i[x,w,\lambda,\mu,\rho,\beta] = F_i(x,w) - \lambda^t g(x,w) - \mu^t h(x,w) - \rho^t \varphi(x,w) - \beta^t \psi(x,w) \tag{47}$$

for each player $i \in T_1 = \{1,2,\ldots,m\}$ and

$$L_j[x,w,\lambda,\mu,\rho,\beta] = F_j(x,w) - \lambda^t g(x,w) - \mu^t h(x,w) - \rho^t \varphi(x,w) - \beta^t \psi(x,w) \tag{48}$$

for each player $j \in T_2 = \{m+1,\ldots,r\}$.
While the partial derivatives of L_j are evaluated using the two sets of multipliers $\rho(j),\beta(j)\forall j \in \tau \subset \{m+1,\ldots, r\}$ and $\bar{\rho}(j),\bar{\beta}(j)\forall j \notin \tau$ depending on the minimization or maximization criteria for the player j, $\Re^{n_1} \times \Re^{n_2} \to \Re^n$ and $\Re^{q_1} \times \Re^{q_2} \to \Re^q$.

Proof. Consider the following cones

$$\kappa_i = \left\{ y \in \Re^M \,\middle|\, \begin{array}{l} y^t = \mu^t \left[\frac{\partial h}{\partial u_i} - \frac{\partial h}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1} \frac{\partial g}{\partial u_i}\right], \\ \mu^t h(\hat{x},\hat{w}) = 0, \mu(i) \geq 0, i = 1,\ldots,m \end{array} \right\} \tag{49}$$

$$\kappa_j = \left\{ y \in \Re^N \,\middle|\, \begin{array}{l} y^t = \beta^t \left[\frac{\partial \psi}{\partial v_j} - \frac{\partial \psi}{\partial x}\left[\frac{\partial \varphi}{\partial x}\right]^{-1} \frac{\partial \varphi}{\partial v_j}\right], \\ \beta^t \psi(\hat{x},\hat{w}) = 0, \beta(j) \geq 0, j \in \tau \end{array} \right\} \tag{50}$$

$$\kappa_j = \left\{ y \in \Re^Q \,\middle|\, \begin{array}{l} y^t = \beta^t \left[\frac{\partial \psi}{\partial v_j} - \frac{\partial \psi}{\partial x}\left[\frac{\partial \varphi}{\partial x}\right]^{-1} \frac{\partial \varphi}{\partial v_j}\right], \\ \beta^t \psi(\hat{x},\hat{w}) = 0, \beta(j) \geq 0, j \notin \tau \end{array} \right\} \tag{51}$$

where $\tau \subset \{m+1,\ldots,r\}$ and the polar cones are

$$\kappa_i^* = \left\{ v \in \Re^M \,\middle|\, \begin{array}{l} y^t v \geq 0\ \forall\ y \in \kappa_i, \\ i = 1,2,\ldots,m \end{array} \right\} \tag{52}$$

$$\kappa_j^* = \left\{ v \in \Re^N \,\middle|\, \begin{array}{l} y^t v \geq 0\ \forall\ y \in \kappa_j, \\ j \in \tau \subset \{m+1,\ldots,r\} \end{array} \right\} \tag{53}$$

$$\kappa_j^* = \left\{ v \in \Re^Q \,\middle|\, \begin{array}{l} y^t v \geq 0\, \forall\, y \in \kappa_j \\ j \notin \tau \end{array} \right\} \tag{54}$$

where all equations are evaluated at $(\hat{x},\hat{w})$. By the same proof of **Lemma 2.1**, since $\hat{w} = (\hat{u},\hat{v})$ is a regular point of U_i the tangent cones T_i and T_j, given by (16), we have

$$T_i = \kappa_i^*\ \forall\ i = 1,2,\ldots,m$$
$$T_j = \kappa_j^*\ \forall\ j \in \tau \subset \{m+1,\ldots,r\}$$
$$T_j = \kappa_j^*\ \forall\ j \notin \tau$$

