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Multistage Optimal Homotopy Asymptotic Method for the Nonlinear Riccati Ordinary Differential Equation in Nonlinear Physics

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Abstract: The present paper addresses the approximate analytical solution of the nonlinear Riccati differential equations using the Multistage homotopy asymptotic method which is used in nonlinear physics. The suggested algorithm is accurate, effective and simple to utilize semi-analytic tool for nonlinear problems. To test accuracy of the recommended algorithm serval test problems are considered and the obtained results are compared with those of the recent literature. These results revealed accuracy and efficiency of the suggested method.

Keywords: Multistage Optimal Homotopy Asymptotic Method, Riccati Differential Equation, Series solution

1 Introduction

Nonlinear Ordinary differential equations (ODEs) are utilized to understand and model several realism matters in applied science and material science. One of the indispensable nonlinear ODE is the Riccati equation which has the following form

$$\frac{dy}{d\hat{\tau}} = P(\hat{\tau}) + Q(\hat{\tau})y + R(\hat{\tau})y^2, \tag{1}$$

where P, Q, R and y are real functions of the $\hat{\tau}$.
Model equation (1) has applications in diffusion problems, random processes and engineering science, including network synthesis, optimal control and robust stabilization. Reid [1] featured a portion of the essential theoretical concepts identified with the Riccati equation. Because of its significance, its efficient and accurate solution is essential. In most cases, it is hard to tackle nonlinear problems analytically. To find the approximate analytical solution numerous techniques have been recently used for Riccati model equation. In [2, 3], the authors utilized the Adomian decomposition method (ADM) while in [4], the variational iteration method (VIM) was utilized. Similarly, the Homotopy analysis method (HAM) was implemented in [5] whereas optimal homotopy asymptotic method (OHAM) has been utilized in [6] and Iterative reproducing kernel Hilbert spaces method has been implemented in [7].

The OHAM is an approximate analytical method which can be easily used and has a built in convergence criteria, like homotopy analysis method (HAM), but with a higher degree of flexibility. One of the advantages of OHAM is independency whether small parameters exist or do not exist in the governing model equation. Many authors have shown that the suggested procedure is accurate and reliable, and calculated the solutions of complex problems which have significant applications in science and technology, see [8–14] and the references theirin. It is noticeable that OHAM solution with the easiest auxiliary function of the form $H_i(q)$ for the initial value problems is valid for a short time span. Accordingly, to circumvent this limitation, a new

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modification based on the standard OHAM is made. It is an easy way to insure the validity of the approximations of large time T is dividing the interval $[0,T]$ by subinterval as $[\hat{\tau}_0,\hat{\tau}_1],...,[\hat{\tau}_{j-1},\hat{\tau}_j]$ where $\hat{\tau}_j = T$ and implementing the Multistage homotopy asymptotic method (MOAHM) solution on each subintervals. In the proposed procedure, previous interval's solution is used for the initial approximation in each interval, so one can calculate a continuous approximate analytical solution.

Currently, the researchers work competently to develop methods for the approximate and exact solutions for such sort of complex PDEs [15–19]. In this regard, numerous methods have been employed for the solution like finite difference method [20], homotopy analysis method [21, 22], variational iteration method (VIM) and modified VIM algorithms [23–30], meshless method [31–41] and Adomian decomposition method [42]. The present paper aims to utilize MOHAM to find out the approximate solutions for the nonlinear Riccati ordinary differential equations.

The rest of this paper is organized as follow: Section 2 addresses the principles of the proposed algorithm. In Section 3, we employ the suggested method for solving several examples of the model equations. Results are presented and compared with those in the previous pieces of literature in Section 4. The last section is dedicated to conclusion.

