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Spectral Legendre-Chebyshev Treatment of 2D Linear and Nonlinear Mixed Volterra-Fredholm Integral Equation

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Abstract: A numerical algorithm to handle the 2D mixed Volterra-Fredholm integral equations (MV-FIEs) is introduced. The fundamentals of shifted Legendre and shifted Chebyshev polynomials are first mentioned. These fundamentals jointly with the shifted Gauss-Legendre and Chebyshev nodes are then used to transform the integral equations to matrix equation. The main advantages of the presented scheme is its reasonable accuracy. Sequentially, fine numerical results can be obtained via a relatively few number of collocation points. The convergence and error analyses of the method were discussed in detail. Numerical test experiments are offered to demonstrate the validity, efficiency, and applicability of the scheme.

Keywords: Volterra-Fredholm integral equations; Shifted Legendre polynomials; Shifted Chebyshev polynomials; Gauss-Legendre nodes; Gauss-Chebyshev nodes

1 Introduction

In many disciplines of sciences and engineering, many high-accurate approximate solvers are offered to handle the relevant problems. At the forepart of these schemes, weighted residual methods are extensively considered as powerful techniques in the finding semi-analytic solutions for different kinds of differential, integral and integro-differential equations (see, ([1–7])

Recently, several works Were focussed on developing of more new and reliable methods for the linear/nonlinear MV-FIEs. The spline collocation method (SCM) [8] and the successive iterative method (SIM) [9] were introduced for obtaining the approximation solution of these equations. Behzadi [10] used a scheme based on Homotopy analysis method (HAM) for solving nonlinear V-FIEs of the first type, while Abdou et al. [11] was reduced V-FIE of the second type to a system of Fredholm integral equations using Toeplitz matrix method (TMM) and Product Nystrm method (PNM). Shekarabi, Maleknejad and Ezzati [12] applied the 2D Bernstein operational matrices method to solve mixed Volterra-Fredholm integral equations, while in [13], Dastjerdi et al. used the radial basis function approximation for the numerical solution of MV-FIEs. Hafez et al. [1] solved classes of two-dimensional linear and nonlinear MV-FIEs using Bernoulli collocation method. Also, Taylor polynomial method (TPM) has been applied for finding numerical solution of the MV-FIEs in [14], Paripour and Kamyar [15] introduced novel basis functions for approximating the solution of nonlinear V-FIEs via a direct method. Also, the homotopy perturbation method (HPM) and the modified HPM have been used for finding approximate solution of nonlinear V-FIEs in [16], [17], and [18], respectively.

It's established that Legendre and Chebyshev polynomials play a vital role numerical analysis, especially in spectral approximations of differential, integral and integro-differential equations, (see, Refs. [19–24]). Our work aims to build some perfect collocation algorithms to treat 2D linear and nonlinear MV-FIEs. One great performance of such algorithms is that it transforms the underlying problems to solving systems of algebraic/transcendental equations by using the Legendre-Chebyshev polynomials as basis functions hybrid with the Gauss-Legendre and Chebyshev nodes as the collocation points. The collocation algorithm has been good tackled many problems [25–28].

The outline of this paper is as follows: In the next section, we introduce some notations and some few

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mathematical facts needed in the rest the paper. In Section 3, the procedures of constructing the collocation methodology for 2D linear MV-FIEs are described via the shifted Legendre and shifted Chebyshev polynomials. In Section 4, the procedures of constructing the collocation methodology for 2D nonlinearMV-FIEs are described via the shifted Legendre and shifted Chebyshev polynomials. Moreover, in Section 6 the presented technique is used to some various test experiments. Finally, concluding remarks are reported in Section 7.

2 Some properties of polynomials

This section is devoted to some fundamental information of orthogonal shifted Legendre and shifted Chebyshev polynomials.

2.1 Some properties of shifted Legendre polynomials

The known Legendre polynomials $L_i(\eta)$ are defined on the usual interval $(-1,1)$. Firstly, some of the properties of regular Legendre polynomials (LP) have been mentioned in this section. LP satisfy the following

$$L_0(\eta)=1,\ L_1(\eta)=\eta,$$

$$L_{k+2}(\eta)=\frac{2k+3}{k+2}\eta L_{k+1}(\eta)-\frac{k+1}{k+2}L_k(\eta)$$

and the orthogonality relation

$$(L_k(\eta),L_l(\eta))_w=\int_{-1}^{1}L_k(\eta)L_l(\eta)\,w(\eta)\,d\eta=h_k\delta_{lk}.$$

where $w(\eta)=1$, $h_k=\frac{2}{2k+1}$. The Legendre-Gauss-Lobatto (LGL) quadrature was used to compute the previous integrals accurately. For $\phi\in S_{2N-1}[-1,1]$, one have

$$\int_{-1}^{1}\phi(\eta)d\eta=\sum_{j=0}^{N}\varpi_{N,j}\phi(\xi_{N,j}).$$

Consider the following discrete inner product

$$(u,v)_w=\sum_{j=0}^{N}u(\xi_{N,j})\,v(\xi_{N,j})\,\varpi_{N,j}.$$

For Legndre Gauss-Lobatto, [29].
Let the shifted LP $L_i(2\xi-1)$ be denoted by $P_i(\xi)$. Then $P_i(\xi)$ can be resulted with the aid of the following recursive relation:

$$(i+1)P_{i+1}(\xi)=(2i+1)(2\xi-1)P_i(\xi)-iP_{i-1}(\xi),i=1,2,\cdots$$

The analytic power form of the shifted LP $P_i(\xi)$ of degree i is defined by

$$P_i(\xi)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!\,(k!)^2\,L^k}\,\xi^k, \qquad (1)$$

The orthogonality condition is

$$\int_0^1 P_j(\xi)P_k(\xi)w^*(\xi)d\xi=h_k\,\delta_{jk},$$

where $w^*(\xi)=1\,and\,\bar{h}_j=\frac{1}{2j+1}$.

