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Properties of the Caputo-Fabrizio Fractional Derivative

G. A. Mboro Nchama [1,2,*], *L. D. Lau-Alfonso* [2], *A. M. León Mecías* [2] *and M. Rodríguez Ricard* [2]

[1]Departamento de Ciencias Técnicas, Universidad Nacional de Guinea Ecuatorial (UNGE), Malabo, Guinea Ecuatorial
[2]Departamento de Matemática Aplicada, Universidad de La Habana, La Habana, Cuba

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Abstract: In this paper, we investigate some properties related Caputo-Fabrizio (CF) fractional derivative. We prove some regularity properties and bounds characterizing the Caputo-Fabrizio derivative operator. Using the method of Laplace transform, we found explicit solutions of some differential equations containing the Caputo-Fabrizio fractional derivative. Different types of inequalities generated by using the Caputo-Fabrizio derivative are also presented.

Keywords: Caputo-Fabrizio Fractional derivative, Fractional integral inequalities, Functions with the same sense of variation.

1 Introduction

A fractional derivative D^α is an operator, which generalizes the ordinary derivative. The origins of the fractional derivatives date back to 1695 when L'Hopital raised, by a letter to Leibniz, the question of how the expression

$$D^n f(t) = \frac{d^n}{dt^n} f(t),$$

should be understood if n was a fraction [1]. This question is commonly accepted as the first occurrence of what is currently known as fractional calculus. Since then, this branch has been addressed by eminent mathematicians, such as Euler, Laplace, Fourier, Liouville, Riemann, Laurent, Weyl, and Abel, who first applied it in physics to solve the integral equation arising from the tautochrone problem [2]. Fractional derivative has various definitions, which do not generally coincide. This is possible since different researchers attempt to preserve different properties of the classical integer order derivative. Fractional operators are used in various fields of science and engineering to describe some natural phenomena in [3-9], to enhance the contrast in an image [10-13], as well as to prove the existence and uniqueness of fractional differential equations in [14-19].

In 2015, Caputo and Fabrizio introduced a new fractional approach [20]. The interest in this definition existed due to the necessity to describe a class of non-local systems which cannot be well described by classical local theories or by fractional models with singular kernel [20]. In this paper, we present some interesting properties of Caputo-Fabrizio derivative, as follows: In Section 2, we briefly review the basic concepts and definitions. In Section 3, we obtain some regularity properties. Section 4 presents the solutions of some ordinary fractional differential equations with the Caputo-Fabrizio derivative. In Section 5, we deduce new integral inequalities. Section 6 is dedicated to conclusion.

2 Preliminaries

Here, we present basic definitions and theorems, which are used in our subsequent discussion.

Definition 1.*We postulate that two functions f and g have the same sense of variation (synchronous) on $[0,\infty)$ if [21]*

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho)) \geq 0 \quad \tau,\rho \in (0,t), t>0.$$

For example, one can easily see that functions $f(t)=t$ and $g(t)=t^2$ are synchronous on $[0,\infty)$, i. e.

$$\begin{aligned}&(\tau-\rho)(\tau^2-\rho^2)\\ &=(\tau-\rho)(\tau-\rho)(\tau+\rho)=(\tau-\rho)^2(\tau+\rho)\geq 0.\end{aligned}$$

Definition 2.*Let $f(t) \in L^1_{loc}(\mathbb{R})$. The Laplace transform of $f(t)$ is defined by [22]*

$$\mathfrak{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)dt.$$

* Corresponding author e-mail: becquerrr10@hotmail.com

Definition 3. *The Sobolev space $W^{1,p}(a,b)$ is defined by [23]*

$$W^{1,p}(a,b) = \Big\{ f \in L^p(a,b) \; ; \exists g \in L^p(a,b) \text{ such that } \int_a^b f\varphi' = -\int_a^b g\varphi, \quad \forall \varphi \in C_0^\infty(a,b) \Big\}.$$

We set

$$H^1(a,b) = W^{1,2}(a,b).$$

Note 1. Let $I = [a,b]$, then $C^1(I) \subset W^{1,p}(I)$ for all $1 \le p \le \infty$.