2 Description of MOHAM

This section is devoted to the principles of the OHAM as given in [43–45]. Consider the initial-value problem

$$L_i(y_i(\hat{\tau}))+N_i(y_i(\hat{\tau}))=0,\ \ i=1,2,...,N, \tag{2}$$

Initial condition as

$$y_i(\alpha)=\alpha_i, \tag{3}$$

where $y_i(\hat{\tau})$ represents the unknown function, whereas L_i and N_i denote the linear operator and nonlinear operator, and $\hat{\tau}$ is the independent variable. By means of the MOHAM, one can construct the homotopy

$$(1-q)[L_i(v_i(\hat{\tau},q))-y_{i,0}(\hat{\tau})]=Hi(q,\hat{\tau})[L_i(v_i(\hat{\tau},q)) +N_i(v_i(\hat{\tau},q))], \tag{4}$$

Here, $q\in[0,1]$ and $\hat{\tau}\in R$ where $H_i(q)\neq 0$ is an auxiliary function. $H_i(0)=0$ for $q?=0$, and $v_i(\hat{\tau},q)$ is an unknown function. It is understood that $v_i(\hat{\tau},0)=y_{i,0}(\hat{\tau})$ holds for $q=0$ and $v_i(\hat{\tau},0)=y_{i,0}(\hat{\tau})$ holds for $q=1$. Similarly, q changes from 0 to 1, and the solution $v_i(\hat{\tau},q)$ changes from $y_{i,0}(\hat{\tau})$ to $y_i(\hat{\tau})$ where $y_{i,0}(\hat{\tau})$ is the initial guess which is known and calculated from Eq. (3) for $q=0$ as

$$L_i(y_{i,0}(\hat{\tau}))=0. \tag{5}$$

Now, $H_i(q)$ has been chosen in the following manner

$$H_i(q)=C_{1,j}q+C_{2,j}q^2+C_{3,j}q^3+\dots \tag{6}$$

or

$$H_i(q,\hat{\tau})=C_{1,j}q+C_{2,j}\hat{\tau}q^2+C_{3,j}\hat{\tau}^2q^3+\dots \tag{7}$$

where $C_{1,j},C_{2,j},C_{3,j},\dots$ denote convergence control parameters (CCPs). To compute the required approximate solution, the Taylor's series are utilized in the accompanying form to expand $v_i(\hat{\tau},q,C_k)$ about q

$$v_i(\hat{\tau},q,C_k)=y_0(\hat{\tau})+\sum_{k=1}^{\infty}y_{i,k}(\hat{\tau},C_1,C_2,\dots,C_k)q^k. \tag{8}$$

Define the vectors

$$\mathbf{C}_i=\{C_1,C_2,\dots,C_i\},$$

$$\mathbf{y}_{i,s}=\{y_{i,0}(\hat{\tau}),y_{i,1}(\hat{\tau},C_1),\dots,y_{i,s}(\hat{\tau},\mathbf{C}_s)\},$$

where $s=1,2,3,\dots$, setting Eq. (8) into Eq. (4) and to the linear equations which are given below, we proceed by comparing coefficient q, Also, Eq. (5) gives the zeroth-order problem, whereas the first- and second-order problems are given, as follows:

$$L_i(y_{i,1}\hat{\tau}))=C_1N_0(\mathbf{y}_{i,0}),\quad y_{i,1}(a)=0, \tag{9}$$

and

$$L_i(y_{i,2}(\hat{\tau}))-L_i(y_{i,1}(\hat{\tau}))=C_2N_{i,0}(\mathbf{y}_{i,0})+C_{1,j}[L_i(y_{i,1}(\hat{\tau})) +N_{i,1}(\mathbf{y}_{i,1})],y_{i,2}(a)=0. \tag{10}$$

The general equations for $y_{i,k}(\hat{\tau})$ are

$$L_i(y_{i,k}(\hat{\tau}))-L_i(y_{i,k-1}(\hat{\tau}))=C_{k,j}N_{i,0}(y_{i,0}(\hat{\tau})) +\sum_{m=1}^{k-1}C_{i,m}[L_i(y_{i,k-m}(\hat{\tau})) +N_{i,1}(\mathbf{y}_{i,k-1})],\ \ y_{i,k}(a)=0, \tag{11}$$

where $k=2,3,\dots$ and $N_{i,m}(y_0(\hat{\tau}),y_{i,1}(\hat{\tau}),\dots,y_{i,m}(\hat{\tau}))$ is the coefficient of q^m in the expansion of $N_i(v_i(\hat{\tau},q))$ about q which is known as embedding parameter,

$$N_i(v_i(\hat{\tau},q))=N_{i,0}(y_{i,0}(\hat{\tau}))+\sum_{m=1}^{\infty}N_{i,m}(\mathbf{y}_{i,m})q^m. \tag{12}$$