2.2 Some properties of shifted Chebyshev polynomial

The famous Chebyshev polynomials (CP) are defined on the usual interval $(-1,1)$ and can be generated with the aid of the following recurrence formula:

$$T_{k+1}(\eta)=2\eta T_k(\eta)-T_{k-1}(\eta),\quad k=1,2,\cdots,$$

where $T_0(\eta)=1$ and $T_1(\eta)=\eta$. The Chebyshev polynomials are satisfying the following relations

$$T_k(\pm1)=(\pm1)^k,\quad T_k(-\eta)=(-1)^kT_k(\eta).$$

Let $w(\eta)=\dfrac{1}{\sqrt{1-\eta^2}}$, then define the weighted space L^2_w as usual. The inner product and the norm of L^2_w w.r.t. the weight function are defined as:

$$(u,v)_w=\int_{-1}^{1}u(\eta)\,v(\eta)w(\eta)d\eta,\qquad \|u\|_w=(u,u)_w^{\frac{1}{2}}.$$

The set of CP forms a complete L^2_w-orthogonal system, and

$$\|T_k\|_w=h_k=\begin{cases}\frac{C_k}{2}\pi, & k=j,\\ 0, & k\neq j,\end{cases}\qquad C_0=2,\ C_k=1,\ k\geq 1.$$

To tame these polynomials over $(0,1)$ define the shifted CP by the map $\eta=2\xi-1$. Denote the shifted CP $T_i(2\xi-1)$ by $T_i^*(\xi)$. Then $T_i^*(\xi)$ can be generated as follows:

$$T^*_{i+1}(\xi)=2(2\xi-1)T^*_i(\xi)-T^*_{i-1}(\xi),\quad i=1,2,\cdots,$$

where $T^*_0(\xi)=1$ and $T^*_1(\xi)=2\xi-1$. The analytic power form of the shifted CP $T^*_i(\xi)$ of degree i is given by

$$T^*_i(\xi)=i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!\,2^{2k}}{(i-k)!\,(2k)!\,L^k}\,\xi^k, \qquad (2)$$

where $T^*_i(0)=(-1)^i$ and $T^*_i(1)=1$.

The orthogonality condition is

$$\int_0^1 T_j^*(\xi)T_k^*(\xi)w^*(\xi)d\xi = h_k,$$

where $w^*(\xi) = \dfrac{1}{\sqrt{\xi-\xi^2}}$ and $h_k = \begin{cases} \dfrac{\epsilon_j}{2}\pi, & k=j, \\ 0, & k\neq j, \end{cases}$ $\epsilon_0 = 2,\ \epsilon_i = 1,\ i\geq 1.$

3 Linear MV-FIEs

Here, we are focused in using the Legendre-Chebyshev collocation method (L-CCM) for solving linear 2D MV-FIEs:

$$u(\xi,\eta) = g(\xi,\eta) + \int_0^\xi \int_0^1 k(\xi,s,\eta,t)u(s,t)dtds, \quad (3)$$

where, $u(\xi,\eta)$ is an unknown function, $g(\xi,\eta)$ and $k(\xi,s,\eta,t)$ are analytical functions on $[0,1]^2$ and $[0,1]^4$, respectively.

The aim of our methodology is to get solution using the procedures listed in the previous section as:

$$u(\xi,\eta) \backsimeq \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}P_i(\xi)T_j^*(\eta) = \psi(\xi,\eta)C, \quad (4)$$

where, $c_{ij},\ i=0,1,\dots,N,\ j=0,1,\dots,M$ are the unknown coefficients,

$$C = [c_{00}, c_{10},\dots,c_{N0}; c_{01}, c_{11},\dots,c_{N1}; c_{0M}, c_{1M}, \dots, c_{NM}]^T,$$

N and M are any arbitrary positive integers, $P_i(\xi),\ i=0,1,\dots,N$ and $T_j^*(\eta),\ j=0,1,\dots,M$ are shifted legendre polynomials and shifted Chebyshev polynomials defined in Eqs. (1) and (2), respectively. Also $\psi(\xi,\eta)$ is $1\times(N+1)(M+1)$ matrix introduced as follows

$$\psi(\xi,\eta) = [r_{00}(\xi,\eta), r_{10}(\xi,\eta),\dots,r_{N0}(\xi,\eta); r_{01}(\xi,\eta), r_{11}(\xi,\eta),\dots,r_{N1}(\xi,\eta); r_{0M}(\xi,\eta), r_{1M}(\xi,\eta), \dots, r_{NM}(\xi,\eta)],$$

where

$$r_{ij}(\xi,\eta) = P_i(\xi)T_j^*(\eta),\ i=0,1,\dots,N,\ j=0,1,\dots,M.$$

Substituting Eq. (4) into Eq. (3) yields:

$$\sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}P_i(\xi)T_j^*(\eta) = g(\xi,\eta) + \int_0^\xi\int_0^1 k(\xi,s,\eta,t)\sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}P_i(s)T_j^*(t)dtds. \quad (5)$$

Suppose that:

$$f_{ij}(\xi,\eta) = P_i(\xi)T_j^*(\eta) - \int_0^\xi\int_0^1 k(\xi,s,\eta,t)P_i(s)T_j^*(t)dtds,$$

then, Eq. (5) can be rewritten as:

$$\sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}f_{ij}(\xi,\eta) = g(\xi,\eta). \quad (6)$$