Definition 4. *Let $a,b,\alpha \in \mathbb{R} (a < b)$, $\alpha \in (0,1)$, $f \in H^1(a,b)$. The Caputo-Fabrizio fractional integral of order α is defined by [24]*

$$I_{at}^{\alpha} f(t) = (1-\alpha) f(t) + \alpha \int_a^t f(\tau) d\tau.$$

Definition 5. *Let $a,b,\alpha \in \mathbb{R} (a < b)$, $\alpha \in (0,1)$, $f \in H^1(a,b)$. The Caputo-Fabrizio fractional derivative of order α respect to time variable is defined by [25]*

$$\begin{aligned} D_{at}^{\alpha} f(t) &= \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f'(\tau) d\tau \\ &= \frac{1}{1-\alpha}\Big(f(t) - e^{-\frac{\alpha}{1-\alpha}t} f(a) \Big) \\ &\quad - \frac{\alpha}{(1-\alpha)^2} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau) d\tau. \end{aligned} \tag{1}$$

Definition 6. *Let $[a,b] \subset \mathbb{R}$. The space of continuous functions from the subset $[a,b]$ to $\mathbb{R}$ is defined by*

$$C([a,b],\mathbb{R}) = \{ f : [a,b] \to \mathbb{R} | \; f \text{ is continuous} \},$$

with the norm

$$\| f(x) \|_{C([a,b])} = \max_{x \in [a,b]} |f(x)|. \tag{2}$$

Definition 7. *Let $a,b \in \mathbb{R} (a < b)$. We denote by $C^1([a,b])$ the space of real-valued functions $f(x)$ whose derivative f' is continuous, with the norm*

$$\| f(x) \|_{C^1([a,b])} = \max_{x \in [a,b]} |f(x)| + \max_{x \in [a,b]} |f'(x)|. \tag{3}$$

Theorem 1. *[26] Let $n \in \mathbb{N} - \{0\}$, $a,b \in \mathbb{R} (a < b)$ and $f \in C^n([a,b])$. Then, the equality*

$$\frac{d^n}{dt^n}(D_{at}^{\alpha} f(t)) = \sum_{i=1}^{n} (-1)^{n-i} \frac{\alpha^{n-i}}{(1-\alpha)^{n+1-i}} f^{(i)}(t) + (-1)^n \Big(\frac{\alpha}{1-\alpha} \Big)^n D_{at}^{\alpha} f(t),$$

holds true.

3 Some regularity properties of the Caputo-Fabrizio derivative operator

In this section, we introduce some theorems that characterize the Caputo-Fabrizio fractional derivative in certain spaces, such as C (space of all continuous functions) or C^1 (space of functions which first derivative is continuous),... Although this class of spaces are considered very restricted, their importance for practical applications is great because the character of the majority of dynamic processes is smooth and has no discontinuities.

Theorem 2. *Let $f \in C^1[a,b]$. Then $D_{at}^{\alpha} f(t) \in C^1[a,b]$.*

Proof. As the function

$$y_\tau(t) = \frac{1}{1-\alpha} \cdot e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \cdot f'(\tau),$$

is continuous and integrable for all $t, \tau \in [a,b]$, we conclude that the function

$$F(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \cdot f'(\tau) d\tau,$$

is differentiable in $[a,b]$. This means that $D_{at}^{\alpha} f(t) \in C^1[a,b]$.

Theorem 3. *The operator $D_{at}^{\alpha} : C^1[a,b] \to C^1[a,b]$ is bounded and*

$$\| D_{at}^{\alpha} f \|_{C^1[a,b]} \le \frac{1}{\alpha} \Big(1 - e^{-\frac{\alpha}{1-\alpha}(b-a)} \Big) \| f \|_{C^1[a,b]}. \tag{4}$$

Proof. Considering the norm defined in (3), we obtain

$$\begin{aligned} & \| D_{at}^{\alpha} f(t) \|_{C^1[a,b]} \\ &= \Big\| \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} f'(\xi) d\xi \Big\|_{C^1[a,b]} \\ &\le \Big\| \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} |f'(\xi)| d\xi \Big\|_{C^1[a,b]} \\ &\le \Big\| \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} \big(|f(\xi)| + |f'(\xi)| \big) d\xi \Big\|_{C^1[a,b]} \\ &\le \frac{1}{1-\alpha} \| f \|_{C^1[a,b]} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} d\xi \\ &\le \frac{1}{\alpha} \| f \|_{C^1[a,b]} \Big(1 - e^{-\frac{\alpha}{1-\alpha}(t-a)} \Big) \\ &\le \frac{1}{\alpha} \| f \|_{C^1[a,b]} \Big(1 - e^{-\frac{\alpha}{1-\alpha}(b-a)} \Big). \end{aligned} \tag{5}$$

The inequality (4) follows from (5).