Since the convergence of Eq. (12) depends on the CCPs $C_1,C_2,C_3,\dots$, if it is convergent at $q=1$, then

$$v_i(\hat{\tau},C_k)=y_{i,0}(\hat{\tau})+\sum_{k=1}^{\infty}y_k(\hat{\tau},C_1,C_2,\dots,C_k). \tag{13}$$

The result of the m^{th}-order approximation is as follows

$$\tilde{y}(\hat{\tau},C_1,C_2,C_3,\dots,C_k)=y_0(\hat{\tau})+\sum_{k=1}^{\infty}y_k(\hat{\tau},C_1,C_2,\dots,C_k). \tag{14}$$

Substituting Eq. (14) into Eq. (2) gives the accompanying residual

$$\begin{aligned} &R_i(\widehat{\tau}, C_{1,j}, C_{2,j}, C_{3,j}, ..., C_{m,j}) \\ &= L(\tilde{y}_i(\widehat{\tau}, C_{1,j}, C_{2,j}, C_{3,j}, ..., C_{m,j})) \\ &+ N(\tilde{y}_i(\widehat{\tau}, C_{1,j}, C_{2,j}, C_{3,j}, ..., C_{m,j})), \end{aligned} \tag{15}$$

where $y(\widehat{\tau})$ represents the exact solution when $R_i = 0$. It is noticeable that such type of case will not happen for nonlinear problems, yet we can limit the function

$$\begin{aligned} &J_i(C_{1,j}, C_{2,j}, C_{3,j}, ..., C_{m,j}) \\ &= \int_{\widehat{\tau}_j}^{\widehat{\tau}_{j+h}} R_i^2(y, \widehat{\tau}, C_{1,j}, C_{2,j}, C_{3,j}, ..., C_{m,j}) d\widehat{\tau}, \end{aligned} \tag{16}$$

where the length and the number of subintervals $[\widehat{\tau}_j, \widehat{\tau}_{j+1}]$ are denoted by h and $N = [\frac{T}{h}]$ respectively. Next, in each subinterval, changing the initial approximation from the previous one, we can solve Eq. (16) at $j = 0, 1, \ldots, N$. For instance, we define $\alpha = y(\widehat{\tau}_j)$ in $[\widehat{\tau}_j, \widehat{\tau}_{j+1}]$. The unknown CCPs $C_{i,j}(i = 1, 2, 3, \ldots, m, \quad j = 1, 2, \ldots, N)$ can be defined from the solution of the system of equations given below

$$\frac{\partial J}{\partial C_{1,j}} = \frac{\partial J}{\partial C_{2,j}} = \ldots = \frac{\partial J}{\partial C_{m,j}} = 0, \tag{17}$$

so the approximate solution is given, as follows:

$$\tilde{y}(\widehat{\tau}) = \begin{cases} y_1(\widehat{\tau}), & \widehat{\tau}_0 \le \widehat{\tau} < \widehat{\tau}_1, \\ y_2(\widehat{\tau}), & \widehat{\tau}_1 \le \widehat{\tau} < \widehat{\tau}_2, \\ . \\ . \\ . \\ y_N(\widehat{\tau}), & \widehat{\tau}_{N-1} \le \widehat{\tau} < T. \end{cases} \tag{18}$$

We effectively calculate the initial value problems' solution analytically for large value of T. MOHAM converts to the standard OHAM when $j = 0$. It also essential to mention that MOHM gives an easy way to adjust and control the convergence region by means of the auxiliary function $H_i(q)$ involving many convergent control parameters (CCPs) $C_{i,j}$'s. Then, the proposed method overcomes the main difficulty, due to the large computational domain, in calculating the solution of problems.