Collocating Eq. (6) in $N+1$ and $M+1$ roots of the shifted legendre polynomial $P_{N+1}(\xi)$ and shifted Chebyshev polynomial $T_{M+1}(\xi)$, respectively, the shifted Legendre-Gauss in combination with shifted Chebyshev-Gauss nodes, then:

$$\sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}f_{ij}(\xi_n,\eta_m) = g(\xi_n,\eta_m), \quad for\ n=0,1,\dots,N,\ m=0,1,\dots,M, \quad (7)$$

which can be written in the following matrix form:

$$F^TC = G,$$

where

$$G = [g(\xi_0,\eta_0), g(\xi_1,\eta_0),\dots,g(\xi_N,\eta_0); g(\xi_0,\eta_1), g(\xi_1,\eta_1), \dots, g(\xi_N,\eta_1); g(\xi_0,\eta_M), g(\xi_1,\eta_M),\dots,g(\xi_N,\eta_M)]^T,$$

and

$$F = (f_{ijnm}),\ i,n=0,1,\dots,N,\ j,m=0,1,\dots,M,$$

in which the entries of the matrix F are defined as:

$$f_{ijnm} = f_{ij}(\xi_n,\eta_m),\ i,n=0,1,\dots,N,\ j,m=0,1,\dots,M.$$

In our calculations, this system has been solved using the *Mathematica* package "FindRoot" with vanishing initial approximation.

4 Nonlinear MV-FIEs

The 2D nonlinear MV-FIEs is given by:

$$u(\xi,\eta) = \int_0^\xi\int_\Omega k(\xi,\eta,s,t,u(s,t))dtds + g(\xi,\eta), \quad (8)$$

where $u(\xi,\eta)$ is an unknown function defined on

$$D = [0,T]\times\Omega,$$

and Ω is a closed compact subset of $\mathbb{R}^n$, $n=1,2,3$. The functions $k(\xi,\eta,s,t,u)$ and $g(\xi,\eta)$ are given functions defined on D and

$$S = \{(\xi,\eta,s,t,u) : 0\leq s\leq\xi\leq T,\ \eta\in\Omega,\ t\in\Omega\},$$

respectively [30]. Note that any interval $[0,T]$ can be shifted to $[0,1]$ by a linear dilation, so suppose that $[0,T]=[0,1]$ and $\Omega=[0,1]$ WLOG. Now, approximate the solution $u(\xi,\eta)$ using Eq. (4); then one get,

$$\psi(\xi,\eta)\mathcal{C}=\int_0^{\xi}\int_0^1 k(\xi,\eta,s,t,\psi(s,t)\mathcal{C})+g(\xi,\eta)dtds. \tag{9}$$

Now, collocate Eq. (9) at points (ξ_n,η_m) *for* $n=0,1,\dots,N,\ m=0,1,\dots,M$. Hence, we have

$$\psi(\xi_n,\eta_m)\mathcal{C}=\int_0^{\xi_n}\int_0^1 k(\xi_n,\eta_m,s,t,\psi(s,t)\mathcal{C})dtds + g(\xi_n,\eta_m), \tag{10}$$

for $n=0,1,\dots,N,\ m=0,1,\dots,M$. From (10), one obtain a nonlinear system of algebraic equations that can be solved by Newton iteration method to obtain the unknown vector $\mathcal{C}$.

5 Convergence and Error Analysis

In this section, we find an upper estimate for both the truncation and global error of the algorithm, for this purpose we state and prove three theorems. The following theorem is needed,

Theorem 1. *[31] The repeated integration of shifted Legendre polynomials is given by*

$$\underbrace{\int\int\cdots\int}_{r-times} P_i(\xi)(d\xi)^r = \frac{1}{4^r}\sum_{j=0}^{r}\binom{r}{j}(-1)^j\frac{(i+r-2j+\frac{1}{2})}{\Gamma(i+r-j+\frac{3}{2})}P_{i+r-2j}(\xi).$$

and the following lemma.

Theorem 2. *[32] The following inequality is valid:*

$$|\ P_i(\xi)\ |\le[2\pi(\xi-\xi^2)]^{-\frac{1}{2}},$$

$$u(\xi,\eta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P_i(\xi)T_j^*(\eta),$$

$$\int_0^1\int_0^1\frac{u(\xi,\eta)}{\sqrt{\eta-\eta^2}}P_i(\xi)T_j^*(\eta)d\xi d\eta=\frac{\varepsilon_j\pi}{4i+2}c_{ij}.$$

Therefore

$$c_{ij}=\frac{4i+2}{\pi\varepsilon_j}\int_0^1\int_0^1\frac{u(\xi,\eta)P_i(\xi)T_j^*(\eta)}{\sqrt{\eta-\eta^2}}d\xi d\eta.$$

Theorem 3. *If $u(\xi,\eta)$ is separable i.e. $u(\xi,\eta)=u_1(\xi)u_2(\eta)$ and if there exists $M_1,\ M_2$ such that $u_1^{(p)}(\xi)\le M_1$ and $u_2^{(q)}(\eta)\le M_2$, for some p,q positive integers, assume that $u(\xi,\eta)$ is approximated by $u(\xi,\eta)\simeq\sum_{i=0}^{N}\sum_{j=0}^{M}P_i(\xi)T_j^*(\eta)$, then the expansion coefficients c_{ij} satisfy the following estimate*

$$|\ c_{ij}\ |=\mathcal{O}(i^{\frac{1}{2}-p}j^{-q}),\ \forall\, i>p>\frac{3}{2},\ j>q>1.$$

Proof. Following [33] and [34].
Starting from

$$c_{ij}=\frac{4i+2}{\pi\varepsilon_j}\int_0^1 u_1(\xi)P_i(\xi)d\xi\int_0^1\frac{u_2(\eta)T_j^*(\eta)}{\sqrt{\eta-\eta^2}}d\eta.$$

Let

$$I_1=\int_0^1 u_1(\xi)P_i(\xi)d\xi,\ I_2=\int_0^1\frac{u_2(\eta)T_j^*(\eta)}{\sqrt{\eta-\eta^2}}d\eta.$$

For the integral I_1, Integrating by parts p-times and applying Theorem 1 and inequality in Theorem 2 and after some algebraic manipulations we get

$$|\ I_1\ |\le c_1(i^{\frac{1}{2}}-p). \tag{11}$$

For the integral I_2, using the substitution $2\eta-1=\cos\theta$ and integration by parts q-times and after some lengthy manipulations we get

$$|\ I_2\ |\le c_2(j^{-q}). \tag{12}$$

Joining (11) and (12), the Theorem is proved.