Lemma 1. *Let $f(t) \in H^1(a,b)$. Then $D_{at}^{\alpha} f(t) \in L^2(a,b)$.*

Proof. One can easily see that

$$\begin{aligned}\|D^{\alpha}_{at}f(t)\|^2_{L^2(a,b)} &= \frac{1}{1-\alpha}\int_a^b \left|\int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f'(\tau)d\tau\right|^2 dt \\ &\le \frac{1}{1-\alpha}\int_a^b \left(\int_a^t |f'(\tau)|^2 d\tau\right) dt \\ &\le \frac{1}{1-\alpha}\int_a^b \left(\int_a^b |f'(\tau)|^2 d\tau\right) dt \\ &= \frac{b-a}{1-\alpha}\cdot \|f'\|^2_{L^2(a,b)} < \infty,\end{aligned}$$

as required.

Theorem 4. *Let $f \in C^1[a,b]$. Then $D^{\alpha}_{at} f \in W^{1,p}[a,b], 1 \le p \le \infty$.*

Proof. As $f \in C^1[a,b]$, we obtain from Theorem 2 that $D^{\alpha}_{at} f \in C^1[a,b]$. Considering the Note 1, we know that $C^1[a,b] \subset W^{1,p}[a,b], \forall p \ge 1$. Therefore, $D^{\alpha}_{at} f \in W^{1,p}[a,b]$.

Theorem 5. *The Caputo-Fabrizio operator $D^{\alpha}_{at} : C^1[a,b] \to W^{1,1}[a,b]$ satisfies*

$$\|D^{\alpha}_{at} f\|_{W^{1,1}[a,b]} \le \frac{1+(b-a)[1+\alpha(b-a)]}{(1-\alpha)^2}\|f\|_{C^1[a,b]}.$$

Proof. Using the Theorem 1 and (1), we obtain

$$\begin{aligned}&\|D^{\alpha}_{at} f\|_{W^{1,1}[a,b]} \\ &= \|D^{\alpha}_{at} f\|_{L^1[a,b]} + \|\frac{d}{dt}(D^{\alpha}_{at} f)\|_{L^1[a,b]} \\ &= \|D^{\alpha}_{at} f\|_{L^1[a,b]} + \left\|\frac{1}{1-\alpha}f'(t) - \frac{\alpha}{1-\alpha}D^{\alpha}_{at} f\right\|_{L^1[a,b]} \\ &\le \frac{1}{1-\alpha}\|f'(t)\|_{L^1[a,b]} + \frac{1}{1-\alpha}\|D^{\alpha}_{at} f\|_{L^1[a,b]} \\ &= \frac{1}{1-\alpha}\|f'(t)\|_{L^1[a,b]} + \frac{1}{(1-\alpha)^2}\Big\| f(t) \\ &\quad - e^{-\frac{\alpha}{1-\alpha}(t-a)} f(a) - \frac{\alpha}{1-\alpha}\int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)d\tau\Big\|_{L^1[a,b]} \\ &\le \frac{1}{1-\alpha}\|f'(t)\|_{L^1[a,b]} + \frac{1}{(1-\alpha)^2}\|f(t)\|_{L^1[a,b]} \\ &\quad + \frac{1}{(1-\alpha)^2}\|e^{-\frac{\alpha}{1-\alpha}(t-a)} f(a)\|_{L^1[a,b]} \\ &\quad + \frac{\alpha}{(1-\alpha)^3}\int_a^b \left|\int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)d\tau\right| dt. \qquad (6)\end{aligned}$$

Considering the fact that

$$\begin{aligned}&\|e^{-\frac{\alpha}{1-\alpha}(t-a)} f(a)\|_{L^1[a,b]} \\ &= \int_a^b |e^{-\frac{\alpha}{1-\alpha}(t-a)} f(a)|dt \\ &\le \int_a^b |f(a)|dt \le \int_a^b \max_{t\in[a,b]}|f(t)|dt = (b-a)\max_{t\in[a,b]}|f(t)|,\end{aligned}$$

$$\begin{aligned}&\int_a^b \left|\int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau)d\tau\right| dt \\ &\le \int_a^b \left(\int_a^t |f(\tau)|d\tau\right) dt \\ &\le \int_a^b \left(\int_a^b \max_{\tau\in[a,b]}|f(\tau)|d\tau\right) dt = (b-a)^2 \max_{t\in[a,b]}|f(t)|,\end{aligned}$$