3 Implementation of proposed scheme

The suggested MOHAM is implemented in Ricaati nonlinear differential equations to show the effectiveness and validity of the algorithm. Furthermore, the initial-boundary conditions can be computed easily in accordance with the exact solution throughout the paper.

Test Problem 1*First, let's consider the nonlinear initial-value problem [2, 4, 6, 7, 46, 47]*

$$\frac{dy}{dt} + y^2 - 1 = 0, \quad y(0) = 0, \tag{19}$$

with the exact solution

$$y(\widehat{\tau}) = \frac{e^{2\widehat{\tau}} - 1}{e^{2\widehat{\tau}} + 1}. \tag{20}$$

To solve Test Problem 1 utilizing MOHAM, we proceed as follows

$$L[y(\widehat{\tau}, q)] = \frac{dy(\widehat{\tau}, q)}{d\widehat{\tau}}, \; g(\widehat{\tau}) = -1, \quad N[y(\widehat{\tau}, q)] = y^2(\widehat{\tau}, q). \tag{21}$$

Choose the auxiliary function $H_i(q)$ as $H_i(q) = (C_{1,j}q + C_{2,j}q^2 + C_{3,j}q^3)$, where $C_{1,j}, C_{2,j}, C_{3,j}$ are unknown constants to be computed.
Using the method as discussed in Section 1 with step-size $h = 0.1$ and starting with $\widehat{\tau}_0 = 0$ to $\widehat{\tau}_{10} = T = 1$ various order problems and their solutions for the first subintervals are given, as follow:
Zeroth-order problem

$$\acute{y}_0(\widehat{\tau}) = 0, \quad y_0(0) = 0, \tag{22}$$

Their solution

$$y_0 = 0. \tag{23}$$

First-order problem

$$\acute{y}_1(\widehat{\tau}, C_{1,j}) = C_{1,j}(y_0^2 - 1) + (1 + C_{1,j})\acute{y}_0, \; y_1(0) = 0. \tag{24}$$

Solution

$$y_1 = -C_{1,j}\widehat{\tau}. \tag{25}$$

Second-order problem

$$\begin{aligned} \acute{y}_2(\widehat{\tau}, C_{1,j}, C_{2,j}) &= C_{2,j}(y_0^2 - 1) + 2C_{1,j}y_0y_1 + C_{2,j}\acute{y}_0 \\ &+ (1 + C_{1,j})\acute{y}_1, \; y_2(0) = 0. \end{aligned} \tag{26}$$

Solution

$$y_2 = -(C_{1,j} + C_{1,j}^2 + C_{2,j})\widehat{\tau}. \tag{27}$$

Third-order problem

$$\begin{aligned} \acute{y}_3(t, C_{1,j}, C_{2,j}, C_{3,j}) &= C_{3,j}(y_0^2 - 1) + (2C_{2,j}y_0 + C_{1,j}y_1)y_1 \\ &+ 2C_{1,j}y_0y_1 + C_{3,j}\acute{y}_0 + C_{2,j}\acute{y}_1 \\ &+ (1 + C_{1,j})\acute{y}_2, \quad y_3(0) = 0. \end{aligned} \tag{28}$$

Solution

$$\begin{aligned} y_3 = \frac{1}{3}(&-3C_{1,j}\widehat{\tau} - 6C_{1,j}^2\widehat{\tau} - 3C_{1,j}^3\widehat{\tau} - 3C_{2,j}\widehat{\tau} \\ &- 6C_{1,j}C_{2,j}\widehat{\tau} - 3C_{3,j}\widehat{\tau} + C_{1,j}^3\widehat{\tau}^3). \end{aligned} \tag{29}$$

Thus, the third-order solution in the first subinterval is

$$\begin{aligned} \tilde{y} &= y_0(\widehat{\tau}) + y_1(\widehat{\tau}, C_{1,j}) + y_2(\widehat{\tau}, C_{1,j}, C_{2,j}) \\ &+ y_3(\widehat{\tau}, C_{1,j}, C_{2,j}, C_{3,j}). \end{aligned} \tag{30}$$

