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Parameter Estimation for the Generalized Linear Exponential Distribution Based on Progressively Type-*II* Hybrid Censored Data

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Abstract: Estimations of the unknown parameters of the generalized linear exponential distribution (GLED) using type-*II* progressive hybrid censored (HC) data are obtained. The maximum likelihood estimation (MLE) and Bayes estimation for all parameters and some lifetime functions (reliability, hazard function and reversed hazard function) are obtained. Also, we apply Markov chain Monte Carlo (MCMC) technique and Lindely's approximation technique to carry out a Bayesian estimation. Under the assumptions of informative and non-informative priors, estimates of Bayes and credible intervals are obtained. Different methods have been compared using Mont Carlo simulations. Real data set has been investigated for illustrative purpose.

Keywords: Bayes Estimators, Generalized Linear Exponential Distribution (GLE), Hybrid censoring scheme(HCS), Lindely approximation, Markov Chain Monte Carlo (MCMC), Maximum Likelihood Estimation (MLE).

1 Introduction

In parametric estimation problems, we often assume that a set of randomly selected samples of pre-specified number of units, say n, is available. It is further assumed that random observations follow a specified distribution. However, in practice observations on all the units may be unavailable because observations corresponding to some units are either not recorded or lost during intermediate transmission. The resulting data are termed as censored sample. The two most common and popular censoring schemes are Type-I and type-II censoring schemes. The mixture of type-I and type-II censoring schemes is known as the hybrid censoring scheme (HCS). The HCS was first introduced by [1] and became quite popular in reliability and life testing experiments.

[2] presented details and additional references on some of the statistical inferences and applications for the exponential distribution with HCS. [1]introduced the type-I HCS in which the experimental time is $T^* = \min\{x_{m:n},\ T\}$for a pre-fixed value of T. The main disadvantage of type-I HCS is that most of the inferential results need to be developed in this case under the condition that the number of observed failures is at least one. Moreover, there may be very few failures occurring up to the pre-fixed time T which results in the estimator(s) of the model parameter(s) with low efficiency. For this reason, [3] introduced an alternative HCS that would terminate the experiment at the random time $T^* = \max\{x_{m:n},\ T\}$. This HCS is called type-II hybrid censoring scheme (type-II HCS), and it has the advantage of guaranteeing at least m failures to be observed by the end of the experiment. If m failures occur before time T, the experiment will continue up to time T which may end up yielding more than m failures in the data.

On the other hand, if the $m-$th failure does not occur before time T, the experiment will continue until the time when the mth failure occurs. In this case, we would exactly observe m failures in the data. [4] combined type-I HCS and type-II HCS schemes and introduced the so-called unified HCS. [5] presented detailed description on hybrid and adaptive censoring schemes as well as inferential methods and pointed out their advantages and disadvantages. Type-II progressively hybrid censoring

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scheme, which is a mixture of Type-II progressive and hybrid censoring schemes was introduced by [6]. It can be described as follows: Consider n identical items which put on a test. Each unit in a randomly selected sample is subjected under identical environmental conditions.

The lifetimes of the sample units are independent and identically distributed random variables . The integer $m < n$ and $R_1, R_2, \ldots, R_m$ satisfying $R_1 + R_2 + \ldots + R_m + m = n$ are fixed at the beginning of the experiment. The time point T is also fixed before hand. At the time of first failure, $X_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure, say $X_{2:m:n}$, R_2 of the remaining units are randomly removed and so on. If the mth failure , say $X_{m:m:n}$ occurs before the time point T, the experiment stops at the time point $X_{m:m:n}$. On the other hand if the m-th failure does not occur before time point T and only J failure occurs before the time point T, where $m \geqslant 0$, then at the time point T all the remaining R_J^* units are removed and the experiment terminates at the time point T. Note that $R_J^* = n - R_1 - R_2 - \ldots - R_J - J$ We denote these two cases as Case I and Case II, respectively,and call these censoring schemes as progressively Type-II hybrid censoring schemes. Therefore, in the presence of progressively Type-II hybrid censoring schemes,we have one of the following types of observations:

case I: $\{X_{1:m:n}, \ldots, X_{m:m:n}\}$ if $X_{m:m:n} < T$

case II: $\{X_{1:m:n}, \ldots, X_{m:m:n}\}$ if $X_{J:m:n} < T < X_{J+1:m:n}$.

Note that for Case II ,$X_{J:m:n} < T < X_{J+1:m:n} < \ldots < X_{m:m:n}$ and $X_{J+1:m:n}, \ldots, X_{m:m:n}$ are not observed.

The exponential distribution is the most widely used lifetime model in reliability theory, because of its simplicity and mathematical feasibility. The exponential distribution was generalized by introducing a shape parameter, and extensively explored by [7] and [8]. [9] studied generalized exponential distribution from a Bayesian point of view. Similarly, [10] introduced a shape parameter in the Inverted exponential distribution to obtain generalized inverted exponential distribution. They derived many distributional properties and reliability characteristics of generalized inverted exponential distribution. [11] addressed generalized inverted exponential distribution based on progressively Type-II hybrid censored data. [12] covered the statistical inference on Weibull parameters when the observed data are progressively Type-II hybrid censored.They derived the maximum likelihood estimators (MLEs) and the approximate maximum likelihood estimators (AMLEs) of the Weibull parameters. Then, they used the asymptotic distributions of the maximum likelihood estimators to construct approximate confidence intervals. Bayes estimates and the corresponding highest posterior density credible intervals of the unknown parameters were obtained under suitable priors on the unknown parameters and using the Gibbs sampling procedure. [13] derived the maximum likelihood estimators of Weibull distribution parameters and the acceleration factors were discussed based on two different types of progressively hybrid censoring schemes under step-stress partially accelerated life test model. [14] considered the statistical inference of a two-parameter exponential distribution under the Type-II progressively hybrid censoring scheme. Distributions of the maximum likelihood estimators (MLEs) in Type-II (progressive) hybrid censoring based on two-parameter exponential distributions were obtained using a moment generating function approach by [15]. The statistical inference for the Gompertz distribution based on Type-II progressively hybrid censored data was addressed by [16].

Recently, [17] has investigated the point and interval estimates of the parameters from Weibull distribution based on adaptive Type-II progressively hybrid censored data in constant-stress accelerated life test using maximum likelihood estimation (MLE) methods and Bayesian estimation (BE). [18] introduced the generalized linear exponential distribution. They derived some statistical properties such as moments, modes and quartiels.

The paper is organized as follows: Section 2 presents the importance of GLED and some statistical functions. The ML estimators of the unknown parameters, reliability and hazard functions are presented. The corresponding approximate confidence intervals for the parameters are presented in Section 3. Section 4 comprises description of the priors, posteriors, Gibbs sampling and Metropolis-Hasting (MH) algorithm. Moreover, the proposed hybrid algorithm with resulting Bays estimators are discussed. Numerical example and real data sets have been analyzed in Sections 5 and 6. Conclusion is presented in Section 7.

2 Generalized Linear Exponential Distribution

The generalized linear exponential distribution can be used for modeling bathtub, increasing and decreasing hazard rate behavior. It was first proposed by [18]. This distribution is important because it involves some widely well-known distributions such as exponential distribution, Rayleigh distribution, the linear exponential distribution and the Weibull distribution. The three parameter generalized linear exponential distribution has a probability density function (pdf)

$$f(x;\alpha,\theta,\lambda) = \alpha\left(\lambda x + \frac{\theta}{2}x^2\right)^{\alpha-1}(\lambda+\theta x)\,e^{-\left(\lambda x+\frac{\theta}{2}x^2\right)^{\alpha}}, \quad \alpha,\theta>0 \text{ and } \lambda \geq 0, \tag{1}$$

cumulative distribution function (cdf)

$$F(x,\alpha,\theta,\lambda) = 1 - e^{-\left(\lambda x+\frac{\theta}{2}x^2\right)^{\alpha}}, \alpha,\theta>0 \text{ and } \lambda \geq 0, \tag{2}$$

the survival function $R(t)$

$$R(t) = e^{-\left(\lambda t+\frac{\theta}{2}t^2\right)^{\alpha}}, t > 0, \quad (3)$$

the hazard function $H(t)$

$$H(t) = \alpha(\lambda+\theta t)\left(\lambda t+\frac{\theta}{2}t^2\right)^{\alpha-1}, t > 0, \quad (4)$$

and the reversed hazard function $Q(t)$

$$Q(t) = \frac{\alpha(\lambda+\theta t)\left(\lambda t+\frac{\theta}{2}t^2\right)^{\alpha-1} e^{-\left(\lambda t+\frac{\theta}{2}t^2\right)^{\alpha}}}{1-e^{-\left(\lambda t+\frac{\theta}{2}t^2\right)^{\alpha}}}, t > 0, \quad (5)$$