$$\begin{aligned}&\|f'(t)\|_{L^1[a,b]} \\ &= \int_a^b |f'(t)|dt \le \int_a^b \max_{t\in[a,b]}|f'(t)|dt = (b-a)\max_{t\in[a,b]}|f'(t)|,\end{aligned}$$

and

$$\begin{aligned}&\|f(t)\|_{L^1[a,b]} \\ &= \int_a^b |f(t)|dt \le \int_a^b \max_{t\in[a,b]}|f(t)|dt = (b-a)\max_{t\in[a,b]}|f(t)|,\end{aligned}$$

we obtain from (6) the following

$$\begin{aligned}&\|D^{\alpha}_{at} f\|_{W^{1,1}[a,b]} \\ &\le \frac{b-a}{1-\alpha}\max_{t\in[a,b]}|f'(t)| + \frac{b-a}{(1-\alpha)^2}\max_{t\in[a,b]}|f(t)| \\ &\quad + \frac{b-a}{(1-\alpha)^2}\max_{t\in[a,b]}|f(t)| + \frac{\alpha}{(1-\alpha)^3}(b-a)^2\max_{t\in[a,b]}|f(t)| \\ &= \frac{1}{1-\alpha}(b-a)\max_{t\in[a,b]}|f'(t)| \\ &\quad + \left\{\frac{(b-a)}{(1-\alpha)^2} + \frac{1}{(1-\alpha)^2} + \frac{\alpha(b-a)^2}{(1-\alpha)^3}\right\}\max_{t\in[a,b]}|f(t)| \\ &\le \left\{\frac{(b-a)}{(1-\alpha)^2} + \frac{1}{(1-\alpha)^2} + \frac{\alpha(b-a)^2}{(1-\alpha)^3}\right\}\max_{t\in[a,b]}|f'(t)| \\ &\quad + \left\{\frac{(b-a)}{(1-\alpha)^2} + \frac{1}{(1-\alpha)^2} + \frac{\alpha(b-a)^2}{(1-\alpha)^3}\right\}\max_{t\in[a,b]}|f(t)| \\ &= \left\{\frac{(b-a)}{(1-\alpha)^2} + \frac{1}{(1-\alpha)^2} + \frac{\alpha(b-a)^2}{(1-\alpha)^3}\right\}\cdot \\ &\quad \cdot\left[\max_{t\in[a,b]}|f'(t)| + \max_{t\in[a,b]}|f(t)|\right] \\ &= \frac{(b-a+1)(1-\alpha)+\alpha(b-a)^2}{(1-\alpha)^3}\|f(t)\|_{C^1[a,b]},\end{aligned}$$

as required.

Theorem 6. *The subspace $C^1[a,b] \subset H^1$ is invariant with respect to the Caputo-Fabrizio operator D^{α}_{at}.*

Proof. We want to show that for all $f \in C^1[a,b]$, then $D^{\alpha}_{at} f \in C^1[a,b]$. From the Note 1, we know that $C^1[a,b] \subset H^1$. Let $f \in C^1[a,b]$. Then using Theorem 2 we conclude that $D^{\alpha}_{at} f \in C^1[a,b]$.

Theorem 7. *Let $\alpha \in (0,1)$, $a,b \in \mathbb{R}(a<b)$, $\Omega = [a,b]$ and $D(\Omega)$ the space of test functions. The operator $T^{\alpha} : D(\Omega) \to \mathbb{R}$ given by*

$$T^{\alpha}(u) = \frac{1}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} u'(\tau)d\tau,$$

is a distribution.

Proof. We have to prove that operator T^{α} is linear and continuous. For any $u,v \in D(\Omega)$, we obtain the linearity, as follows:

$$\begin{aligned} &T^{\alpha}(\beta u + \gamma v) \\ &= \frac{1}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} (\beta u + \gamma v)'(\tau)d\tau \\ &= \frac{\beta}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} u'(\tau)d\tau \\ &\quad + \frac{\gamma}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} v'(\tau)d\tau \\ &= \beta T^{\alpha}(u) + \gamma T^{\alpha}(v). \end{aligned}$$