Setting Eqs. (23), (25), (27), (29) in equation Eq. (30), we have

$$\tilde{y} = -C_{1,j}\widehat{\tau} + (-C_{1,j} - C_{1,j}^2 - C_{2,j})\widehat{\tau} + \frac{1}{3}(-3C_{1,j}\widehat{\tau} - 6C_{1,j}^2\widehat{\tau} - 3C_{1,j}^3\widehat{\tau} - 3C_{2,j}\widehat{\tau} - 6C_{1,j}C_{2,j}\widehat{\tau} - 3C_{3,j}\widehat{\tau} + C_{1,j}^3\widehat{\tau}^3). \quad (31)$$

Setting Eq. (31) into Eq. (19) yields the residual and the functional J, respectively.

$$R_i(\widehat{\tau}, C_{1,1}, C_{2,1}, C_{3,1}) = \frac{d\tilde{y}}{d\widehat{\tau}} + \tilde{y}^2 - 1. \quad (32)$$

$$J_i(C_{1,1}, C_{2,1}, C_{3,1}) = \int_0^{0.1} R_i^2(\widehat{\tau}, C_{1,1}, C_{2,1}, C_{3,1}). \quad (33)$$

From the conditions in equation (17), we have

$$\frac{\partial J}{\partial C_{1,k}} = 0, \quad k = 1, 2, 3. \quad (34)$$

Similar procedure is adopted for the remaining subintervals. The values of the CCPs $C_{i,j}$, $i, j = 1, 2, 3$ are presented in Table 1.
Using values of the CCPs, the third-order MOHAM approximate solution (31) is

$$\tilde{y} = \begin{cases} 0. + 0.996671336163\widehat{\tau}, \quad 0 \le \widehat{\tau} \le 0.1 \\ -0.000029418844814 + 1.0007606282\widehat{\tau} - 0.0069983\widehat{\tau}^2 \\ -0.31070241\widehat{\tau}^3 + 0.0100618\widehat{\tau}^4 + 0.003425\widehat{\tau}^5 \\ -0.00003167\widehat{\tau}^6 - 0.00000503\widehat{\tau}^7, \quad 0.1 \le \widehat{\tau} \le 0.2 \\ -0.0002888 + 1.0043734\widehat{\tau} - 0.0229839\widehat{\tau}^2 \\ -0.2928424\widehat{\tau}^3 + 0.0292025\widehat{\tau}^4 + 0.0112269\widehat{\tau}^5 \\ -0.00040991\widehat{\tau}^6 - 0.0000732\widehat{\tau}^7, \quad 0.2 \le \widehat{\tau} \le 0.3 \\ -0.00110897 + 1.01194 - 0.0443849\widehat{\tau}^2 - 0.280614\widehat{\tau}^3 \\ +0.0448625\widehat{\tau}^4 + 0.0195707\widehat{\tau}^5 - 0.00154347\widehat{\tau}^6 \\ -0.000314995\widehat{\tau}^7, \quad 0.3 \le \widehat{\tau} \le 0.4 \\ -0.00405472 + 1.03475\widehat{\tau} - 0.105588\widehat{\tau}^2 - 0.219526\widehat{\tau}^3 \\ +0.0348786\widehat{\tau}^4 + 0.0153235\widehat{\tau}^5 - 0.00137746\widehat{\tau}^6 \\ -0.000327967\widehat{\tau}^7, \quad 0.4 \le \widehat{\tau} \le 0.5 \\ -0.00405534 + 1.03658 - 0.111662\widehat{\tau}^2 - 0.216119\widehat{\tau}^3 \\ +0.0368791^4 + 0.0169487\widehat{\tau}^5 \\ -0.00105361\widehat{\tau}^6 - 0.000301032\widehat{\tau}^7, \quad 0.5 \le \widehat{\tau} \le 0.6 \\ -0.00186323 + 1.0128 - 0.0277962\widehat{\tau}^2 - 0.33006\widehat{\tau}^3 \\ +0.067117\widehat{\tau}^4 + 0.058378\widehat{\tau}^5 - 0.0123766\widehat{\tau}^6 \\ -0.0044202\widehat{\tau}^7, \quad 0.6 \le \widehat{\tau} \le 0.7 \\ -0.00125685 + 1.00944\widehat{\tau} - 0.0214852\widehat{\tau}^2 - 0.331413\widehat{\tau}^3 \\ +0.0564445\widehat{\tau}^4 + 0.0707526\widehat{\tau}^5 - 0.0141525\widehat{\tau}^6 \\ -0.00673927\widehat{\tau}^7, \quad 0.7 \le \widehat{\tau} \le 0.8 \\ -0.000631917 + 1.00406\widehat{\tau} - 0.0100671\widehat{\tau}^2 - 0.333289\widehat{\tau}^3 \\ +0.0396243\widehat{\tau}^4 + 0.0847778\widehat{\tau}^5 - 0.0133528\widehat{\tau}^6 \\ -0.00953772\widehat{\tau}^7, \quad 0.8 \le \widehat{\tau} \le 0.9 \\ -0.0018064 + 0.999734\widehat{\tau} + 0.00515828\widehat{\tau}^2 - 0.333541\widehat{\tau}^3 \\ +0.0155664\widehat{\tau}^4 + 0.0979867\widehat{\tau}^5 - 0.00885416\widehat{\tau}^6 \\ -0.0126488\widehat{\tau}^7, \quad 0.9 \le \widehat{\tau} \le 1. \end{cases}$$