3 Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) is often the most feasible method, which is used when doing statistical inference. In this section, we study MLEs of the GLE distribution based on Type-II progressively hybrid censored scheme with pdf given in (1). Also, we construct Approximate confidence intervals (ACIs) of the parameters of GLE distribution based on Type-II progressively hybrid censored scheme. The likelihood function in case I is given

$$L(\alpha,\theta,\lambda) = k_1 \prod_{\imath=1}^{m} f(x_{i:m:n})\left[1-F(x_{i:m:n})\right]^{R_i}, \quad (6)$$

for case II,

$$L(\alpha,\theta,\lambda) = k_2 \prod_{\imath=1}^{d} f(x_{i:m:n})\left[1-F(x_{i:m:n})\right]^{R_i}\left[1-F(T)\right]^{R_j^*}, \quad (7)$$

where $k_1 = \prod_{\imath=1}^{n}[n - \sum_{k=1}^{i-1}(1+R_k)]$ and $k_2 = \prod_{\imath=1}^{j}[n - \sum_{k=1}^{i-1}(1+R_k)]$ both are constant. Here f(x) is presented in (1). We present likelihood function (6) and (7) by:

$$L(\alpha,\theta,\lambda) = C\prod_{\imath=1}^{r} f(x_i)\left[1-F(x_i)\right]^{R_i}\left[1-F(T)\right]^{R^*_{(j)}}, \quad (8)$$

where

$$r = \begin{cases} m & \text{for case I} \\ j & \text{for case II} \end{cases}, \quad (9)$$

and

$$C = \begin{cases} k_1 & \text{for case I} \\ k_2 & \text{for case II} \end{cases}. \quad (10)$$

$$R^*_{(j)} = \begin{cases} 0 & \text{for case I} \\ R^*_j & \text{for case II} \end{cases}. \quad (11)$$

By substituting (1) and (2) in Equation (8) and taking the logarithm, the log likelihood function can be written as

$$\begin{aligned} log\, L(\alpha,\theta,\lambda) =& const. + r\log\alpha \\ &+ (\alpha-1)\sum_{i=1}^{r}\log\left(\lambda x_i+\frac{\theta}{2}x_i^2\right) \\ &- \sum_{i=1}^{r}(R_i+1)\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha} \\ &+ \sum_{i=1}^{r}\log(\lambda+\theta x_i) - R^*_{(j)}\left(\lambda T+\frac{\theta}{2}T^2\right)^{\alpha}. \end{aligned} \quad (12)$$

Applying the first derivative with respect to α, θ and λ and equating by zero, we get the normal equations as follows:

$$\begin{aligned} &\frac{r}{\hat{\alpha}} + \sum_{i=1}^{r}\log\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right) - \sum_{i=1}^{r}(R_i+1)\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)^{\hat{\alpha}} \\ &\times \log\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right) - R^*_{(j)}\left(\hat{\lambda}T+\frac{\hat{\theta}}{2}T^2\right)^{\alpha} \\ &\times \log\left(\hat{\lambda}T+\frac{\hat{\theta}}{2}T^2\right)^{\alpha} = 0, \end{aligned} \quad (13)$$

$$\begin{aligned} &(\hat{\alpha}-1)\sum_{i=1}^{r}\frac{x_i^2}{2\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)} + \sum_{i=1}^{r}\frac{x_i}{\left(\hat{\lambda}+\hat{\theta}x_i\right)} \\ &- \hat{\alpha}\sum_{i=1}^{r}(R_i+1)\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)^{\hat{\alpha}-1}\frac{x_i^2}{2} \\ &- \hat{\alpha}R^*_{(j)}\left(\hat{\lambda}T+\frac{\hat{\theta}}{2}T^2\right)^{\hat{\alpha}-1}\frac{T^2}{2} = 0, \end{aligned} \quad (14)$$

and

$$(\hat{\alpha}-1)\sum_{i=1}^{r}\frac{x_i}{\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)}+\sum_{i=1}^{r}\frac{1}{\left(\hat{\lambda}+\hat{\theta}x_i\right)}$$

$$-\hat{\alpha}\sum_{i=1}^{r}(R_i+1)\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)^{\hat{\alpha}-1}x_i \quad (15)$$

$$-\hat{\alpha}R^*_{(j)}\left(\hat{\lambda}T+\frac{\hat{\theta}}{2}T^2\right)^{\hat{\alpha}-1}T=0.$$

Because Equations (13), (14) and (15) are nonlinear equations in three parameters $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\lambda}$, it is difficult to compute the exact solution. Thus, some numerical methods must be employed.

By invariance property of ML estimators, the ML estimators of reliability function R(t), hazard rate function H(t) and reversed hazard rate function Q(t) can be obtained by substituting the MLE's of the parameters α, θ and λ in (3), (4) and (5), respectively. Hence,

$$\hat{R}(t,\hat{\alpha},\hat{\theta},\hat{\lambda})=e^{-\left(\hat{\lambda}t+\frac{\hat{\theta}}{2}t^2\right)^{\hat{\alpha}}},t>0, \quad (16)$$

$$\hat{H}(t,\hat{\alpha},\hat{\theta},\hat{\lambda})=\hat{\alpha}(\hat{\lambda}+\hat{\theta}t)\left(\hat{\lambda}t+\frac{\hat{\theta}}{2}t^2\right)^{\hat{\alpha}-1},$$

$$t>0, \quad (17)$$

and

$$\hat{Q}(t,\hat{\alpha},\hat{\theta},\hat{\lambda})=\frac{\hat{\alpha}(\hat{\lambda}+\hat{\theta}t)\left(\hat{\lambda}t+\frac{\hat{\theta}}{2}t^2\right)^{\hat{\alpha}-1}e^{-\left(\hat{\lambda}t+\frac{\hat{\theta}}{2}t^2\right)^{\hat{\alpha}}}}{1-e^{-\left(\hat{\lambda}t+\frac{\hat{\theta}}{2}t^2\right)^{\hat{\alpha}}}},$$

$$t>0, \quad (18)$$

3.1 Approximate confidence interval

The asymptotic variance-covariance matrix of the estimators of the parameters $\varphi=(\varphi_1,\dots\varphi_n)$ is obtained by inverting the Fisher information matrix (given by taking the expectation of the second derivative of the log-likelihood functions) in which elements are negatives. In the present situation, it seems appropriate to approximate the expected values by their maximum likelihood (ML) estimates. Accordingly, the approximate variance-covariance matrix is given by [19]

$$\begin{pmatrix} -\frac{\partial^2 l}{\partial^2\varphi_1^2} & \cdots & -\frac{\partial^2 l}{\partial^2\varphi_1\varphi_n} \\ \vdots & \vdots & \vdots \\ -\frac{\partial^2 l}{\partial^2\varphi_n} & \cdots & -\frac{\partial^2 l}{\partial^2\varphi_n^2} \end{pmatrix}^{-1}_{(\hat{\varphi_1},\dots,\hat{\varphi_n})}$$