Theorem 4. *Under the hypotheses of Theorem 3, one have the following error estimate*

$$e_{N,M}=|\ u(\xi,\eta)-u_{N,M}(\xi,\eta)\ |=\mathcal{O}(N^{\frac{3}{2}-p}M^{1-q})$$

Proof.

$$\begin{aligned}e_{N,M}&=|\ u(\xi,\eta)-u_{N,M}(\xi,\eta)\ |\\&=|\sum_{i=N+1}^{\infty}\sum_{j=M+1}^{\infty}c_{ij}P_i(\xi)T_j^*(\eta)\ |,\end{aligned}$$

using the result of Theorem 3 and the inequalities $|\ P_i(\xi)\ |\le 1,\ |\ T_j^*(\eta)\ |\le 1,$

$$e_{N,M}\le\iota\sum_{i=N+1}^{\infty}i^{\frac{1}{2}-p}\sum_{j=M+1}^{\infty}j^{-q}.$$

Now the reminder of any convergent series $\sum_{k=1}^{\infty}f(k)$ satisfy $\sum_{k=N+1}^{\infty}f(k)<\int_N^{\infty}f(\xi)d\xi$, therefore

$$\begin{aligned}e_{N,M}&\le\iota\int_N^{\infty}\xi^{\frac{1}{2}-p}d\xi\int_M^{\infty}\eta^{-q}d\eta\\&=\iota\frac{2N^{\frac{3}{2}-p}}{2p-3}\frac{M^{1-q}}{q-1}.\end{aligned}$$

Which completes the proof of the Theorem.

Theorem 5.*If ε is the residual of Eq. (3) then one have the following estimate $|\ \varepsilon\ |\leq (1+k)e_{N,M}$, where k is upper bound of the kernel function $k(\xi,t)$.*

Proof.

$$u(\xi,\eta)=g(\xi,\eta)+\int_0^\xi\int_0^1 k(\xi,s,\eta,t)u(s,t)dtds,$$

$$u_{N,M}(\xi,\eta)=\varepsilon+g(\xi,\eta)$$

$$+\int_0^\xi\int_0^1 k(\xi,s,\eta,t)u_{N,M}(s,t)dtds,$$

$$e_{N,M}=\varepsilon+\int_0^\xi\int_0^1 k(\xi,s,\eta,t)e_{N,M}dtds,$$

hence

$$|\ \varepsilon\ |\leq e_{N,M}+\int_0^\xi\int_0^1 |\ k(\xi,s,\eta,t)\ |\ e_{N,M}dtds,$$

which yields

$$|\ \varepsilon\ |\leq e_{N,M}(1+k\xi)\leq(1+k)e_{N,M}.$$

6 Numerical results

This part is devoted to some numerical examples to show the accuracy, quality and applicability of the proposed method.

The distance between the value of approximate solution and its exact value, is determined by

$$E(\xi,\eta)=|u(\xi,\eta)-\widetilde{u}(\xi,\eta)|, \tag{13}$$

where $u(\xi,\eta)$ and $\widetilde{u}(\xi,\eta)$ are the exact solution and the numerical solution at the point (ξ,η), respectively. Moreover, the maximum AEs is given by

$$\text{MAEs}=\text{Max}\{E(\xi,t):\forall(\xi,t)\in[0,1]\times[0,1]\}=L^\infty. \tag{14}$$

Also we can denote to L^2 by

$$L^2=\sqrt{\frac{\sum_{i=0}^{M}\sum_{j=0}^{M}(approximate-exact)^2}{(N+1)(M+1)}} \tag{15}$$

*Example 1.*Let us first consider the following linear MV-FIEs equation [12]

$$u(\xi,\eta)=g(\xi,\eta)+\int_0^\xi\int_0^1 \eta^2e^{-t}u(s,t)dtds, \quad \xi,\eta\in[0,1), \tag{16}$$

where, $g(\xi,\eta)=\xi^2-\frac{1}{3}\xi^3\eta^2$ with the exact solution $u(\xi,\eta)=\xi^2e^\eta$, for $0\leq\xi,\eta<1$.

Table 1: Comparison of the AEs with various choices of ξ,η,N,M, for Example 1.

$\xi=\eta=$	$N=M=4$		$N=M=8$	
	[12]	CLM	[12]	CLM
0.0	$2.26.10^{-8}$	$1.04.10^{-17}$	$6.39.10^{-10}$	$7.39.10^{-17}$
0.1	$4.27.10^{-5}$	$3.74.10^{-8}$	$7.06.10^{-5}$	$1.50.10^{-13}$
0.2	$2.00.10^{-4}$	$7.58.10^{-7}$	$3.41.10^{-4}$	$1.18.10^{-12}$
0.3	$4.71.10^{-4}$	$2.16.10^{-6}$	$8.36.10^{-4}$	$1.36.10^{-12}$
0.4	$7.69.10^{-4}$	$6.25.10^{-7}$	$1.47.10^{-3}$	$5.44.10^{-12}$
0.5	$8.78.10^{-4}$	$6.47.10^{-6}$	$2.03.10^{-3}$	$8.49.10^{-12}$
0.6	$4.21.10^{-4}$	$1.45.10^{-5}$	$2.13.10^{-3}$	$1.14.10^{-11}$
0.7	$1.18.10^{-3}$	$1.06.10^{-5}$	$1.17.10^{-3}$	$2.58.10^{-11}$
0.8	$4.75.10^{-3}$	$1.62.10^{-5}$	$1.65.10^{-3}$	$1.15.10^{-12}$
0.9	$1.14.10^{-3}$	$4.16.10^{-5}$	$7.48.10^{-3}$	$3.73.10^{-11}$
1.0	$2.25.10^{-3}$	$5.62.10^{-5}$	$9.77.10^{-3}$	$7.14.10^{-11}$

Table 2: L^2 and L^∞-error for Example 1.