Suppose that u is any element of the space of test functions $D(\Omega)$. Then we know that $u \in W_0^{1,p}(\Omega)(1 \le p < \infty)$. Thus, $u(a) = u(b) = 0$ and

$$\begin{aligned} &T^{\alpha}(u) \\ &= \frac{1}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} u'(\tau)d\tau \\ &= \frac{1}{1-\alpha}\Big[u(b) - e^{-\frac{\alpha}{1-\alpha}(b-a)}u(a) \\ &\quad - \frac{\alpha}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} u(\tau)d\tau\Big] \\ &= -\frac{\alpha}{(1-\alpha)^2}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} u(\tau)d\tau. \end{aligned} \tag{7}$$

Now, to show the continuity of T^{α}, let take $u, u_n \in D(\Omega)$ such that $u_n \to u$. Using (7), we obtain

$$\begin{aligned} &|T^{\alpha}(u) - T^{\alpha}(u_n)| \\ &= \frac{\alpha}{(1-\alpha)^2}\left|\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} [u(\tau) - u_n(\tau)]d\tau\right| \\ &\le \frac{\alpha}{(1-\alpha)^2} \sup_{\tau \in [a,b]} |u(\tau) - u_n(\tau)| \int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)}d\tau, \end{aligned}$$

which tends to zero by the uniform convergence of the u_n and grace to that $e^{-\frac{\alpha}{1-\alpha}(b-\tau)} \in L^1(\Omega)$. Thus, the opertor T^{α} is a distribution.

4 Ordinary fractional differential equations

Fractional differential equations frequently appear in various areas of engineering applications. Some examples of differential equations containing fractional derivative have been explored [27]. In [28], authors considered the following linear fractional differential equation

$$D^{\beta}_{at} x(t) - \lambda \cdot x(t) = f(t),$$

where $\beta = \alpha + 1$ such that $0 < \alpha \le 1$, $a \in (-\infty, t)$. In this section, we present some examples of the solution of differential equations containing the Caputo-Fabrizio fractional derivative.

Example 1. Let us consider the equation

$$D^{\alpha}_{0t} x(t) + c_2 \cdot D^{\beta}_{0t} x(t) - c_3 \cdot x(t) = 0, \tag{8}$$
$$x(0) = x_0, \tag{9}$$

where $c_2, c_3, \alpha, \beta, x_0 \in \mathbb{R}$ with $0 < \alpha, \beta < 1$.

Applying the definicion of fractional Caputo-Fabrizio derivative, we rewrite equation (8) as

$$\begin{aligned} &\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} x'(\tau)d\tau \\ &+ \frac{c_2}{1-\beta}\int_0^t e^{-\frac{\beta}{1-\beta}(t-\tau)} x'(\tau)d\tau = c_3 x(t). \end{aligned} \tag{10}$$

Applying the Laplace transform to (10) and taking into account the condition (9), we get

$$\mathfrak{L}\{x(t)\} = \frac{p(s)}{q(s)}, \tag{11}$$

where

$$\begin{aligned} p(s) &= [x_0(1-\beta) + c_2 x_0(1-\alpha)]s + x_0\beta + c_2 x_0 \alpha, \\ q(s) &= [(1-\beta) + c_2(1-\alpha) - c_3(1-\alpha)(1-\beta)]s^2 \\ &\quad + [\beta + c_2\alpha - c_3(1-\alpha)\beta - c_3\alpha(1-\beta)]s - c_3\alpha\beta. \end{aligned}$$

The function $q(s)$ has two roots

$$\gamma_1 = \frac{-\Delta_0 + \sqrt{\Delta}}{2\Delta_1}, \qquad \gamma_2 = \frac{-\Delta_0 - \sqrt{\Delta}}{2\Delta_1},$$

with

$$\begin{aligned} \Delta_0 &= \beta + c_2\alpha - c_3(1-\alpha)\beta - c_3\alpha(1-\beta), \\ \Delta_1 &= (1-\beta) + c_2(1-\alpha) - c_3(1-\alpha)(1-\beta), \\ \Delta &= \Delta_0^2 + 4c_3 \cdot \alpha \cdot \beta\Delta_1. \end{aligned}$$

Case 1. When $\Delta > 0$: we have from (11) that

$$\begin{aligned} \mathfrak{L}\{x(t)\} &= \frac{[x_0(1-\beta) + c_2 x_0(1-\alpha)]s + x_0\beta + c_2 x_0\alpha}{(s-\gamma_1)(s-\gamma_2)} \\ &= \frac{A}{s-\gamma_1} + \frac{B}{s-\gamma_2} = \frac{A(s-\gamma_2) + B(s-\gamma_1)}{(s-\gamma_1)(s-\gamma_2)}. \end{aligned} \tag{12}$$