From the log-likelihood equation (12), we get

$$\frac{\partial^2 l}{\partial\alpha^2}=-\frac{r}{\alpha^2}-\sum_{i=1}^{r}[(R_i+1)\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}\times\left(\log\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)\right)^2-R^*_{(j)}\left(\lambda T+\frac{\theta}{2}T^2\right)^{\alpha}\left(\log\left(\lambda T+\frac{\theta}{2}T^2\right)\right)^2], \quad (19)$$

$$\frac{\partial^2 l}{\partial\theta^2}=-(\alpha-1)\sum_{i=1}^{r}\frac{1}{(\lambda x_i+\frac{\theta}{2}x_i^2)^2}\left(\frac{x_i^4}{4}\right)-\sum_{i=1}^{r}\frac{x_i^2}{(\lambda+\theta x_i^2)^2}-\sum_{i=1}^{r}[\alpha(\alpha-1)(R_i+1)\times\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-2}\left(\frac{x_i^4}{4}\right)-\alpha(\alpha-1)R^*_{(j)}\left(\lambda T+\frac{\theta}{2}T^2\right)^{\alpha-2}\left(\frac{T^4}{4}\right)], \quad (20)$$

$$\frac{\partial^2 l}{\partial\lambda^2}=-(\alpha-1)\sum_{i=1}^{r}\frac{x_i^2}{(\lambda x_i+\frac{\theta}{2}x_i^2)^2}-\sum_{i=1}^{r}\frac{1}{(\lambda+\theta x_i)}-\sum_{i=1}^{r}[\alpha(\alpha-1)(R_i+1)\times\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-2}(x_i^2)-\alpha(\alpha-1)R^*_{(j)}\left(\lambda T+\frac{\theta}{2}T^2\right)^{\alpha-2}T^2], \quad (21)$$

$$\frac{\partial^2 l}{\partial\alpha\theta}=\sum_{i=1}^{r}\frac{1}{(\lambda x_i+\frac{\theta}{2}x_i^2)}\left(\frac{x_i^2}{2}\right)-\sum_{i=1}^{r}[(R_i+1)\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}\left(\frac{x_i^2}{2}\right)\times[\alpha\left(\log\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)\right)+1]+R^*_{(j)}\left(\lambda T+\frac{\theta}{2}T^2\right)^{\alpha-1}\left(\frac{T^2}{2}\right)\times\left(\alpha\log\left(\lambda T+\frac{\theta}{2}T^2\right)+1\right)], \quad (22)$$

$$\frac{\partial^2 l}{\partial\alpha\lambda}=\sum_{i=1}^{r}\frac{x_i}{(\lambda x_i+\frac{\theta}{2}x_i^2)}-\sum_{i=1}^{r}[(R_i+1)$$

$$\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1} x_i \times\left(\alpha\log\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)+1\right)$$

$$+R_{(j)}^{*}\left(\lambda T+\frac{\theta}{2}T^2\right)^{\alpha-1} T\left(\alpha\log\left(\lambda T+\frac{\theta}{2}T^2\right)+1\right)], \tag{23}$$

$$\frac{\partial^2 l}{\partial\theta\lambda}=-(\alpha-1)\sum_{i=1}^{r}\frac{x_i^3}{2\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)}-\sum_{i=1}^{r}\frac{x_i}{(\lambda+\theta x_i)^2}$$
$$-\sum_{i=1}^{r}\alpha(\alpha-1)(R_i+1)\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-2}\left(\frac{x_i^3}{2}\right)$$
$$-\alpha(\alpha-1)R_{(j)}^{*}\left(\lambda T+\frac{\theta}{2}T^2\right)^{\alpha-2}\left(\frac{T^3}{2}\right). \tag{24}$$

Then, the asymptotic variance-covariance matrix of the estimators of the parameters α,θ and λ is obtained by inverting the Fisher information matrix given by taking the expectation of Equations (19), (20), (21), (22), (23) and (24) in which elements are negatives. In the present situation, it seems appropriate to approximate the expected values by their ML estimates. Accordingly, the approximate variance -covariance matrix is given as

$$\begin{pmatrix} \sigma_{\hat{\alpha}\alpha} & \sigma_{\hat{\alpha}\theta} & \sigma_{\hat{\alpha}\lambda} \\ \sigma_{\hat{\alpha}\theta} & \sigma_{\hat{\theta}\theta} & \sigma_{\hat{\theta}\lambda} \\ \sigma_{\alpha\lambda} & \sigma_{\theta\lambda} & \sigma_{\lambda\lambda} \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\alpha^2} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\alpha\theta} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\alpha\lambda} \\ -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\alpha\theta} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\theta^2} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\theta\lambda} \\ -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\alpha\lambda} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\theta\lambda} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\lambda^2} \end{pmatrix}^{-1}_{(\hat{\alpha},\hat{\theta},\hat{\lambda})}. \tag{25}$$

The ACIs for the parameters α,θ and λ are respectively given as:

$$\hat{\alpha}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_{\alpha\alpha}},\quad \hat{\theta}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_{\theta\theta}}\quad\text{and}\quad \hat{\lambda}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_{\lambda\lambda}},$$

where $Z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\alpha}{2}$.

4 Bayesian Estimation of the Parameters

In Bayesian estimation, we study two types of loss functions. The first is the squared error loss (quadratic loss) function which is know as a symmetric function and associates equal importance to the losses for overestimation and underestimation of equal magnitude. However, in most practical circumstances such a limitation may be impractical. For instance, in the estimation of reliability and failure rate function , an overestimation is usually much more serious than an underestimation. In this situation, the use of symmetrical loss function might be inappropriate as highlighted by [20]. The second is LINEX (linear-exponential) loss function which is asymmetric and was introduced by [21]. These loss functions were widely used by several authors, including [20], [22], [23], [24], [25]. [26] and [27]. This function rises roughly exponentially on one side of zero and roughly linearly on the other hand. It can also be noted that for a specific choice of the loss function parameter, the squared error loss function can be obtained as a specific member of the LINEX loss function. The squared error loss function and the LINEX loss function for a parameter δ are as follows:

$$L_{BS}\left(\delta,\hat{\delta}\right)=\left(\delta-\hat{\delta}\right)^2, \tag{26}$$

and

$$L_{BL}(\Delta)=a\left(e^{h\Delta}-h\Delta-1\right)\qquad, a>0\ ,\ h\neq 0, \tag{27}$$

where a and h are shape and scale parameters of the loss function, respectively and $\Delta=\left(\delta-\hat{\delta}\right)$ denotes the scalar estimation error using $\hat{\delta}$ to estimate δ. The sign and magnitude of h in LINEX loss function affect the direction and degree of asymmetry. This was introduced by [28] and further properties of this loss function were investigated by [29]. For small values of h (near to zero), the LINEX loss function is almost the same as the squared error loss function and for the choice of negative or positive values of h , the LINEX loss function gives more weight to overestimation or underestimation (for details, see [29]).

Bayesian estimates of δ against the squared error loss function and the LINEX loss function are as follows:

$$\hat{\delta}_{BS}=E\left[\delta\mid x\right], \tag{28}$$

and

$$\hat{\delta}_{BL}=-\frac{1}{h}\log\{E\left[e^{-h\delta}\mid x\right]\}. \tag{29}$$

Now we propose the Bayesian estimator of parameters $(\alpha,\theta,\ \lambda)$ as well as reliability R(t), hazard H(t) and reversed hazard Q(t) functions of the GLE. It is assumed that α, θ and λ have the following independent prior:

$$\pi_1(\alpha)\propto\alpha^{w_2-1}e^{-\alpha w_1}\quad ,w_1>0,w_2>0\quad \alpha>0,$$

$$\pi_2(\theta)\propto\theta^{w_4-1}e^{-\theta w_3}\quad ,w_3>0,w_4>0,\ \ \theta>0, \tag{30}$$

and

$$\pi_3(\lambda) \propto \lambda^{w_6-1} e^{-\lambda w_5} \quad , w_6 > 0, w_5 > 0,\ \lambda \geq 0,$$

where $w_1, w_2, w_3, w_4, w_5, w_6$ are chosen to reflect prior knowledge on α, θ and λ. When $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 0$, non-informative priors of α, θ and λ exist.