Table 1: values of the CCPs $C_{i,j}$ for Test Problem 1.

j	$C_{1,j}$	$C_{2,j}$	$C_{3,j}$
1	0.	-0.765845968670	0.535020601178
2	-0.981011273584	-0.005678127993	0.000191514720
3	-0.963243543160	-0.005864337183	0.000356952046
4	-0.932350143076	-0.010096794169	0.001028931060
5	-0.663132015693	-0.204090050582	0.099132469520
6	-0.496386056003	-0.490046974348	0.366210085802
7	-0.994825946937	0.0001697579701	0.000002877803
8	-0.978187220353	0.0000944362786	0.000004034184
9	-0.968675169304	-0.000010388908	0.000017971951
10	-0.962006041350	-0.000101003368	0.000003724602

Table 2: Comparison of the 3rd MOHAM with [2,4,7,46,47] for Test Problem 1.

$\widehat{\tau}$	ADM [2]	VIM [4]	IDM [47]	MHPM [46]	IRKHSM [7]	MOHAM
0	0	0	0	0	0	0
0.1	8.82×10^{-14}	5.0×10^{-11}	1.00×10^{-11}	0	9.1×10^{-6}	8.6×10^{-7}
0.2	1.78×10^{-10}	4.3×10^{-9}	1.0×10^{-12}	0	1.7×10^{-5}	9.7×10^{-7}
0.3	1.51×10^{-8}	1.5×10^{-7}	2.50×10^{-9}	1.0×10^{-6}	2.4×10^{-5}	1.3×10^{-6}
0.4	3.49×10^{-7}	1.9×10^{-6}	5.61×10^{-8}	5.0×10^{-6}	2.9×10^{-5}	1.5×10^{-6}
0.5	3.92×10^{-6}	1.3×10^{-5}	6.03×10^{-7}	3.9×10^{-5}	3.1×10^{-5}	2.4×10^{-6}
0.6	2.80×10^{-5}	6.6×10^{-5}	4.09×10^{-6}	1.9×10^{-4}	3.2×10^{-5}	1.7×10^{-6}
0.7	1.46×10^{-4}	2.4×10^{-4}	2.01×10^{-5}	7.4×10^{-4}	3.1×10^{-5}	1.3×10^{-6}
0.8	6.04×10^{-4}	7.3×10^{-4}	7.78×10^{-5}	2.3×10^{-3}	2.8×10^{-5}	1.1×10^{-6}
0.9	2.09×10^{-3}	1.9×10^{-3}	2.50×10^{-4}	6.3×10^{-3}	2.3×10^{-5}	1.3×10^{-6}
1.0	6.30×10^{-3}	4.4×10^{-3}	6.99×10^{-4}	1.5×10^{-3}	1.2×10^{-5}	1.0×10^{-7}