Using elementary calculations, we obtain

$$B = \frac{[c_1x_0(1-\beta)+c_2x_0(1-\alpha)]\gamma_2 + c_1x_0\beta + c_2x_0\alpha}{\gamma_2-\gamma_1},$$
$$A = \frac{[c_1x_0(1-\beta)+c_2x_0(1-\alpha)]\gamma_1 + c_1x_0\beta + c_2x_0\alpha}{\gamma_1-\gamma_2}.$$

Applying inverse Laplace transform to (12), one easily obtain

$$x(t) = A\cdot e^{\gamma_1 t} + B\cdot e^{\gamma_2 t}. \tag{13}$$

Caso 2. $\Delta < 0$: we obtain from (11) that

$$\begin{aligned}
&\mathfrak{L}\{x(t)\}\\
&= \frac{p(s)}{q(s)}\\
&= \frac{1}{\Delta_1}\cdot\frac{[x_0(1-\beta)+c_2x_0(1-\alpha)]s - [-x_0\beta - c_2x_0\alpha]}{\left(s+\frac{\Delta_0}{2\Delta_1}\right)^2 + \left[\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta - \frac{\Delta_0^2}{4\Delta_1}\right)}\right]^2}\\
&= \frac{x_0(1-\beta)+c_2x_0(1-\alpha)}{\Delta_1}\cdot\\
&\quad\cdot\frac{s+\frac{\Delta_0}{2\Delta_1}}{\left(s+\frac{\Delta_0}{2\Delta_1}\right)^2 + \left[\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta - \frac{\Delta_0^2}{4\Delta_1}\right)}\right]^2}\\
&\quad - \frac{x_0(1-\beta)+c_2x_0(1-\alpha)}{\Delta_1}\cdot\\
&\quad\cdot\frac{\frac{-x_0\beta - c_2x_0\alpha}{x_0(1-\beta)+c_2x_0(1-\alpha)} + \frac{\Delta_0}{2\Delta_1}}{\left(s+\frac{\Delta_0}{2\Delta_1}\right)^2 + \left[\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta - \frac{\Delta_0^2}{4\Delta_1}\right)}\right]^2}.
\end{aligned} \tag{14}$$

Using inverse Laplace transform, we obtain

$$\begin{aligned}
&x(t)\\
&= \frac{c_1x_0(1-\beta)+c_2x_0(1-\alpha)}{\Delta_1}\cdot e^{-\frac{\Delta_0}{2\Delta_1}t}\cdot\\
&\cdot\cos\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta - \frac{\Delta_0^2}{4\Delta_1}\right)}\cdot t -\\
&- \frac{c_1x_0(1-\beta)+c_2x_0(1-\alpha)}{\Delta_1}\cdot\\
&\cdot\frac{\frac{-c_1x_0\beta - c_2x_0\alpha}{c_1x_0(1-\beta)+c_2x_0(1-\alpha)} + \frac{\Delta_0}{2\Delta_1}}{\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta - \frac{\Delta_0^2}{4\Delta_1}\right)}}\cdot e^{-\frac{\Delta_0}{2\Delta_1}t}\cdot\\
&\cdot\sin\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta - \frac{\Delta_0^2}{4\Delta_1}\right)}\cdot t
\end{aligned} \tag{15}$$

Case 3. $\Delta = 0$: we obtain from (11), the following

$$\begin{aligned}
\mathfrak{L}\{x(t)\} &= \frac{[c_1x_0(1-\beta)+c_2x_0(1-\alpha)]s + c_1x_0\beta + c_2x_0\alpha}{\left(s+\frac{\Delta_0}{2\Delta_1}\right)^2}\\
&= \frac{A_2}{s+\frac{\Delta_0}{2\Delta_1}} + \frac{B_2}{\left(s+\frac{\Delta_0}{2\Delta_1}\right)^2}.
\end{aligned}$$

where A and B are constants given by

$$B_2 = -[c_1x_0(1-\beta)+c_2x_0(1-\alpha)]\cdot\frac{\Delta_0}{2\Delta_1} + c_1x_0\beta + c_2x_0\alpha,$$

and

$$A_2 = \frac{c_1x_0\beta + c_2x_0\alpha - B_2}{\Delta_0}\cdot 2\cdot\Delta_1,$$

respectively. Applying the inverse Laplace transform, we obtain

$$x(t) = A_2\cdot e^{\frac{\Delta_0}{2\Delta_1}t} + B_2\cdot t\cdot e^{\frac{\Delta_0}{2\Delta_1}t}. \tag{16}$$

This result can be formulated in the following Theorem.