From Equations (12) and (30) , the joint prior density function of α, θ and λ is of the form

$$\pi(\alpha,\theta,\lambda) \propto \alpha^{w_2-1} e^{-\alpha w_1}\, \theta^{w_4-1} e^{-\theta w_3}\, \lambda^{w_6-1} e^{-\lambda w_5}, \quad \alpha,\theta > 0,\ \lambda \geq 0,$$

$$w_1, w_2, w_3, w_4, w_5, w_6 > 0.$$

Then the posterior distribution α, θ and λ can be written as

$$\pi^*(\alpha,\theta,\lambda \mid x) = \frac{1}{k} \alpha^{w_2+r-1} e^{-\alpha w_1} \theta^{w_4-1} e^{-\theta w_3} \lambda^{w_6-1} e^{-\lambda w_5} \times \prod_{\imath=1}^{r} \left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha-1} (\lambda+\theta x_i) e^{-\left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha}} e^{-(n-r)\left(\lambda c + \frac{\theta}{2} c^2\right)^{\alpha}}, \tag{31}$$

where

$$k = \int_0^\infty \int_0^\infty \int_0^\infty \Big[\alpha^{w_2+r-1} e^{-\alpha w_1} \theta^{w_4-1} e^{-\theta w_3} \lambda^{w_6-1} e^{-\lambda w_5} \times \prod_{\imath=1}^{r} \left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha-1} (\lambda+\theta x_i) e^{-\left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha}} \times e^{-(n-r)\left(\lambda c + \frac{\theta}{2} c^2\right)^{\alpha}}\Big] d\alpha d\theta d\lambda.$$

In Bayesian statistics the posterior distribution $\pi^*(\alpha,\theta,\lambda \mid x)$ contains all information on the unknown parameters given the observed data. From the posterior distribution, all statistical inference can be deduced. We observe that equation (31) can not be solved explicitly, so two different procedures are introduced Lindley's approximation and MCMC technique that can be used to obtain the Bayes estimator for α, θ and λ and the corresponding credible intervals. For any function $u(\alpha,\theta,\lambda)$ of α, θ and λ, the Bayes estimates are given by

$$\hat{u}(\alpha,\theta,\lambda) = \frac{1}{k} \int_0^\infty \int_0^\infty \int_0^\infty u(\alpha,\theta,\lambda) L(\alpha,\theta,\lambda) \times \pi(\alpha,\theta,\lambda)\, d\alpha d\theta d\lambda. \tag{32}$$

4.1 Lindely approximation

Lindley's approximation, which was introduced by [30] can approximate the Bayes estimators into a form containing no integral. For our estimation problem we describe this method below. As noticed the Bayesian estimates involve the ratio of two integrals, we consider $I(x)$ defined as

$$I(x) = E\left[u(\gamma_1,\gamma_2,\gamma_3)\right] = \frac{\int_{\gamma_3}\int_{\gamma_2}\int_{\gamma_1} u(\gamma_1,\gamma_2,\gamma_3)\, e^{L(\gamma_1,\gamma_2,\gamma_3)+\rho(\gamma_1,\gamma_2,\gamma_3)} d\gamma_1 d\gamma_2 d\gamma_3}{\int_{\gamma_3}\int_{\gamma_2}\int_{\gamma_1} e^{L(\gamma_1,\gamma_2,\gamma_3)+\rho(\gamma_1,\gamma_2,\gamma_3)} d\gamma_1 d\gamma_2 d\gamma_3},$$

where $u(\gamma_1,\gamma_2,\gamma_3)$ is a function of γ_1,γ_2 or γ_3 only.
$L(\gamma_1,\gamma_2,\gamma_3)$ is log of likelihood function.
$\rho(\gamma_1,\gamma_2,\gamma_3)$ is log joint prior of γ_1,γ_2 and γ_3.

Utilizing the Lindley's method $I(x)$ can be approximated as

$$\begin{aligned} I(x) =& u(\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3) + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5) \\ &+ \frac{1}{2}\left[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13})\right. \\ &+ B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) \\ &\left.+ C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})\right], \end{aligned} \tag{33}$$

where $\hat{\gamma}_1, \hat{\gamma}_2$ and $\hat{\gamma}_3$ are the MLE of of γ_1, γ_2 and γ_3 respectively.

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3},\ i = 1,2,3,$$

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23},$$

$$a_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}),$$

$$\begin{aligned} A =& \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} \\ &+ \sigma_{22}L_{221} + \sigma_{33}L_{331}, \end{aligned}$$

$$\begin{aligned} B =& \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} \\ &+ \sigma_{22}L_{222} + \sigma_{33}L_{332}, \end{aligned}$$

$$\begin{aligned} C =& \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} \\ &+ \sigma_{22}L_{223} + \sigma_{33}L_{333}, \end{aligned}$$

and subscripts 1,2,3 on the right-hand sides refer to γ_1, γ_2 , γ_3 respectively and

$$\rho_i = \frac{\partial \rho}{\partial \gamma i}, \quad u_i = \frac{\partial u(\gamma_1,\gamma_2,\gamma_3)}{\partial \gamma i},\ i = 1,2,3,$$

$$u_{ij} = \frac{\partial^2 u(\gamma_1,\gamma_2,\gamma_3)}{\partial \gamma_i \partial \gamma_j},\ i,j = 1,2,3,$$

$$L_{ij}=\frac{\partial^2 L(\gamma_1,\gamma_2,\gamma_3)}{\partial\gamma i\partial\gamma_j},\ i,j=1,2,3\ ,$$
$$L_{ijk}=\frac{\partial^3 L(\gamma_1,\gamma_2,\gamma_3)}{\partial\gamma_i\partial\gamma_j\partial\gamma_k},\ i,j,k=1,2,3,$$

and σ_{ij} is the $(i,j)-th$ element of the inverse of the matrix $\{L_{ij}\}$. All are evaluated at the MLE of parameters. With the above-mentioned defined expressions, we can obtain the values of the Bayes estimates of various parameters .

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=\hat{\alpha}$, the Bayes estimate of the parameter α under the squared error loss (SEL) function from (33) is

$$\hat{\alpha}_{BS}=\hat{\alpha}+a_1+\frac{1}{2}\left[A\sigma_{11}+B\sigma_{21}+C\sigma_{31}\right] \tag{34}$$

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=\hat{\theta}$, the Bayes estimate of the parameter θ under the squared error loss (SEL) function is

$$\hat{\theta}_{BS}=\hat{\theta}+a_2+\frac{1}{2}\left[A\sigma_{12}+B\sigma_{22}+C\sigma_{32}\right]. \tag{35}$$

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=\hat{\lambda}$, the Bayes estimate of the parameter λ under the squared error loss (SEL) function is

$$\hat{\lambda}_{BS}=\hat{\lambda}+a_3+\frac{1}{2}\left[A\sigma_{13}+B\sigma_{23}+C\sigma_{33}\right]. \tag{36}$$

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=e^{-h\hat{\alpha}}$,the Bayes estimate of the parameter α under the LINEX loss function from (33) is

$$\hat{\alpha}_{BL}=-\frac{1}{h}\log\left\{e^{-h\hat{\alpha}}\left[1-ha_1+\frac{1}{2}h^2\sigma_{11}-\frac{1}{2}h\left(A\sigma_{11}+B\sigma_{21}+C\sigma_{31}\right)\right]\right\}. \tag{37}$$

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=e^{-h\hat{\theta}}$, the Bayes estimate of the parameter θ under the LINEX loss function is

$$\hat{\theta}_{BL}=-\frac{1}{h}\log\left\{e^{-h\hat{\theta}}\left[1-ha_2+\frac{1}{2}h^2\sigma_{22}-\frac{1}{2}h\left(A\sigma_{12}+B\sigma_{22}+C\sigma_{32}\right)\right]\right\}. \tag{38}$$

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=e^{-h\hat{\lambda}}$, the Bayes estimate of the parameter λ under the LINEX loss function is

$$\hat{\lambda}_{BL}=-\frac{1}{h}\log\left\{e^{-h\hat{\lambda}}\left[1-ha_3+\frac{1}{2}h^2\sigma_{33}-\frac{1}{2}h\left(A\sigma_{13}+B\sigma_{23}+C\sigma_{33}\right)\right]\right\}. \tag{39}$$

The estimators of reliability function R(t), hazard rate function H(t) and reversed hazard rate function Q(t) can be obtained by substituting the Lindely's approximate under the squared error loss (SEL) function and LINEX loss function of the parameters α, θ and λ in (3), (4) and (5).

The approximate Bayes estimator of α, θ and λ can be obtained using Lindley approximation , but it is impossible to construct highest posterior density(HPD) confidence intervals using this method. Thus, we use the following Markov Chain Mont Carlo (MCMC) method to generate samples from the posterior density function, and obtain the Bayes estimators as well as HPD confidence intervals.