$N=M$	L^2-error	L^∞-error
4	$7.960.10^{-8}$	$2.740.10^{-16}$
8	$2.093.10^{-14}$	$1.598.10^{-16}$

To prove that our methodology is more power-full than the operational matrix using Bernstein polynomials [12], in Table 1, we give the AE with several choices of N, M, ξ, and η and compare the obtained results with those obtained using the operational matrix using Bernstein polynomials [12]. Moreover, Table 2 display the L^2-error and L^∞-error using our methodology (Chebyshev-Legendre Method CLM) with several choices of N, M. We see in these tables that the results are very accurate for small values of N and M. we depicted

Fig. 1), show the exact solution of Example 1 at $N=M=8$.

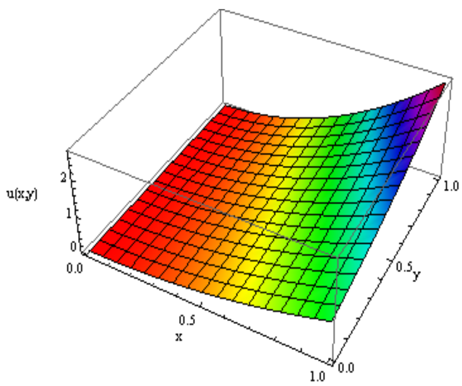

Fig. 1: The exact $u(\xi,\eta)$ solution for Example 1 where $N=M=8$.

Fig. 2), show the numerical solution of Example 1 at $N=M=8$.

Fig. 3) The AE for Example 1 where $N=M=8$.

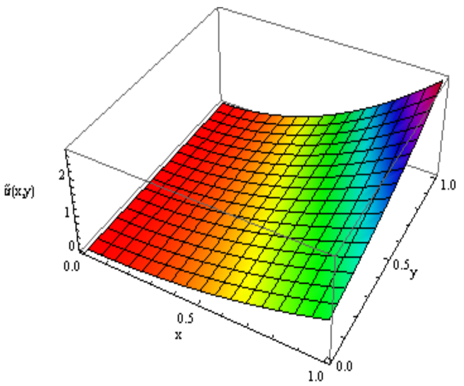

Fig. 2: The numerical $\widetilde{u}(\xi,\eta)$ solution for Example 1 where $N = M = 8$.

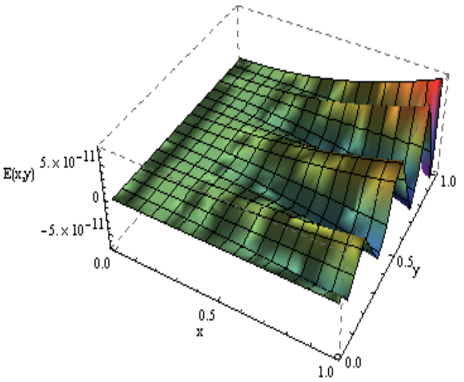

Fig. 3: The AE for Example 1 where $N = M = 8$.

Fig. 4) The graph of AE for Example 1 where $N = M = 8$ at $\xi = 0.5$.

Fig. 5) The graph of exact $u(\xi,\eta)$ solution for Example 1 where $N = M = 8$ at three different values of η.

Fig. 6) The graph of numerical $\widetilde{u}(\xi,\eta)$ solution for Example 1 where $N = M = 8$ at three different values of ξ.

*Example 2.*Consider the following linear mixed Volterra-Fredholm integral equation [12]

$$u(\xi,\eta) = g(\xi,\eta) - \int_0^{\xi}\int_0^{1}(2t-1)e^{s}u(s,t)dtds, \qquad (17)$$
$$\xi,\eta \in [0,1),$$

where, $g(\xi,\eta) = \sin(\xi) + \eta + \frac{1}{6}e^{\xi} - \frac{1}{6}$ with the exact solution $u(\xi,\eta) = \sin(\xi) + \eta$, for $0 \leq \xi,\eta < 1$.

Table 3 shows that the AE obtained by the our method is significantly better than that obtained by the operational matrix using Bernstein polynomials [12]. Moreover,

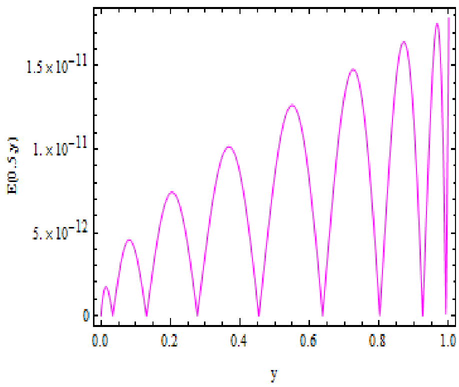

Fig. 4: The graph of AE for Example 1 where $N = M = 8$ at $\xi = 0.5$.

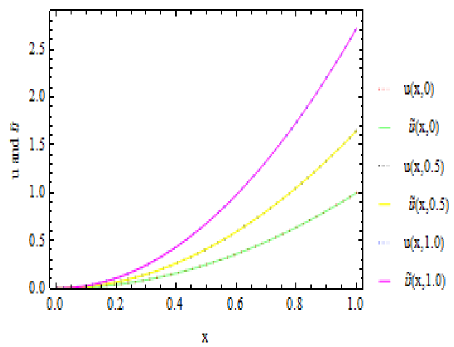

Fig. 5: The graph of exact $u(\xi,\eta)$ and numerical $\widetilde{u}(\xi,\eta)$ solutions for Example 1 where $N = M = 8$ at three different values of ξ.