from this result and **Lemma 4.1**, we get

$$\frac{\partial J_i[\hat{x},\hat{w},\gamma(i),\gamma(j)]}{\partial u_i} \in \kappa_i\, \forall\, i = 1,2,\ldots,m \tag{55}$$
$$\frac{\partial J_j[\hat{x},\hat{w},\gamma(i),\gamma(j)]}{\partial v_j} \in \kappa_j\, \forall\, j \in \tau \subset \{m+1,\ldots,r\} \tag{56}$$
$$\frac{\partial J_j[\hat{x},\hat{w},\gamma(i),\gamma(j)]}{\partial v_j} \in \kappa_j\, \forall\, j \notin \tau \tag{57}$$

where $\gamma(i)$ and $\gamma(j)$ are defined by

$$\frac{\partial J_i[\hat{x},\hat{w},\gamma(i),\gamma(j)]}{\partial x} = 0\, \forall\, i = 1,2,\ldots,m\,, \tag{58}$$
$$\frac{\partial J_j[\hat{x},\hat{w},\gamma(i),\gamma(j)]}{\partial x} \in \kappa_j\, \forall\, j = m+1,\ldots,r \tag{59}$$

where

$$J_i[x,w,\gamma(i),\gamma(j)] \triangleq F_i(x,w) - \gamma^t(i) g(x,w) - \gamma^t(j)\varphi(x,w),\ i = 1,2,\ldots,m \tag{60}$$

and

$$J_j[x,w,\gamma(i),\gamma(j)] \triangleq F_j(x,w) - \gamma^t(i) g(x,w) - \gamma^t(j)\varphi(x,w),\ j = m+1,\ldots,r \tag{61}$$

Thus, from (49), we have

$$\frac{\partial J_i}{\partial u_i} = \mu^t \left[\frac{\partial h}{\partial u_i} - \frac{\partial h}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1} \frac{\partial g}{\partial u_i}\right],\ i = 1,2,\ldots,m \tag{62}$$

where $\mu(i)$ satisfies (39), (41) and (44). Then, we define

$$\lambda^t(i) = \gamma^t(i) - \mu^t \left[\frac{\partial h}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1} \right]\ \forall\, i = 1,2,\ldots,m, \tag{63}$$

$$\rho^t(j) = \gamma^t(j) - \beta^t \left[\frac{\partial \psi}{\partial x}\left[\frac{\partial \varphi}{\partial x}\right]^{-1} \right]\ \forall\, j \in \tau, \tag{64}$$

and

$$\bar{\rho}^t(j) = \gamma^t(j) - \beta^t \left[\frac{\partial \psi}{\partial x}\left[\frac{\partial \varphi}{\partial x}\right]^{-1} \right]\ \forall\, j \notin \tau \tag{65}$$

where $\tau \subset \{m+1,\ldots,r\}$, $\gamma(i)$ and $\gamma(j)$ are defined by (58) and (60).

Combining (12) and (63), for all $i = 1,2,\ldots,m$ and $\tau \subset \{m+1,\ldots,r\}$, yields

$$0 = \frac{\partial J_i}{\partial x} = \frac{\partial F_i}{\partial x} - \left[\lambda^t(i) + \mu^t(i)\left[\frac{\partial h}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1}\right]\right]_{(i=1,2,\ldots,m)} \frac{\partial g}{\partial x} - \left[\rho^t(j) + \beta^t(j)\left[\frac{\partial \psi}{\partial x}\left[\frac{\partial \varphi}{\partial x}\right]^{-1}\right]\right]_{(j\in\tau)} \frac{\partial \varphi}{\partial x} - \left[\bar{\rho}^t(j) + \bar{\beta}^t(j)\left[\frac{\partial \psi}{\partial x}\left[\frac{\partial \varphi}{\partial x}\right]^{-1}\right]\right]_{(j\notin\tau)} \frac{\partial \varphi}{\partial x} \quad (66)$$

which is equivalent to (31). Combining (60), (62) and (63) yields (32). From (48) and choosing $\beta(j) \geq 0 \; \forall \; j \in \tau \subset m+1,\ldots,r$, then (33) follows from (59) by substituting for $\gamma(i)$ and $\gamma(j)$ from (63) and (64), respectively. Similarly, (35), (42) and (45) easily follows from (50), (56) and (64). By choosing $\bar{\beta}(j) \leq 0 \; \forall j \notin \tau$, and substituting for $\gamma(i)$ from (63) and $\gamma(j)$ from (65), then (34) follows from (59). Also, (36), (43) and (46) can easily follow from (51) and (57) which complete the theorem proof. ■