4.2 MCMC Method

We study the Markov Chain Monte Carlo (MCMC) methods to draw samples from the posterior density function, then compute the Bayes estimators and construct HPD credible intervals of α, θ and λ.Several conventional ways define such Markov Chains exist, including Gibbs sampling, Metropolis-Hastings (MH) and reversible jump. Using these algorithms it is possible to implement posterior simulation, which allows point wise evaluation of the prior distribution and likelihood function in any issue. From Equation (31), the marginal posterior density of α is proportional to

$$\pi_1^*(\alpha\mid\theta,\lambda,\ x)\propto\alpha^{w_2+r-1}e^{-\alpha w_1}\prod_{\imath=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}\times(\lambda+\theta x_i)\,e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}e^{-\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha(n-r)}} \tag{40}$$

Similarly, the posterior conditional distributions for θ and λ are respectively

$$\pi_2^*(\theta\mid\alpha,\lambda,\ x)\propto\theta^{w_4+r-1}e^{-\theta w_3}\prod_{\imath=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}\times(\lambda+\theta x_i)\,e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}e^{-\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha(n-r)}} \tag{41}$$

and

$$\pi_3^*(\lambda\mid\alpha,\theta,\ x)\propto\theta^{w_6+r-1}e^{-\theta w_5}\prod_{\imath=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}\times(\lambda+\theta x_i)\,e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}e^{-\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha(n-r)}} \tag{42}$$

We can see the posteriors conditional distributions for α, θ and λ are log-concave and can not be reduced analytically to well-known distributions. Consequently, as suggested by [31], a common way to solve this problem is to use the hybrid algorithm by combining a Metropolis-Hasting (MH) sampling with Gibbs sampling scheme using normal distribution. Hence, the algorithm works as follows:

1-Set the initial values of α, θ and λ say (α_0, θ_0,λ_0).
2-Set $j=1$.

3-Using MH, generate α_1^j from $\pi_1^*(\alpha^{j-1} \mid \theta^{j-1}, \lambda^{j-1}, x)$ with normal distribution, $N(\alpha^{j-1}, K\alpha V\alpha)$.
4-Using MH, generate θ_1^j from $\pi_2^*(\theta^{j-1} \mid \alpha^{j-1}, \lambda^{j-1}, x)$ with normal distribution, $N(\theta^{j-1}, K_\theta V_\theta)$.
5-Using MH, generate λ_1^j from $\pi_3^*(\lambda^{j-1} \mid \alpha^{j-1}, \theta^{j-1}, x)$ with normal distribution, $N(\lambda^{j-1}, K_\lambda V_\theta)$, where $K\alpha, K_\theta$ and K_λ are scaling factor and $V\alpha$, V_θ and V_θ are variances-co variances matrix.
6-Set $j = j+1$.
7-Repeat steps from 1 to 5 N times.
8-The Bayes estimators of u(α, θ, λ) can be approximated as:

$$\hat{u}_{Mc} \approx \frac{\frac{1}{N}\sum_{i=1}^{N} u\left(\alpha_j, \theta_j, \lambda_j\right) f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{i=1}^{N} f_1(\alpha_j, \theta_j, \lambda_j)} \tag{43}$$

where

$$f_1(\alpha_j, \theta_j, \lambda_j) = \prod_{\imath=1}^{r}\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1}(\lambda+\theta x_i) \times e^{-\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha}} e^{-\left(\lambda c + \frac{\theta}{2}c^2\right)^{\alpha(n-r)}},$$

9-Order α_j , θ_j and λ_j , $j = 1, \ldots N$ and suppose that we would like to construct the HPD credible intervals of α, θ and λ Now, we construct all 100(1-α)% credible intervals of α say $(\alpha_{[1]}, \alpha_{[N(1-\alpha)]}), \ldots (\alpha_{[N\alpha]}, \alpha_{[N]})$.

Here [X] denotes the largest integer less than or equal to X. Then the HPD credible interval of α is the interval which has the shortest duration. Similarly, the HPD credible interval of θ and λ can also be constructed. Now, two loss functions define the Bayes estimates based on MCMC method from (35).

4.2.1 Bayes estimate based on MCMC under LINEX loss function

1-For estimating α, consider $u\left(\alpha_j, \theta_j, \lambda_j\right) = \exp[-h\alpha_j]$, so

$$\hat{\alpha}_{BL} = \frac{-1}{h}\log\left[\frac{\frac{1}{N}\sum_{i=1}^{N} u\left(\alpha_j, \theta_j, \lambda_j\right) f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{i=1}^{N} f_1(\alpha_j, \theta_j, \lambda_j)}\right]. \tag{44}$$

2-For estimating θ, consider $u\left(\alpha_j, \theta_j, \lambda_j\right) = \exp[-h\theta_j]$, so

$$\hat{\theta}_{BL} = \frac{-1}{h}\log\left[\frac{\frac{1}{N}\sum_{i=1}^{N} u\left(\alpha_j, \theta_j, \lambda_j\right) f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{i=1}^{N} f_1(\alpha_j, \theta_j, \lambda_j)}\right]. \tag{45}$$

3-For estimating λ, consider $u\left(\alpha_j, \theta_j, \lambda_j\right) = \exp[-h\lambda_j]$, then

$$\hat{\lambda}_{BL} = \frac{-1}{h}\log\left[\frac{\frac{1}{N}\sum_{i=1}^{N} u\left(\alpha_j, \theta_j, \lambda_j\right) f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{i=1}^{N} f_1(\alpha_j, \theta_j, \lambda_j)}\right]. \tag{46}$$

The estimators of reliability function R(t), hazard rate function H(t) and reversed hazard rate function Q(t) can be obtained by substituting the MCMC under the LINEX loss function of the parameters α, θ and λ in (3), (4) and (5).

4.2.2 Bayes estimate based on MCMC under Squared Error loss function

1-For estimating α, consider $u\left(\alpha_j, \theta_j, \lambda_j\right) = \alpha_j$, so

$$\hat{\alpha}_{SL} = \frac{\frac{1}{N}\sum_{i=1}^{N} u\left(\alpha_j, \theta_j, \lambda_j\right) f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{i=1}^{N} f_1(\alpha_j, \theta_j, \lambda_j)}. \tag{47}$$

2-For estimating θ, consider $u\left(\alpha_j, \theta_j, \lambda_j\right) = \theta_j$, so

$$\hat{\theta}_{SL} = \frac{\frac{1}{N}\sum_{i=1}^{N} u\left(\alpha_j, \theta_j, \lambda_j\right) f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{i=1}^{N} f_1(\alpha_j, \theta_j, \lambda_j)}. \tag{48}$$

3-For estimating λ, consider $u\left(\alpha_j, \theta_j, \lambda_j\right) = \lambda_j$, then

$$\hat{\lambda}_{SL} = \frac{\frac{1}{N}\sum_{i=1}^{N} u\left(\alpha_j, \theta_j, \lambda_j\right) f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{i=1}^{N} f_1(\alpha_j, \theta_j, \lambda_j)}. \tag{49}$$

The estimators of reliability function R(t), hazard rate function H(t) and reversed hazard rate function Q(t) can be obtained by substituting the MCMC under the Squared Error loss(SEL) function of the parameters α, θ and λ in (3), (4) and (5).

5 Simulation Study

In this section, we carry out a simulation study to compare the performance of ML estimators and Bays estimators. We estimate the unknown parameters using the ML estimate and Bayes estimators obtained by Lindley approximations and MCMC method. The performances of different estimators with MSE are compared. Also, the average length of the asymptotic confidence intervals and the HPD confidence intervals are obtained. The comparison between the estimates occurs according to the following steps:

1-For the given parameters $w_1, w_2, w_3, w_{4,} w_5$ and w_6 generate random values of α, θ and λ .
2-For given values of n and r with initial values of α, θ and λ given in step 1, we generate random samples from inverse cumulative distribution function of GLE distribution, then we order them.
3-The ML estimates of α, θ and λ are then obtained by solving the nonlinear equations (13), (14) and (15).
4-The ML estimates of the hazard function, survival function and reversed hazard function are obtained from equations (16),(17) and (18) with t=0.25.
5-The Bayes estimates of α, θ and λ using Lindley's approximation forms under Secured Error (SE) loss function are given by (34) - (36), and under LINEX loss function are given by (37) - (39).

6-The Bayes estimates of α, θ , λ, hazard function, survival function and reversed hazard function are computed by applying MCMC method with 1100 observations under Secured Error (SE) loss function and LINEX loss function .