Table 4 display the L^2-error and L^∞-error using our method with several choices of N, M. The Graph of analytical solution and approximate solution at $N = M = 8$ is depicted in Fig 7 to make it easier to compare with analytical solution. Fig. 9, allows us to see the absolute error $E(\xi,t)$ at $N = M = 8$.

Fig. 10) The graph of exact $u(\xi,\eta)$ and numerical $\widetilde{u}(\xi,\eta)$ solutions for Example 2 where $N = M = 8$ at three different values of η.

Fig. 11)] The graph of exact $u(\xi,\eta)$ and numerical $\widetilde{u}(\xi,\eta)$ solutions for Example 2 where $N = M = 8$ at three different values of ξ.

Fig. 12) The graph of AE for Example 2 where $N = M = 8$ at $\eta = 0.5$.

Fig. 13) The graph of AE for Example 2 where $N = M = 8$ at $\xi = 0.5$.

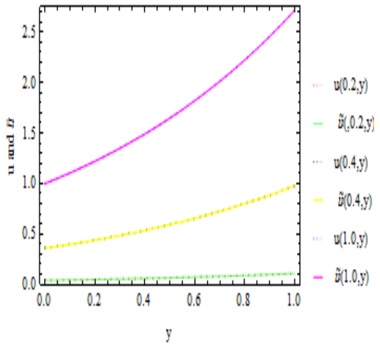

Fig. 6: The graph of exact $u(\xi,\eta)$ and numerical $\widetilde{u}(\xi,\eta)$ solutions for Example 1 where $N = M = 8$ at three different values of ξ.

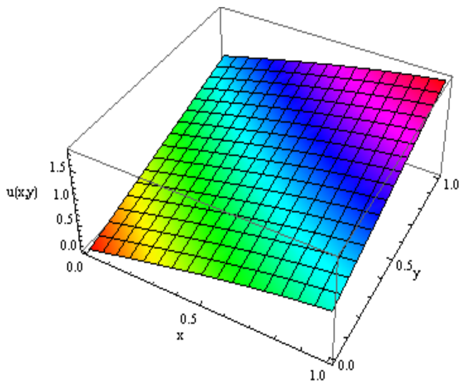

Fig. 7: The exact $u(\xi,\eta)$ solution for Example 2 where $N = M = 8$.

Table 3: Comparison of the AEs with various choices of ξ,η,N,M, for Example 2.

ξ,η	$N=M=4$		$N=M=8$	
	[12]	CLM	[12]	CLM
0.0	$1.142.10^{-8}$	$8.247.10^{-17}$	$7.432.10^{-9}$	$3.219.10^{-17}$
0.1	$6.482.10^{-4}$	$1.424.10^{-5}$	$6.512.10^{-4}$	$2.157.10^{-11}$
0.2	$1.257.10^{-3}$	$6.116.10^{-6}$	$5.992.10^{-4}$	$1.852.10^{-11}$
0.3	$1.826.10^{-3}$	$2.080.10^{-5}$	$9.213.10^{-5}$	$8.707.10^{-12}$
0.4	$2.356.10^{-3}$	$1.705.10^{-5}$	$1.376.10^{-3}$	$2.683.10^{-11}$
0.5	$2.858.10^{-3}$	$1.665.10^{-16}$	$3.219.10^{-3}$	$3.885.10^{-16}$
0.6	$3.347.10^{-3}$	$1.674.10^{-5}$	$5.601.10^{-3}$	$2.653.10^{-11}$
0.7	$3.840.10^{-3}$	$2.005.10^{-5}$	$8.597.10^{-3}$	$8.518.10^{-12}$
0.8	$4.358.10^{-3}$	$5.790.10^{-6}$	$1.193.10^{-2}$	$1.792.10^{-11}$
0.9	$4.926.10^{-3}$	$1.324.10^{-5}$	$1.588.10^{-2}$	$2.065.10^{-11}$
1.0	$5.568.10^{-3}$	$3.306.10^{-16}$	$2.036.10^{-2}$	$4.440.10^{-16}$

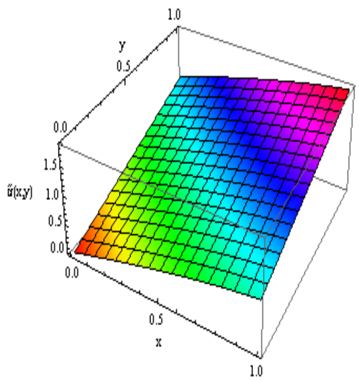

Fig. 8: The numerical $\widetilde{u}(\xi,\eta)$ solution for Example 2 where $N = M = 8$.

Table 4: L^2 and L^∞-error for Example 2.

$N=M$	L^2-error	L^∞-error
4	$1.660.10^{-16}$	$1.549.10^{-5}$
8	$3.905.10^{-16}$	$2.461.10^{-11}$

Table 5: Comparison of the AEs with various choices of ξ,η, for Example 3.