Table 2 contains the approximate results computed by the third-order MOHAM and compared with those results obtained by various methods such as ADM [2], VIM [4], IDM [47], MHPM [46] and IRKHSM [7]. The tabulated values revealed that the MOHAM produced better results compared to the methods in [2, 4, 7, 46, 47]. Moreover, the error in the solution of MOHAM does not grow exponentially as the time progresses compared to other methods reported in this table.

Test Problem 2

$$\frac{dy}{d\widehat{\tau}} - 2y + y^2 - 1 = 0, \quad y(0) = 0, \quad (35)$$

This problem was also considered in [2, 4, 6, 7, 46, 47]. The exact solution is:

$$y(\widehat{\tau}) = 1 + \sqrt{2}\tanh[\sqrt{2}\widehat{\tau} + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})]. \quad (36)$$

Now, to solve the problem (34) using MOHAM, we consider linear and nonlinear operators as

$$L[y(\widehat{\tau}, q)] = \frac{dy(\widehat{\tau}, q)}{d\widehat{\tau}}, \quad g(\widehat{\tau}) = -1, \quad N[y(\widehat{\tau}, q)] = y^2(\widehat{\tau}, q). \quad (37)$$

The auxiliary function $H_i(q)$ is selected in the form $H_i(q) = C_{1,j}p + C_{2,j}p^2$ where $C_{1,j}, C_{2,j}$ are unknown CCPs .
Using the algorithm discussed in Section 2 by taking step-size $h = 0.2$ and a start with $\widehat{\tau}_0 = 0$ to $\widehat{\tau}_5 = T = 1$, various order problems and their solution are Zeroth-order problem

$$\acute{y}_0(q) = 0, \quad y_0(0) = 0. \quad (38)$$

Solution

$$y_0 = 0. \tag{39}$$

First-order problem

$$\acute{y}_1(\widehat{\tau}, C_{1,j}) = (1 + C_{1,j})\acute{y}_0 + C_{1,j}y_0^2 - C_{1,j} - 2C_{1,j}y_0, \quad y_1(0) = 0. \tag{40}$$

Solution

$$y_1 = -C_{1,j}\widehat{\tau}. \tag{41}$$

Second-order problem

$$\acute{y}_2(\widehat{\tau}, C_{1,j}, C_{2,j}) = (1 + C_{1,j})\acute{y}_1 + C_{2,j}\acute{y}_1 + 2C_{1,j}(y_0 - 1)y_1 - C_{2,j}(1 - (y_0 - 2)y_0), \quad y_2(0) = 0. \tag{42}$$

Solution

$$y_2 = -C_{1,j}\widehat{\tau} - C_{1,j}^2\widehat{\tau} - C_{2,j}\widehat{\tau} + C_{1,j}^2\widehat{\tau}^2. \tag{43}$$

The approximate solution of the problem (35) can be written as

$$\tilde{y} = y_0(\widehat{\tau}) + y_1(\widehat{\tau}, C_{1,j}) + y_2(\widehat{\tau}, C_{1,j}, C_{2,j}). \tag{44}$$

Putting Eqs, (39), (41) and (43) into Eq.(44), we have

$$\tilde{y} = -2C_{1,j}\widehat{\tau} - C_{1,j}^2\widehat{\tau} - C_{2,j}\widehat{\tau} + C_{1,j}^2\widehat{\tau}^2. \tag{45}$$