7-The quantities $(\hat{\kappa} - \kappa)^2$ are computed where $\hat{\kappa}$ stands for an estimate of κ (ML or Bayes).

Steps 1-7 are repeated at least 1000 times for informative prior and non-informative prior and for different sample sizes n and r at T=2 . A simulation data for progressive hybrid type-II censored sample from GLED is generated with $\alpha = 2, \theta = 2.6$, $\lambda = 1.1$ and $T = 1$ for different choices of n and m using the following scheme with informative priors

Table 1: Different Schemes

n	m	R
10	5	(0 1 4 0 0)
50	15	(3*5,0,5*4,0*5)
100	25	(10*3,5*9,0*13)

$w_1 = 0.05, w_2 = 0.05$, $w_3 = 0.01, w_4 = 0.01, w_5 = 0.01$, $w_6 = 0.01$, and non-informative priors $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 0$, the MSE's, the average asymptotic confidence intervals and the HPD confidence intervals length from the MCMC technique are computed with informative priors and non-informative priors. The MSE of the estimates were estimated by

$$MSE(\hat{\kappa}) = \sum_{1}^{1000} \frac{(\hat{\kappa} - \kappa)^2}{1000}.$$

The results are reported in Tables 2- 8 for the informative priors and Tables 9- 15 for non-informative priors.

From Tables 2-4, the following observations are made for MLE and Bayes estimates: as n and m increase, the MSEs decrease for all choices. The MSEs estimates of survival function and hazard function decrease as n and m increase in Tables 5-7. Now we compare different confidence/credible intervals in terms of their average lengths and coverage probability. From Table 8,we observed that the average confidence/credible lengths decrease as n and m increase. The average confidence lengths for HPD confidence intervals are smaller than the asymptotic confidence intervals for all sample sizes. For non-informative priors we can see the same results for MSEs for all cases Table 8-13. The Bayes credible intervals with respect to the non-informative priors also work very well. Table 15 indicates that the average credible lengths are also smaller than the likelihood ratio-based confidence intervals. It is noticeable that both methods work well even for small n and m. The Bayes estimates of the three parameters α, θ and λ using MCMC method are generally make no sense their MLEs and Bayes estimates using Lindley approximation, based on MSEs.

6 Real Data Analysis

In this section, two data analysis are presented for illustrative purpose. EXAMPLE 1: we use the lifetime data set presented in Table 15 to compare between proposed methods. The data set presented in Table 16 represents 50 observed failure times that were initially reported in [32] and later by a number of authors including [33], [34], [35], [31] and [36].

Before progressing, we would like to check whether the GLED fits this data or not. The calculated value of the K-S test is 0.18497 for the GLED, is smaller than their corresponding values expected at 5% significance level, which is 0.29407 at $n = 50$. We have just plotted the empirical survival function and the fitted survival functions in Figure 1. Observe that the GLED can be a good model fitting this data. Figure 2 shows that all the points of a Q-Q plot are inside the unit square, so, it is noticeable that the GLED fits the data very well.

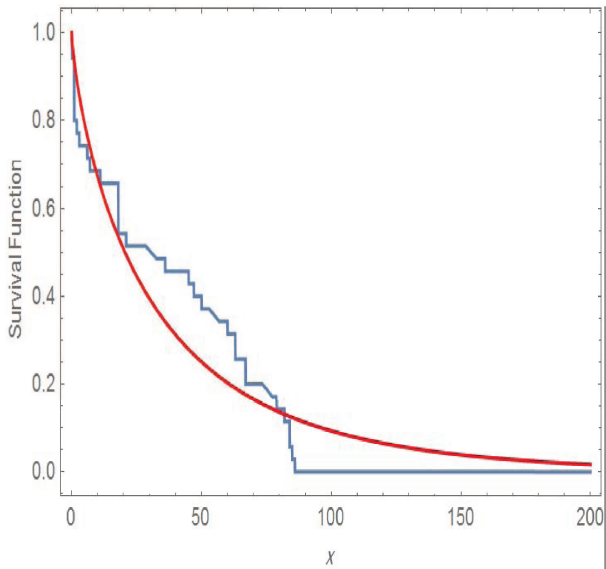

Fig. 1: Empirical and fitted distribution function for completed data set.

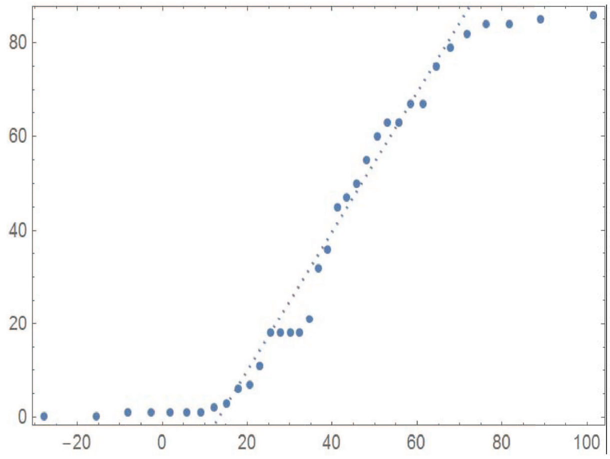

Fig. 2: Q-Q plot compare data to a specific distribution.

The unknown parameters using the ML estimates and the Bayes procedure are estimated. For computing the ML estimates, we use the numerical method and compute the 95% ACIs using the observed Fisher information matrix. For computing the Bayes estimates we consider the SE loss function and LINEX loss function. For comparison purpose, the informative and the non-informative priors were assumed. The Bayes estimates are obtained using importance samples of size N=11000. In all cases $\alpha = 0.7, \theta$ =0.0009, $\lambda = 0.009, c = 1, -1, 4$. are considered. First, the informative priors are considered with $w_1 = 0.05, w_2 = 0.05, w_3 = 0.01, w_4 = 0.01, w_5 = 0.01$ and $w_6 = 0.01$,(Tables 16 and 17). Second, the non informative priors $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 0$,, (Tables 19 and 20).

Using the estimates we obtain the 95% ACIs based on the observed Fisher information matrix. For computing the Bayes estimates we consider the SE loss function and LINEX loss function. (Table 22).

From Tables 17 and 19, The Bayes estimates of the three parameters α, θ and λ using MCMC are similar to MLEs in all cases informative and non informative priors. According to Tables 18 and 20, the length of the HPD credible intervals of α, θ and λ based on informative priors is almost smaller than the corresponding length of the HPD credible intervals based on non-informative priors.

EXAMPLE 2:We use a data set obtained through realistic experiment. The research started when people realized that guinea pigs are highly susceptible to human tuberculosis.Although guinea pigs acquire resistance, a few deadly tubercle bacilli will cause progressive disease and death. The investigation aims to identify the relation between numbers of tubercle bacilli infected individuals and survival time [37].The listed values are the survival times (in days) of the guineas placed in the environment of $4.0 * 106$ bacillary units per 0.5 mL. 72 observations are shown as follows: $12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376$. We will created type-II progressively hybrid censored sample from the aforementioned uncensored data set. Before progressing, we would like to check whether the GLED fits this data or not. The calculated value of the K-S test is 0.146312 for the GLED and this value is smaller than their corresponding values expected at 2% significance level, which is 1.51743 at $n = 72$. In this case $n = 72$ and we take $m = 28$, $T = 60.5$,$R_1 = R_2 = ... = R_{11} = 4$,$R_{12} = R_{13} = ... = R_{28} = 0$. Thus the type-II progressively hybrid censored sample is: $12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60$.

From the above-mentioned sample the unknown parameters using the MLEs and the Bayes procedure are estimated with informative priors $w_1 = 0.1$, $w_2 = 0.6$, $w_3 = 0.2$, $w_4 = 0.1$, $w_5 = 0.6$ and $w_6 = 0.2$, $\alpha = 1, \theta = 1, \lambda = 0.01, c = 2, -2$. (Table 20).

According to Table 20, the Bayes estimates of the three parameters α, θ and λ using MCMC are similar to MLEs. We can see that the length of HDP credible intervals of α, θ and λ is almost smaller than the corresponding length of the ACI intervals.