(ξ,η)	OM [35]	MQs [36]	BCM [1]	L-CCM
(0.0,0.0)	0	$6.43.10^{-3}$	0	0
(0.1,0.1)	$9.39.10^{-4}$	$6.94.10^{-3}$	$1.05.10^{-4}$	$1.61.10^{-8}$
(0.2,0.2)	$6.35.10^{-4}$	$4.31.10^{-2}$	$6.76.10^{-4}$	$3.14.10^{-7}$
(0.3,0.3)	$6.61.10^{-4}$	$9.82.10^{-2}$	$1.51.10^{-3}$	$1.90.10^{-7}$
(0.4,0.4)	$1.02.10^{-3}$	$1.68.10^{-1}$	$1.66.10^{-3}$	$1.33.10^{-6}$
(0.5,0.5)	$1.69.10^{-4}$	$2.63.10^{-1}$	$1.34.10^{-3}$	$2.24.10^{-10}$
(0.6,0.6)	$1.24.10^{-3}$	$3.88.10^{-1}$	$4.05.10^{-3}$	$3.14.10^{-6}$
(0.7,0.7)	$1.13.10^{-3}$	$5.33.10^{-1}$	$9.45.10^{-3}$	$1.14.10^{-6}$
(0.8,0.8)	$1.42.10^{-3}$	$6.91.10^{-1}$	$1.33.10^{-2}$	$5.84.10^{-6}$
(0.9,0.9)	$2.17.10^{-3}$	$8.85.10^{-1}$	$1.13.10^{-2}$	$1.58.10^{-6}$

*Example 3.*Consider the following nonlinear mixed Volterra-Fredholm integral equation [35]

$$u(\xi,\eta) = g(\xi,\eta) - \int_0^\xi \int_0^1 \eta^2 e^{-4s}[u(s,t)]^2 dtds, \qquad (18)$$
$$\xi,\eta \in [0,1),$$

where, $g(\xi,\eta) = \xi^2 e^{2s} - \frac{1}{2}\xi^2 + \eta^2$ with the exact solution $u(\xi,\eta) = \xi^2 e^{2\eta}$, for $0 \le \xi,\eta < 1$.

In this example (Table 5), we compare our results obtained by the L-CCM for the different choices ξ,η, with the method of OM [35], the method of multiquadric (MQs) radial basis functions [36] and Bernoulli collocation method (BCM). The numerical results in this Table demonstrates that the AEs obtained by the L-CCM method is significantly better than those in [35], [36] and Bernoulli collocation method [1]. Fig. 14, allows us to see the AE $E(\xi,t)$ at $N = 2,\ M = 6$.

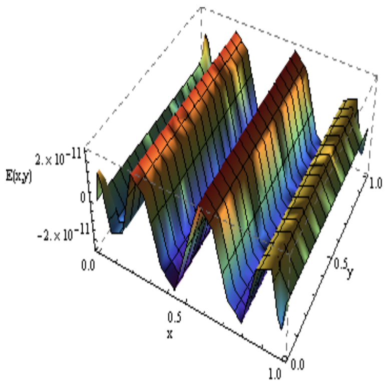

Fig. 9: The AE for Example 2 where $N = M = 8$.

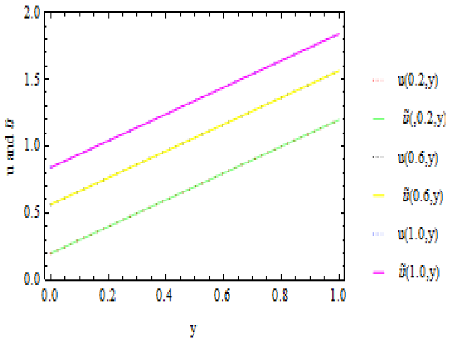

Fig. 11: The graph of exact $u(\xi,\eta)$ and numerical $\widetilde{u}(\xi,\eta)$ solutions for Example 2 where $N = M = 8$ at three different values of ξ.

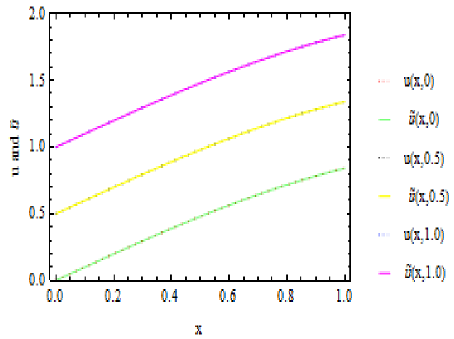

Fig. 10: The graph of exact $u(\xi,\eta)$ and numerical $\widetilde{u}(\xi,\eta)$ solutions for Example 2 where $N = M = 8$ at three different values of η.

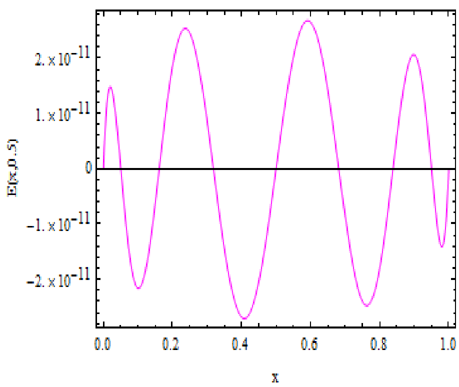

Fig. 12: The graph of AE for Example 2 where $N = M = 8$ at $\eta = 0.5$.

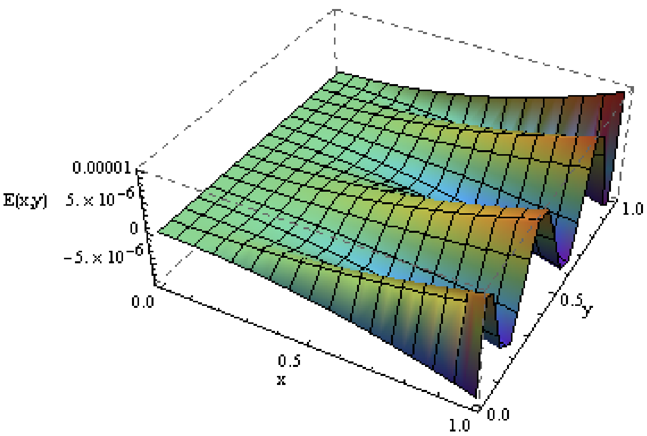

Fig. 14: The AE for Example 3 where $N = 2,\ M = 6$.