The values of CCPs for the first subinterval are computed and discussed in Section 2. Similar procedure is employed for the remaining subintervals. The values of CCPs are given in Table 3. Using values of these constants in Eq. (45), a second-order MOHAM solution of problem (35) is obtained as

$$\tilde{y} = \begin{cases} 0.996256\widehat{\tau} + 1.06815\widehat{\tau}^2, \quad 0 \le \widehat{\tau} \le 0.2 \\ -0.0029499 + 1.014\widehat{\tau} + 1.07187\widehat{\tau}^2 - 0.0877265\widehat{\tau}^3 \\ -0.0290574\widehat{\tau}^4 + 0.00174382\widehat{\tau}^5, \quad 0.2 \le \widehat{\tau} \le 0.4 \\ -0.0484512 + 1.20571\widehat{\tau} + 0.94692\widehat{\tau}^2 - 0.241099\widehat{\tau}^3 \\ +0.0881029\widehat{\tau}^4 + 0.015175\widehat{\tau}^5, \quad 0.4 \le \widehat{\tau} \le 0.6 \\ 0.081225 + 0.505438\widehat{\tau} + 2.11966\widehat{\tau}^2 - 0.596433\widehat{\tau}^3 \\ -0.617155\widehat{\tau}^4 + 0.19027\widehat{\tau}^5, \quad 0.6 \le \widehat{\tau} \le 0.8 \\ -0.0149412 + 0.883726\widehat{\tau} + 1.60129\widehat{\tau}^2 - 0.24679\widehat{\tau}^3 \\ -0.845274\widehat{\tau}^4 + 0.311478\widehat{\tau}^5, \quad 0.8 \le \widehat{\tau} \le 1. \end{cases}$$

The approximate results are tabulated in Table 4 for Test Problem 2. Here, we have compared those of the suggested second-order MOHAM with the results given in MHPM [46], VIM [4], OHAM [6] and IRKHSM [7]. It shows that on one hand MOHAM results are better than those of the cited methods, and on the other hand no fast growth of error occurs with MOHAM compared to other cited methods. Thus, MOHAM provided a two-pronged benefit for this particular test problem.

Table 3: values of the CCPs $C_{i,j}$ s for Test Problem 2.

j	$C_{1,j}$	$C_{2,j}$
1	-1.033511840649	0.002621140292
2	-0.960771065976	-0.00652715816
3	-0.788482838197	-0.05573467233
4	-1.282926388522	-0.07728547770
5	-0.978382586870	0.001991200890

Table 4: Comparison of the second-order MOHAM and the methods in [4, 6, 7, 46] for Test Problem 2.

$\widehat{\tau}$	MHPM [46]	VIM [4]	OHAM [6]	IRKHSM [7]	MOHAM
0	0	0	0	0	0
0.2	1.2×10^{-5}	1.0×10^{-6}	2.9×10^{-4}	7.6×10^{-5}	2.3×10^{-7}
0.4	3.0×10^{-4}	3.3×10^{-5}	2.5×10^{-3}	1.7×10^{-4}	3.6×10^{-6}
0.6	4.7×10^{-3}	9.9×10^{-5}	5.5×10^{-3}	2.5×10^{-4}	1.7×10^{-5}
0.8	1.9×10^{-2}	1.5×10^{-5}	3.8×10^{-3}	3.4×10^{-4}	1.7×10^{-5}
1.0	3.4×10^{-2}	3.4×10^{-3}	3.4×10^{-3}	9.2×10^{-4}	1.4×10^{-5}

4 Conclusion

In this paper, the proposed Multistage Optimal Homotopy Asymptotic Method was utilized for computing the analytical approximate solutions of the nonlinear Riccati differential equations. For this purpose, Mathematica software was utilized. A comparison between the proposed MOHAM and VIM, ADM, IDM, MHPM, IRKHSM and OHAM in the interval [0,1] in light of approximate results, MOHAM proved more effective than these methods. Furthermore, the absolute error of the MOHAM was consistent and had no exponential growth. This shows that the present scheme is more powerful to solve nonlinear differential equations with lower order of approximation. One of the advantages of MOHAM is easy and straightforward calculations and the reduction in the size of computational domain. Moreover, the suggested method helps control the convergence region of the series solution. The results demonstrated that the MOHAM is accurate and efficient for computing approximate analytical solution of the nonlinear differential equations utilized in science and engineering.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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