Note that the necessary optimal conditions stated in **Theorem 4.1** may be simplified for the case of state-independent control constraints, i.e. for $h(u,\upsilon) \geq 0$, the two vectors $\lambda(i)$ and $\gamma(i)$ are identical. Moreover, $\lambda(i)$ and $\rho(j)$ are then uniquely defined as functions of $\hat{x},\hat{u}$ and $\hat{\upsilon}$ from (31), (33) and (34) since $\partial L_i/\partial x$ and $\partial L_j/\partial x$ are independent, control constraints. Similarly, if (40) is state-independent then we can get from (64) that $\rho(j) = \bar{\rho}(j) = \gamma(j)$ and in this case (35) and (36) become identical independent of $\beta(j)$ and $\bar{\beta}(j)$. Furthermore, the necessary optimal conditions are simplified, if the inequality constraints are decoupled; for example, let $\psi_1(v_j) \geq 0, j \in \tau \subset \{m+1,\ldots,r\}$ and $\psi_2(v_j) \geq 0, j \notin \tau$. In this case, the components of $\beta(j)$ multiplying zero gradients $(\partial\psi_2/\partial v_j = 0 \forall j \notin \tau)$ in (35) and the components of $\bar{\beta}(j)$ multiplying zero gradients $(\partial\psi_1/\partial v_j = 0 \forall j \in \tau)$ in (36) may be set equal to zero. Hence, a single β-vector, $\beta(j)$, may be employed with the appropriate signs for the components of $\beta(j)$. That is, $\beta_k(j) \geq 0 (\leq 0)$ if $\psi_k(\cdot) \geq 0$ is applied to the minimizing (maximizing) player. The same case can be obtained with $h_k(\cdot) \geq 0$ that increases the necessary optimal conditions simplification.

5 Algorithm for Solving NMMHCSG [Problem (4)]

The algorithm for solving NMMHCSG, problem (4), can be summarized in the following steps:

***Step* 1:** Form the game between multiple set of players, $i \in T_1 = \{1,2,\ldots,m\} \subset \{1,2,\ldots,m,m+1,\ldots,r\}$ playing independently using NES and others player set $j \in T_2 = \{m+1,\ldots,r\}$ using NMS concept.

***Step* 2:** Formulate the corresponding Nash Min-Max hybrid continuous static game problem (4).

***Step* 3:** Apply the necessary optimal conditions (31) - (46) stated in **Theorem 4.1**.

***Step* 4:** Solve the system of equations (31) - (48) using any computer package, i.e. **[gamultiobj]** MATLAB built-in tool to get the regular local Nash Min-Max solution point to the NMMHCSG problem (4).

Four firm's maximization profit application is given below to illustrate the above-mentioned algorithm steps.

6 Numerical Example (Four Firms Application)

Four firms sell substitutable products and seek to maximize their profits through advertising using the model from [3] [1]. Assuming that the steady-state fractions of the markets x_1,x_2,x_3 and x_4 that each firm receives are given by

$$g_1(\cdot) = 0 = -3x_1 + u_1 - u_1^2 - x_1u_2, \quad (67)$$
$$g_2(\cdot) = 0 = -2x_2 + u_2 - u_2^2 - 2x_2u_1, \quad (68)$$
$$g_3(\cdot) = 0 = -5x_3 + u_3 - u_3^2 - 2x_3u_4, \quad (69)$$
$$g_4(\cdot) = 0 = -4x_4 + u_4 - u_4^2 - 4x_4u_3 \quad (70)$$

where $u_1 \geq 0$, $u_2 \geq 0$, $0 \geq u_3 \leq 1$, and $0 \geq u_4 \leq 1$ are the advertising expenditure rates for firms 1,2,3,4, respectively, and $x_i \geq 0, i = 1,\ldots,4$ with $x_1 + x_2 \leq 1$. The steady-state profits of four firms are taken, respectively, as

$$P_1 = 5x_1 - u_1, \quad (71)$$
$$P_2 = 3x_2 - u_2, \quad (72)$$
$$P_3 = 8x_3 - u_3, \quad (73)$$
$$P_4 = 5x_4 - u_4 \quad (74)$$

Assuming that the game consists of two teams T_1 and T_2. The first team T_1 consists of the first two firms (players) with control vector u_i, $i = 1,2$ each player i seeks to minimize his cost function using NES. On the other hand, the remaining two players, with control vector u_j, $j = 3,4$, form the second team T_2. Each player j seeks to minimize his cost function using MMS.