7 Conclusion

The present paper addressed, MLE and Bayes estimation of the unknown parameter for the progressive Type-II hybrid censored GLE distribution. We provided the maximum likelihood estimators and it was observed that the maximum likelihood estimators of the unknown parameters can not obtained in the closed form and we used the numerical method to compute them. Also, we found the Bayes estimators of the unknown parameters, which can not be obtained in explicit forms. We proposed two approximation methods to compute them. Lindley approximations and the MCMC method were used. We used Monte Carlo simulations to compare the performance of the different methods, it was quite satisfactory. Although we have assumed that the lifetime distributions are generalized linear exponential distribution, most of the methods can be extended to other distributions.

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Table 2: Estimates of the parameters α and the corresponding MSE for informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=0.1	c3=1	c4=4
10	5	2.3191	1.7707	2.0685	1.7543	1.6904	1.7864	2.3158	2.3158	2.3197	2.3158	2.3244
		0.1018	0.0526	0.0047	0.0604	0.0958	0.1	0.0998	0.0998	0.1022	0.0998	0.1052
50	15	1.9614	1.8633	1.9126	1.8588	1.823	1.7644	1.9607	1.9607	1.9615	1.9607	1.9596
		0.0015	0.0187	0.0076	0.0199	0.0313	0.0555	0.0015	0.0015	0.0015	0.0015	0.0016
100	25	1.9985	1.9421	1.9735	1.9391	1.9141	1.8611	2.0102	1.9984	1.9984	1.9984	1.8611
		$2.*10^{-6}$	0.0033	0.0007	0.0073	0.0074	0.0193	0.0001	$3*10^{-6}$	$2.79*10^{-6}$	$3*10^{-6}$	$3.45*10^{-6}$

Table 3: Estimates of the parameters θ and the corresponding MSE for informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=0.1	c3=1	c4=4
10	5	2.1839	1.8735	2.3318	1.8328	0.1038	1.6144	2.1553	2.1554	2.1884	2.1553	2.1909
		0.1731	0.5278	0.072	0.5886	0.9856	0.9713	0.1978	0.1977	0.1695	0.1978	0.1677
50	15	2.5713	2.4387	2.692	2.4139	2.2395	2.136	2.5598	2.5598	2.5664	2.5596	2.56
		0.0008	0.026	0.0085	0.0346	0.13	0.2153	0.0019	0.0019	0.0012	0.002	0.0022
100	25	2.5845	2.4902	2.6581	2.4902	2.3496	2.2313	2.5844	2.5844	2.5928	2.5844	2.6072
		0.0002	0.012	0.0033	0.012	0.0627	0.0185	0.0002	0.0003	0.0001	0.0003	0.0001

Table 4: Estimates of the parameters λ and the corresponding MSE for informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=–0.2	c3=1	c4=4
10	5	0.8688	0.8144	0.855	0.8105	0.7779	0.7123	0.8683	0.8683	0.8725	0.8683	0.8687
		0.1	0.0816	0.06	0.1	0.1038	0.1503	0.1	0.1	0.1	0.1	0.1
50	15	0.8147	0.7929	0.8108	0.7911	0.7757	0.7351	0.815	0.815	0.8179	0.815	0.8133
		0.0814	0.0943	0.0836	0.0954	0.1052	0.1331	0.0812	0.0812	0.0796	0.0812	0.0822
100	25	0.934	0.9181	0.9305	0.9733	0.9061	0.8759	0.9344	0.9344	0.9341	0.9344	0.9339
		0.0276	0.0331	0.0287	0.0161	0.0376	0.0502	0.0274	0.0274	0.0275	0.0274	0.02758

Table 5: Estimates of survival function R(t) and the corresponding MSE for informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=0.1	c3=1	c4=4
10	5	0.9113	0.8699	0.8853	0.8696	0.8738	0.8984	0.9118	0.9118	0.9106	0.9118	0.9116
		0.0085	0.0026	0.0044	0.003	0.003	0.0063	0.0086	0.0086	0.0084	0.0086	0.0086
50	15	0.8738	0.8697	0.865	0.8702	0.8736	0.8768	0.874	0.874	0.8733	0.874	0.8742
		0.003	0.0026	0.0021	0.0026	0.003	0.0034	0.003	0.003	0.003	0.003	0.0031
100	25	0.8543	0.8533	0.8496	0.8537	0.8564	0.8591	0.8542	0.8542	0.8541	0.8542	0.8536
		0.0013	0.0012	0.0009	0.0012	0.0014	0.0016	0.0012	0.0012	0.0012	0.0012	0.0012

Table 6: Estimates of hazard function H(t) and the corresponding MSE for informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=0.1	c3=1	c4=4
10	5	0.9147	1.0338	1.084	1.0237	0.9398	0.7996	0.9062	0.9063	0.9219	0.9062	0.9142
		0.5868	0.4185	0.356	0.4316	0.549	0.7764	0.5998	0.5998	0.5758	0.5999	0.5876
50	15	1.1654	1.1408	1.232	1.132	1.0688	1.0079	1.1622	1.1622	1.1689	1.1622	1.1602
		0.2655	0.2915	0.2013	0.3011	0.3744	0.4526	0.2689	0.2688	0.262	0.2689	0.271
100	25	1.3563	1.3238	1.3933	1.3171	1.2659	1.2023	1.3572	1.3572	1.3596	1.3572	1.3654
		0.105	0.1274	0.0826	0.1322	0.1721	0.2289	0.1047	0.1047	0.1031	0.1047	0.0995

Table 7: Estimates of reversed hazard function Q(t) and the corresponding MSE for informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1			c1=-1	c2=0.1	c3=1	c4=4
10	5	9.3969	6.9116	8.3655	6.8287	6.5086	7.0729	9.3676	9.3677	9.3924	9.3676	9.4247
		3.2317	0.4728	0.5872	0.5937	1.1895	0.2771	3.1276	3.1278	3.2153	3.1273	3.3324
50	15	8.0688	7.6148	7.8934	7.5886	7.3905	7.1737	8.0604	8.0604	8.0597	8.0604	8.0598
		0.2205	0.0002	0.0865	0.0001	0.0436	0.1811	0.2128	0.2128	0.212	0.2128	0.2121
100	25	7.9554	7.7015	7.8738	7.6848	7.5504	7.333	7.9537	7.9537	7.9592	7.9537	7.9621
		0.1269	0.0001	0.0754	0.0001	0.0024	0.005	0.1256	0.1256	0.1296	0.1256	0.1316

Table 8: The 95% ACI's and HPD credible intervals and the corresponding length of α, θ and λ for informative priors

n	m	parameter	ACI	HPD
10	5	α	6.1298	0.0047
		θ	16.8645	0.0324
		λ	5.3752	0.0084
50	15	α	4.1488	0.0047
		θ	16.2651	0.03
		λ	4.9875	0.0038
100	25	α	2.5644	0.0017
		θ	11.0182	0.0212
		λ	3.2047	0.0019

Table 9: Estimates of the parameters α and the corresponding MSE for non-informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=0.1	c3=1	c4=4
10	5	2.3191	1.7893	2.0921	1.7719	1.7004	1.7886	2.3146	2.3146	2.3211	2.3146	2.3157
		0.1018	0.0444	0.0085	0.0521	0.0898	0.1	0.099	0.099	0.1031	0.099	0.0997
50	15	1.9614	1.8658	1.9151	1.8613	1.8251	1.7655	1.9627	1.9627	1.9628	1.9627	1.9611
		0.0015	0.018	0.0072	0.0192	0.0306	0.055	0.0014	0.0014	0.0014	0.0014	0.0015
100	25	1.9985	1.9437	1.9751	1.9407	1.9156	1.862	1.9984	1.9984	1.9985	1.9984	1.9986
		$2.*10^{-6}$	0.0032	0.0006	0.0035	0.0071	0.019	$3.*10^{-6}$	$3.*10^{-6}$	$3.*10^{-6}$	$3.*10^{-6}$	$2.*10^{-6}$