Theorem 8. *Let $0 < \alpha, \beta < 1; c_2, c_3, x_0 \in \mathbb{R}$.*

1) *If $\Delta > 0$, then problem (8)-(9) has a unique solution, which is given by (13).*
2) *If $\Delta < 0$, then a unique solution to problem (8)-(9) exists and is given by (15).*
3) *If $\Delta = 0$, then problem (8)-(9) has a unique solution, which is given by (16).*

Exemple 2. Consider the equation of linear vibrations with the fractional dissipation term of order $\alpha \in (0,1)$.

$$f''(t) + c_1\cdot D_{0t}^{\alpha}f(t) + c_2\cdot f(t) = 0 \quad t>0, \tag{17}$$
$$f'(0) = f_1; f(0) = f_0, \tag{18}$$

where $c_1, c_2, \alpha, f_1, f_0 \in \mathbb{R}$. Inserting the definition of Caputo-Fabrizio derivative into (17), we obtain the equation

$$\begin{aligned}
&f'''(t) + \frac{\alpha}{1-\alpha}f''(t)\\
&+\frac{(1-\alpha)c_2 + c_1}{1-\alpha}f'(t) + \frac{\alpha\cdot c_2}{1-\alpha}f(t) = 0.
\end{aligned} \tag{19}$$

The corresponding characteristic equation of (19) is

$$r^3 + \frac{\alpha}{1-\alpha}r^2 + \frac{(1-\alpha)c_2 + c_1}{1-\alpha}r + \frac{\alpha\cdot c_2}{1-\alpha} = 0,$$

which, by using the Cardano method, roots are

$$\begin{cases}
r_1 = S_1 + S_2 - \frac{a_1}{3}\\
r_2 = -\frac{S_1+S_2}{2} - \frac{a_1}{3} + \frac{i\sqrt{3}}{2}(S_1 - S_2)\\
r_3 = -\frac{S_1+S_2}{2} - \frac{a_1}{3} - \frac{i\sqrt{3}}{2}(S_1 - S_2),
\end{cases}$$

where

$$a_1=\frac{\alpha}{1-\alpha},\quad a_2=\frac{(1-\alpha)c_2+c_1}{1-\alpha},\quad a_3=\frac{\alpha\cdot c_2}{1-\alpha}.$$

$$Q=\frac{3a_2-a_1^2}{9},\quad S_1=\sqrt[3]{R+\sqrt{Q^3+R^2}}.$$

$$R=\frac{9a_1a_2-27a_3-2a_1^3}{54},\quad S_2=\sqrt[3]{R-\sqrt{Q^3+R^2}}.$$

Taking into account the conditions (18), we finally obtain

$$f(t)=\frac{r_2(1-f_0)-f_1+r_3}{r_1-r_2}e^{r_1t}+\frac{r_1(f_0-1)-(f_1-r_3)}{r_1-r_2}e^{r_2t}+e^{r_3t}.$$

5 Fractional integral inequalities involving the Caputo-Fabrizio fractional derivative

In literature few results have been obtained on fractional integral inequalities using Caputo-Fabrizio fractional operators [29]. Motivated by [30], we propose using this operator to establish some new integral inequalities. We write

$$\begin{aligned}D^{\alpha}_{0t}f(t)&=\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}f'(s)ds,\\&=\frac{1}{1-\alpha}\left(f(t)-e^{-\frac{\alpha}{1-\alpha}t}f(0)\right)\\&-\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}f(s)ds.\end{aligned}\tag{20}$$

Let

$$G(f)(t)=\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}f(s)ds.\tag{21}$$

Theorem 9. *Let $\alpha\in(0,1)$, f and g are two functions that have the same sense of variation on $[0,\infty)$. Then*

$$G(f)\cdot G(g)\le\frac{1}{1-\alpha}\cdot\left(1-e^{-\frac{\alpha}{1-\alpha}t}\right)\cdot G(fg),\tag{22}$$

where G is given by (21).