Form the corresponding NMMHCSG, as follows:

$$\textbf{Min} \quad F_1 = -5x_1 + u_1,$$
$$F_2 = -3x_2 + u_2,$$
$$F_3 = -8x_3 + u_3 \text{ for player } P_3 \text{ Min-Max},$$
$$\text{or } F_4 = -5x_4 + u_4 \text{ for player } P_4 \text{ Min-Max}$$

[1] This model has been modified to meet our problem formulation.

Subject to

$$g_1(\cdot)=0=-3x_1+u_1-u_1^2-x_1u_2,$$
$$g_2(\cdot)=0=-2x_2+u_2-u_2^2-2x_2u_1,$$
$$\varphi_1(\cdot)=0=-5x_3+u_3-u_3^2-2x_3u_4,$$
$$\varphi_2(\cdot)=0=-4x_4+u_4-u_4^2-4x_4u_3,$$
$$h_1(\cdot)=x_1\geq 0,\ \ h_2(\cdot)=x_2\geq 0,$$
$$h_3(\cdot)=1-x_1-x_2\geq 0,$$
$$h_4(\cdot)=u_1\geq 0,\ \ h_5(\cdot)=u_2\geq 0,$$
$$\psi_1(\cdot)=x_3\geq 0,\ \ \psi_2(\cdot)=x_4\geq 0,$$
$$\psi_3(\cdot)=u_3\geq 0,\ \ \psi_4(\cdot)=1-u_3\geq 0,$$
$$\psi_5(\cdot)=u_4\geq 0,\ \ \psi_6(\cdot)=1-u_4\geq 0$$

and

Max $F_4=-5x_4+u_4$ for player P_3 Min-Max,
or $F_3=-8x_3+u_3$ for player P_4 Min-Max

Subject to

$$g_1(\cdot)=0=-3x_1+u_1-u_1^2-x_1u_2,$$
$$g_2(\cdot)=0=-2x_2+u_2-u_2^2-2x_2u_1,$$
$$\varphi_1(\cdot)=0=-5x_3+u_3-u_3^2-2x_3u_4,$$
$$\varphi_2(\cdot)=0=-4x_4+u_4-u_4^2-4x_4u_3,$$
$$h_1(\cdot)=x_1\geq 0,\ \ h_2(\cdot)=x_2\geq 0,$$
$$h_3(\cdot)=1-x_1-x_2\geq 0,$$
$$h_4(\cdot)=u_1\geq 0,\ \ h_5(\cdot)=u_2\geq 0,$$
$$\psi_1(\cdot)=x_3\geq 0,\ \ \psi_2(\cdot)=x_4\geq 0,$$
$$\psi_3(\cdot)=u_3\geq 0,\ \ \psi_4(\cdot)=1-u_3\geq 0,$$
$$\psi_5(\cdot)=u_4\geq 0,\ \ \psi_6(\cdot)=1-u_4\geq 0$$

Employing **Theorem 4.1**, the Lagrangian function for the first two players is defined by (47), as follows:

$$\begin{aligned}L_1=&-5x_1+u_1-\lambda_1(1)\left[-3x_1+u_1-u_1^2-x_1u_2\right]-\lambda_2(1)\\&\left[-2x_2+u_2-u_2^2-2x_2u_1\right]-\rho_1(1)\left[-5x_3+u_3-u_3^2\right.\\&\left.-2x_3u_4\right]-\rho_2(1)\left[-4x_4+u_4-u_4^2-4x_4u_3\right]-\mu_1(1)\\&x_1-\mu_2(1)x_2-\mu_3(1)\left[1-x_1-x_2\right]-\mu_4(1)u_1-\mu_5(1)\\&u_2-\beta_1(1)x_3-\beta_2(1)x_4-\beta_3(1)u_3-\beta_4(1)\left[1-u_3\right]-\\&\beta_5(1)u_4-\beta_6(1)\left[1-u_4\right]\end{aligned}$$