Table 10: Estimates of the parameters θ and the corresponding MSE for non-informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=0.1	c3=1	c4=4
10	5	2.1839	1.8786	2.3361	1.8377	1.6101	1.615	2.1801	2.1802	2.1744	2.18	2.1589
		0.1731	0.5204	0.0696	0.5811	0.9799	0.9703	0.1769	0.1768	0.1817	0.177	0.1947
50	15	2.5713	2.4419	2.6948	2.417	2.2418	2.1366	2.579	2.579	2.5753	2.579	2.5496
		0.0008	0.025	0.009	0.0335	0.1283	0.2148	0.0005	0.0005	0.0008	0.0005	0.0025
100	25	2.5845	2.4923	2.66	2.4759	2.3512	2.2318	2.5869	2.5869	2.5833	2.5869	2.5654
		0.0002	0.0116	0.0036	0.0154	0.0619	0.1356	0.0002	0.0002	0.0003	0.0002	0.0012

Table 11: Estimates of the parameters λ and the corresponding MSE for non-informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=0.1	c3=1	c4=4
10	5	0.8688	0.8143	0.8549	0.8104	0.7778	0.7123	0.8694	0.8694	0.8692	0.8694	0.8689
		0.1	0.1	0.1	0.1	0.1038	0.1503	0.1	0.1	0.1	0.1	0.1
50	15	0.8147	0.7928	0.8108	0.791	0.7756	0.7351	0.8172	0.8172	0.8156	0.8172	0.815
		0.0814	0.0944	0.0837	0.0955	0.1025	0.1332	0.08	0.08	0.0809	0.08	0.0812
100	25	0.934	0.9181	0.9305	0.9169	0.9061	0.8759	0.9344	0.9344	0.9335	0.9344	0.9341
		0.0276	0.0331	0.0287	0.0335	0.0376	0.0502	0.0274	0.0274	0.0277	0.0274	0.0275

Table 12: Estimates of survival function R(t) and the corresponding MSE for non-informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=0.1	c3=1	c4=4
10	5	0.9113	0.8723	0.8877	0.8719	0.8752	0.8987	0.9109	0.9109	0.9116	0.9109	0.9116
		0.0085	0.0028	0.0047	0.0028	0.0032	0.0064	0.0085	0.0085	0.0086	0.0085	0.0086
50	15	0.8738	0.8699	0.8652	0.8704	0.8739	0.8769	0.8729	0.8729	0.8737	0.8729	0.8743
		0.003	0.0026	0.0021	0.0027	0.003	0.0034	0.0029	0.0029	0.003	0.0029	0.0031
100	25	0.8543	0.8535	0.8498	0.8538	0.8566	0.8593	0.8542	0.8542	0.8545	0.8542	0.8549
		0.0013	0.0012	0.001	0.0012	0.0014	0.0016	0.0012	0.0012	0.0013	0.0012	0.0013

Table 13: Estimates of hazard rate function H(t) and the corresponding MSE for non- informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=0.1	c3=1	c4=4
10	5	0.9147	1.0248	1.0717	1.0153	0.935	0.7984	0.9169	0.9169	0.9111	0.9169	0.9087
		0.5868	0.4303	0.3709	0.4428	0.5562	0.7785	0.5836	0.5836	0.5924	0.5836	0.596
50	15	1.1654	1.1404	1.2314	1.1315	1.0682	1.0074	1.1763	1.1763	1.1675	1.1763	1.1583
		0.2655	0.292	0.2019	0.3017	0.3751	0.4534	0.2545	0.2544	0.2634	0.2545	0.2729
100	25	1.3563	1.3237	1.3931	1.317	1.2656	1.2019	1.3582	1.3582	1.3551	1.3582	1.349
		0.105	0.1275	0.0827	0.1323	0.1723	0.2293	0.1041	0.1041	0.1061	0.1041	0.1101

Table 14: Estimates of reversed hazard rate function Q(t) and the corresponding MSE for non-informative priors

n	m	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-1	c2=0.1	c3=1	c4=4		c1=-1	c2=0.1	c3=1	c4=4
10	5	9.3969	6.9969	8.4757	6.9092	6.5538	7.0832	9.3728	9.3728	9.3996	9.3727	9.3672
		3.2317	0.3628	0.7682	0.4761	1.0929	0.2663	3.1454	3.1456	3.242	3.1452	3.1258
50	15	8.0688	7.6277	7.9068	7.6014	7.4015	7.1793	8.0766	8.0766	8.0749	8.0766	8.0587
		0.2205	0.0008	0.0946	0.0005	0.0391	0.1764	0.2279	0.2279	0.2263	0.2279	0.2111
100	25	7.9554	7.7095	7.882	7.6095	7.5577	7.3374	7.9546	7.9546	7.9557	7.9546	7.9488
		0.1269	0.0001	0.0799	0.0001	0.0017	0.0686	0.1263	0.1263	0.1271	0.1263	0.1222

Table 15: The 95% ACI's and HPD credible intervals and the corresponding length of α, θ and λ for non -informative priors

n	m	parameter	ACI	HPD
10	5	α	6.1298	0.0059
		θ	16.8645	0.0476
		λ	5.3752	0.0041
50	15	α	4.1488	0.0028
		θ	16.2651	0.0464
		λ	4.9875	0.0037
100	25	α	2.3878	0.0012
		θ	9.8212	0.0168
		λ	2.8937	0.0016

Table 16: Progressively censored sample generated from data in [32].

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$x_{i:35:50}$	0.1	0.2	1	1	1	1	1	2	3	6	7	11	18	18	18	18	21	32
R_i	0	0	0	3	0	0	0	0	0	0	3	0	0	0	0	0	0	3
i	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	
$x_{i:35:50}$	36	45	47	50	55	60	63	63	67	67	75	79	82	84	84	85	86	
R_i	0	0	0	0	0	0	3	0	0	0	0	0	0	3	0	0	0	

Table 17: Estimates of the parameters α, θ , λ for different methods under the informative priors

Parameters	MLEs	BAYES				MCMC			
		SEL	LINEX			SEL	LINEX		
			c=-1	c=4	c=1		c=-1	c=4	c=1
α	0.6185	0.5951	0.5915	0.5821	0.5988	0.6184	0.6184	0.6184	0.6184
θ	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
λ	0.0086	0.009	0.0089	0.0089	0.009	0.0086	0.0086	0.0086	0.0086

Table 18: The 95% ACI and the HPD credible interval of the parameters α, θ , λ under the informative priors

	α	Lenght	θ	Lenght	λ	Lenght
ACI	(0.3903,0.8466)	0.4564	(0.,0.0007)	0.0008	(-0.0015,0.0186)	0.0201
HPD	(0.6184,0.6185)	0.0000358	(0.0004,0.0004)	0.0	(0.0086,0.0086)	$9.*10^{-8}$

Table 19: Estimates of the parameters α, θ , λ for different methods under the non- informative priors

Parameters	MLEs	BAYES				MCMC			
		SEL	LINEX			SEL	LINEX		
			c=-1	c=4	c=1		c=-1	c=4	c=1
α	0.6185	0.5949	0.5913	0.5819	0.5986	0.6185	0.6185	0.6185	0.6185
θ	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
λ	0.0086	0.009	0.0089	0.0089	0.0089	0.0086	0.0086	0.0086	0.0086

Table 20: The 95% ACI and the HPD credible interval of the parameters α, θ , λ under the non-informative priors

	α	Lenght	θ	Lenght	λ	Lenght
ACI	(0.3903,0.8466)	0.4564	(0.,0.0007)	0.0008	(-0.0015,0.0186)	0.0201
HPD	(0.6184,0.6185)	0.000031	(0.0004,0.0004)	0.00	(0.0086,0.0086)	$7.*10^{-8}$

Table 21: Estimates of the parameters α, θ , λ for different methods under the informative priors

Parameters	MLEs	BAYES			MCMC		
		SEL	LINEX		SEL	LINEX	
			c=2	c=-2		c=2	c=-2
α	0.9998	1.0016	1.0016	1.0016	0.9998	0.9998	0.9998
θ	1.3649	1.64	1.5965	1.6388	1.2371	1.2335	1.2407
λ	0.0109	0.0241	0.0248	0.02014	0.02562	0.02595	0.01875

Table 22: The 95% ACI and the HPD credible interval of the parameters α, θ , λ under the informative priors

	α	Lenght	θ	Lenght	λ	Lenght
ACI	(0.9996,1)	0.00042726	(-0.8403,2.5701)	3.4104	(-0.426, 0.404)	0.83
HPD	(0.9996,, 0.9999)	0.00003	(1.1868,1.372)	0.1852	(-0.0629,0.1237)	0.1866

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