Example 4. For the following nonlinear mixed Volterra-Fredholm integral equation [35]

$$u(\xi,\eta) = g(\xi,\eta) + 16\int_0^{\xi}\int_0^1 e^{(\xi+\eta+s+t)}[u(s,t)]^3 dtds,\quad \xi,\eta \in [0,1), \tag{19}$$

where, $g(\xi,\eta) = e^{\xi+\eta+4} - e^{5\xi+\eta} - e^{5\xi+\eta+4}$ with the exact solution $u(\xi,\eta) = e^{\xi+\eta}$.

Maleknejad and JafariBehbahani [35] introduced this problem and applied the operational matrix (OM) for integration in mixed type for obtaining its numerical solution. In order to show that our algorithm is more accurate than the OM [35], in Table 6, we list the AEs with several choices of ξ,η and $N = 2,\ M = 3$ and compare the achieved results with those obtained using the OM [35].

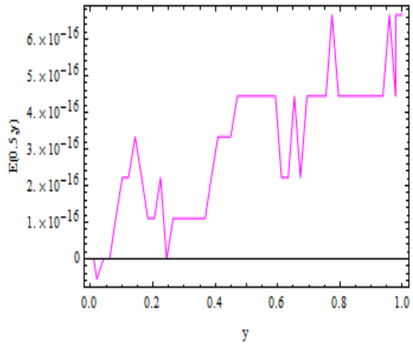

Fig. 13: The graph of AE for Example 2 where $N = M = 8$ at $\xi = 0.5$.

Table 6: Comparison of the AEs with various choices of ξ, η, for Example 4.

(ξ, η)	OM [35] at $m_1 = m_2 = 8$	L-CCM at $N = 2,\ M = 3$
(0.0,0.0)	0	0
(0.1,0.1)	$1.5274.10^{-2}$	$1.6903.10^{-3}$
(0.2,0.2)	$1.0857.10^{-2}$	$3.5678.10^{-3}$
(0.3,0.3)	$1.0937.10^{-2}$	$2.5247.10^{-3}$
(0.4,0.4)	$2.2908.10^{-2}$	$1.0714.10^{-3}$
(0.5,0.5)	$2.5617.10^{-2}$	$6.0552.10^{-3}$
(0.6,0.6)	$3.0525.10^{-2}$	$1.0065.10^{-2}$
(0.7,0.7)	$2.9937.10^{-2}$	$9.0032.10^{-3}$
(0.8,0.8)	$3.5379.10^{-2}$	$3.6852.10^{-3}$
(0.9,0.9)	$5.3123.10^{-2}$	$3.7944.10^{-2}$

*Example 5.*The last example is a nonlinear mixed Volterra-Fredholm integral equation that arises from mathematical modelling of the spatio-temporal development of an epidemic [37], in the form

$$u(\xi,\eta) = g(\xi,\eta) + \int_0^{\xi}\int_0^{1} \frac{\eta(1-t)^2}{(1+\xi)(1+s^2)}(1-exp(-u(s,t)))dtds, \quad \xi,\eta \in [0,1), \tag{20}$$

where, $g(\xi,\eta) = -log(1+\frac{\xi\eta}{1+\xi^2}) + \frac{\eta\xi^2}{24(1+\xi)(1+\xi^2)}$ with the exact solution $u(\xi,\eta) = -log(1+\frac{\xi\eta}{1+\xi^2})$, for $0 \le \xi,\eta < 1$. Fig. 15, Fig. 16 show the exact solution and numerical solutions of Example 5 at $N = M = 8$.

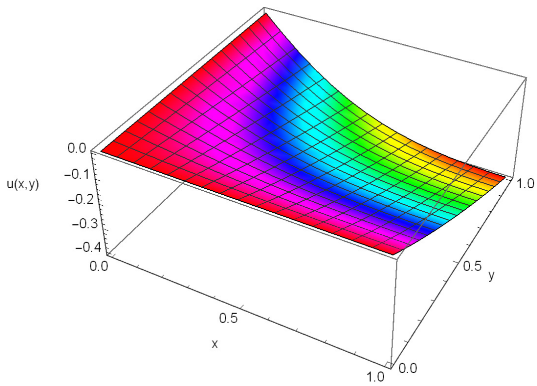

Fig. 15: The exact solution $u(\xi,\eta)$ for Example 5 where $N = M = 8$.

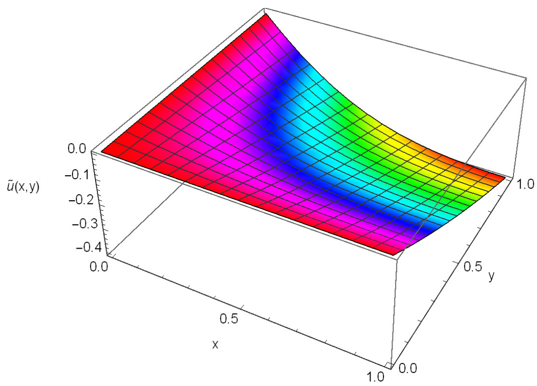

Fig. 16: The numerical solution $\widetilde{u}(\xi,\eta)$ for Example 5 where $N = M = 8$.

7 Concluding Remarks

We have applied a Legendre-Chebyshev collocation method to solve a specific type of two-dimensional mixed V-FIEs. This method uses the shifted Legendre-Gauss combinations with shifted Chebyshev-Gauss points to reduce the considered mixed V-FIEs to the solution of a matrix equation. In the examples given, by selecting relatively few shifted Legendre-Gauss combinations with shifted Chebyshev-Gauss points, we are able to get very accurate approximations, and we are thus able to demonstrate the utility of our approach over other analytical or numerical schemes such as other collocation schemes or perturbation methods. The comparison of the obtained results with those based on other methods shows that the scheme is a powerful solver for these kinds of equations. The only drawback of the proposed scheme is that, raising the values of N and/or M (more than 64) may cause a round of error.

Conflict of Interest

The authors declare that they have no conflict of interest.

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