$$\begin{aligned}L_2=&-3x_1+u_2-\lambda_1(2)\left[-3x_1+u_1-u_1^2-x_1u_2\right]-\lambda_2(2)\\&\left[-2x_2+u_2-u_2^2-2x_2u_1\right]-\rho_1(2)\left[-5x_3+u_3-u_3^2\right.\\&\left.-2x_3u_4\right]-\rho_2(2)\left[-4x_4+u_4-u_4^2-4x_4u_3\right]-\mu_1(2)\\&x_1-\mu_2(2)x_2-\mu_3(2)\left[1-x_1-x_2\right]-\mu_4(2)u_1-\mu_5(2)\\&u_2-\beta_1(2)x_3-\beta_2(2)x_4-\beta_3(2)u_3-\beta_4(2)\left[1-u_3\right]-\\&\beta_5(2)u_4-\beta_6(2)\left[1-u_4\right]\end{aligned}$$

Since the remaining players 3 and 4 use MMS, then

1. To determine the MMS for player 3, define

$$\begin{aligned}L_3=&-8x_3+u_3-\lambda_1(3)\left[-3x_1+u_1-u_1^2-x_1u_2\right]-\lambda_2(3)\\&\left[-2x_2+u_2-u_2^2-2x_2u_1\right]-\rho_1(3)\left[-5x_3+u_3-u_3^2\right.\\&\left.-2x_3u_4\right]-\rho_2(3)\left[-4x_4+u_4-u_4^2-4x_4u_3\right]-\mu_1(3)\\&x_1-\mu_2(3)x_2-\mu_3(3)\left[1-x_1-x_2\right]-\mu_4(3)u_1-\mu_5(3)\\&u_2-\beta_1(3)x_3-\beta_2(3)x_4-\beta_3(3)u_3-\beta_4(3)\left[1-u_3\right]-\\&\beta_5(3)u_4-\beta_6(3)\left[1-u_4\right]\end{aligned}$$

where, $\beta_1(3),\beta_2(3),\beta_3(3),\beta_4(3)\geq 0$ and $\beta_5(3),\beta_6(3)\leq 0$. Applying the necessary optimal conditions, we get

$$\frac{\partial L_1}{\partial x_1}=0=-5+\lambda_1(1)\left[u_2+3\right]-\mu_1(1)+\mu_3(1) \quad (75)$$
$$\frac{\partial L_1}{\partial x_2}=0=2\lambda_2(1)\left[u_1+1\right]-\mu_2(1)+\mu_3(1) \quad (76)$$
$$\frac{\partial L_1}{\partial x_3}=0=\rho_1(1)\left[u_4+5\right]-\beta_1(1) \quad (77)$$
$$\frac{\partial L_1}{\partial x_4}=0=4\rho_2(1)\left[u_3+1\right]-\beta_2(1) \quad (78)$$
$$\frac{\partial L_2}{\partial x_1}=0=\lambda_1(2)\left[u_2+3\right]-\mu_1(2)+\mu_3(2) \quad (79)$$
$$\frac{\partial L_2}{\partial x_2}=0=-3+2\lambda_2(2)\left[u_1+1\right]-\mu_2(2)+\mu_3(2) \quad (80)$$
$$\frac{\partial L_2}{\partial x_3}=0=\rho_1(2)\left[u_4+5\right]-\beta_1(2) \quad (81)$$
$$\frac{\partial L_2}{\partial x_4}=0=4\rho_2(2)\left[u_3+1\right]-\beta_2(2) \quad (82)$$
$$\frac{\partial L_3}{\partial x_1}=0=\lambda_1(3)\left[u_2+3\right]-\mu_1(3)+\mu_3(3) \quad (83)$$
$$\frac{\partial L_3}{\partial x_2}=0=2\lambda_2(3)\left[u_1+1\right]-\mu_2(3)+\mu_3(3) \quad (84)$$
$$\frac{\partial L_3}{\partial x_3}=0=-8+\rho_1(3)\left[2u_4+5\right]-\beta_1(3) \quad (85)$$
$$\frac{\partial L_3}{\partial x_4}=0=4\rho_2(3)\left[u_3+1\right]-\beta_2(3) \quad (86)$$
$$\frac{\partial L_1}{\partial u_1}=0=1+\lambda_1(1)\left[2u_1-1\right]+2x_2\lambda_2(1)-\mu_4(1) \quad (87)$$
$$\frac{\partial L_2}{\partial u_2}=0=1+\lambda_2(2)\left[2u_2-1\right]+x_1\lambda_1(2)-\mu_5(1) \quad (88)$$
$$\frac{\partial L_3}{\partial u_3}=0=1+\rho_1(3)\left[2u_3-1\right]+4x_4\rho_2(3)-\beta_3(3)+\beta_4(3) \quad (89)$$
$$\frac{\partial L_3}{\partial u_4}=0=\rho_2(3)\left[2u_4-1\right]+2x_3\rho_1(3)-\beta_5(3)+\beta_6(3) \quad (90)$$

where, $\mu_e(l)\geq 0$, $\mu_e(l)h_e(l)=0$ and $\beta_k\psi_k(l)=0$, $e=1,\ldots,5$, $l=1,\ldots,3$, $k=1,\ldots,6$. From (75) - (86), we have

$$\lambda_1(1) = \frac{5+\mu_1(1)-\mu_3(1)}{u_2+3}, \quad \lambda_2(1) = \frac{\mu_2(1)-\mu_3(1)}{2(u_1+1)},$$

$$\lambda_1(2) = \frac{\mu_1(2)-\mu_3(2)}{u_2+3}, \quad \lambda_2(2) = \frac{3+\mu_2(2)-\mu_3(2)}{2(u_1+1)},$$
$$\lambda_1(3) = \frac{\mu_1(3)-\mu_3(3)}{u_2+3}, \quad \lambda_2(3) = \frac{\mu_2(3)-\mu_3(3)+0}{2(u_1+1)},$$
$$\rho_1(1) = \frac{\beta_1(1)}{2u_4+5}, \ \rho_2(1) = \frac{\beta_2(1)}{4(u_3+1)}, \ \rho_1(2) = \frac{\beta_1(2)}{2u_4+5},$$
$$\rho_2(2) = \frac{\beta_2(2)}{4(u_3+1)}, \rho_1(3) = \frac{8+\beta_1(3)}{2u_4+5}, \rho_2(3) = \frac{\beta_2(3)}{4(u_3+1)}$$

From (67) - (70), we get

$$x_1 = \frac{u_1(1-u_1)}{u_2+3}, \quad x_2 = \frac{u_2(1-u_2)}{2(u_1+1)},$$
$$x_3 = \frac{u_3(1-u_3)}{2u_4+5}, \quad x_4 = \frac{u_4(1-u_4)}{4(u_3+1)}$$

Substituting these results into (87) - (90), the Nash Min-Max solution for player 3 with $\mu_e(l) = 0$ and $\beta_k(l) = 0$ except $\beta_6(3) \leq 0$ for all $e = 1,...,5$, $l = 1,...,3$ and $k = 1,...,6$ is

$$(u_1^*, u_2^*, u_3^*, u_4^*) = \left(\frac{11}{58}, \frac{3}{29}, \frac{1}{16}, 1\right)$$

Then,

$$x_1^* = \frac{u_1^*(1-u_1^*)}{u_2^*+3} = 0.0495, \quad x_2^* = \frac{u_2^*(1-u_2^*)}{2(u_1^*+1)} = 0.0390,$$
$$x_3^* = \frac{u_3^*(1-u_3^*)}{2u_4^*+5} = 0.0084, \quad x_4^* = \frac{u_4^*(1-u_4^*)}{4(u_3^*+1)} = 0$$

The corresponding profits for each firm are

$$P_1 = 0.0578, \qquad P_2 = 0.0135,$$
$$P_3 = 0.0045, \qquad P_4 = 1$$

2. To determine the MMS for player 4, define

$$\begin{aligned} L_4 = & -5x_3 + u_4 - \lambda_1(4)\left[-3x_1 + u_1 - u_1^2 - x_1u_2\right] - \lambda_2(4) \\ & \left[-2x_2 + u_2 - u_2^2 - 2x_2u_1\right] - \rho_1(4)\left[-5x_3 + u_3 - u_3^2\right. \\ & \left.-2x_3u_4\right] - \rho_2(4)\left[-4x_4 + u_4 - u_4^2 - 4x_4u_3\right] - \mu_1(4) \\ & x_1 - \mu_2(4)x_2 - \mu_3(4)\left[1 - x_1 - x_2\right] - \mu_4(4)u_1 - \mu_5(4) \\ & u_2 - \beta_1(4)x_3 - \beta_2(4)x_4 - \beta_3(4)u_3 - \beta_4(4)\left[1-u_3\right] - \\ & \beta_5(4)u_4 - \beta_6(4)\left[1-u_4\right] \end{aligned}$$

where $\beta_1(4), \beta_2(4), \beta_5(3), \beta_6(3) \geq 0$ and $\beta_3(4), \beta_4(4) \leq 0$. Then,

$$\frac{\partial L_4}{\partial x_1} = 0 = \lambda_1(4)\left[u_2+3\right] - \mu_1(4) + \mu_3(4) \tag{91}$$
$$\frac{\partial L_4}{\partial x_2} = 0 = 2\lambda_2(4)\left[u_1+1\right] - \mu_2(4) + \mu_3(4) \tag{92}$$
$$\frac{\partial L_4}{\partial x_3} = 0 = \rho_1(4)\left[2u_4+5\right] - \beta_1(4) \tag{93}$$
$$\frac{\partial L_4}{\partial x_4} = 0 = -5 + 4\rho_2(4)\left[u_3+1\right] - \beta_2(4) \tag{94}$$
$$\begin{aligned} \frac{\partial L_4}{\partial u_3} = & \ 0 = \rho_1(4)\left[2u_3 - 1\right] + 4x_4\rho_2(4) - \beta_3(4) \\ & + \beta_4(4) \end{aligned} \tag{95}$$
$$\begin{aligned} \frac{\partial L_4}{\partial u_4} = & \ 0 = 1 + \rho_2(4)\left[2u_4 - 1\right] + 2x_3\rho_1(4) - \beta_5(4) \\ & + \beta_6(4) \end{aligned} \tag{96}$$

where $\mu_e(l) \geq 0$, $\mu_e(l)h_e(l) = 0$ and $\beta_k\psi_k(l) = 0$, $e = 1,...,5$, $l = 1,...,3$, $k = 1,...,6$. From (91) - (94), we have

$$\lambda_1(4) = \frac{\mu_1(4)-\mu_3(4)}{u_2+3}, \quad \lambda_2(4) = \frac{\mu_2(4)-\mu_3(4)}{2(u_1+1)},$$
$$\rho_1(4) = \frac{\beta_1(4)}{2u_4+5}, \quad \rho_2(4) = \frac{5+\beta_2(4)}{4(u_3+1)}$$

Substituting these results into (95) - (96), the Nash Min-Max solution for player 4 with $\mu_e(l) = 0$ and $\beta_k(l) = 0$ except $\beta_5(4) \geq 0$ for all $e = 1,...,5$, $l = 1,...,3$ and $k = 1,...,6$ is

$$\left\{ (u_1^*, u_2^*, u_3^*, u_4^*) \in \Re^4 \;\middle|\; \begin{array}{l} u_1^* = \frac{11}{58}, \ u_2^* = \frac{3}{29}, \\ \frac{1}{4} < u_3^* < 1, \ u_4^* = 0 \end{array} \right\}$$

Then,

$$x_1^* = \frac{u_1^*(1-u_1^*)}{u_2^*+3} = 0.0495, \quad x_2^* = \frac{u_2^*(1-u_2^*)}{2(u_1^*+1)} = 0.0390,$$
$$0 < x_3^* = \frac{u_3^*(1-u_3^*)}{2u_4^*+5} < 0.0375, \quad x_4^* = \frac{u_4^*(1-u_4^*)}{4(u_3^*+1)} = 0$$

The corresponding profits for each firm are

$$P_1 = 0.0578, \qquad P_2 = 0.0135,$$
$$0.05 < P_3 < 1, \qquad P_4 = 0$$

7 Conclusion and Recommendations

In this paper, a solving technique for hybrid continuous static games between multiple players playing independently using NES and others playing under a secure concept using MMS was developed. The necessary optimal conditions for obtaining a regular local Nash Min-Max solution point were stated. An algorithm for solving such static optimization problems was presented. A four firm's maximization profit application was introduced as an example to clarify the algorithm steps.

The present paper recommends using the developed approach with its proposed algorithm to solve fuzzy hybrid continuous static games. Also, using the paper approach in industrial static optimization problems that have the same formulation, such as utility and consumer games. Furthermore, developing the procedure to extend the hybrid concept of continuous static games by mixing other types of solutions depending on the application's nature. Moreover, adopting the idea of hybrid games in solving the current vital problems between countries, such as the Renaissance Dam problem.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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