Proof. Suppose that f and g are functions having the same sense of variation on $[0,\infty)$. Then, for all $\tau\ge0,\rho\ge0$, we have

$$\big(f(\tau)-f(\rho)\big)\big(g(\tau)-g(\rho)\big)\ge0.$$

Hence,

$$f(\tau)g(\tau)+f(\rho)g(\rho)\ge f(\tau)g(\rho)+f(\rho)g(\tau).\tag{23}$$

Multiplying both sides of (23) by $\frac{\alpha^2}{(1-\alpha)^4}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}$ and integrating with respect to τ and ρ over $(0,t)\times(0,t)$, we obtain the inequality

$$\begin{aligned}&\frac{\alpha}{(1-\alpha)^2}\int_0^t\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\cdot f(\tau)g(\tau)d\tau\right\}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}d\rho\\&+\frac{\alpha}{(1-\alpha)^2}\int_0^t\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}\cdot f(\rho)g(\rho)d\rho\right\}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau\\&\ge\frac{\alpha}{(1-\alpha)^2}\int_0^t\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\cdot f(\tau)d\tau\right\}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}g(\rho)d\rho\\&+\frac{\alpha}{(1-\alpha)^2}\int_0^t\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}\cdot f(\rho)\right\}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}g(\tau)d\tau,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&G(fg)\cdot\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}d\rho\\&+G(fg)\cdot\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau\\&\ge G(f)\cdot G(g)+G(g)\cdot G(f).\end{aligned}\tag{24}$$

Inequality (22) follows from (24).

Theorem 10. *Let $\alpha\in\mathbb{R}$ with $0<\alpha<1$ and f, g two non negative functions having the same sense of variation on $[0,\infty)$ such that*

1) $f(0)=g(0)=0$,
2) $fg\ge D^{\alpha}_{0t}(fg)$.

Then

$$\begin{aligned}&D^{\alpha}_{0t}(fg)\\&\le f(t)\cdot D^{\alpha}_{0t}(g)+g(t)\cdot D^{\alpha}_{0t}(f)-(1-\alpha)D^{\alpha}_{0t}(f)\cdot D^{\alpha}_{0t}(g).\end{aligned}\tag{25}$$

Proof. From Theorem 9, we get

$$\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}(f)(s)ds\right\}\cdot \cdot\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}(g)(s)ds\right\} \leq \left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}(fg)(s)ds\right\}\cdot \cdot\frac{1}{1-\alpha}\cdot\left(1-e^{-\frac{\alpha}{1-\alpha}t}\right). \tag{26}$$

Combining the expression (20) with the conditions $f(0) = g(0) = 0$, we obtain from (26) that

$$\left\{\frac{1}{1-\alpha}f(t)-D^{\alpha}_{0t}(f)(t)\right\}\left\{\frac{1}{1-\alpha}g(t)-D^{\alpha}_{0t}(g)(t)\right\} \leq\left\{\frac{1}{1-\alpha}fg(t)-D^{\alpha}_{0t}(fg)(t)\right\}\frac{1}{1-\alpha}\cdot\left(1-e^{-\frac{\alpha}{1-\alpha}t}\right). \tag{27}$$

Using the condition 2), one easily obtain

$$0\leq (fg)-D^{\alpha}_{0t}(fg)\leq\frac{1}{1-\alpha}(fg)-D^{\alpha}_{0t}(fg). \tag{28}$$

Combining (28) with (27), we achieve

$$\left\{\frac{1}{1-\alpha}f(t)-D^{\alpha}_{0t}(f)(t)\right\}\left\{\frac{1}{1-\alpha}g(t)-D^{\alpha}_{0t}(g)(t)\right\} \leq\frac{1}{1-\alpha}\left\{\frac{1}{1-\alpha}(fg)(t)-D^{\alpha}_{0t}(fg)(t)\right\}. \tag{29}$$

Inequality (25) follows from (29).

Theorem 11. *Let $\alpha\in(0,1)$, and f, g, f', g' (f' and g' are derivatives of functions f and g, respectively) non-negative functions on $[0,\infty)$. Moreover, let f and g have the same sense of variation on $[0,\infty)$ such that*

1) $f(0) = g(0) = 0$,
2) $fg\geq D^{\alpha}_{0t}(fg)$,

then

$$D^{\alpha}_{0t}(fg)\leq f(t)\cdot D^{\alpha}_{0t}(g)+g(t)\cdot D^{\alpha}_{0t}(f). \tag{30}$$

Proof. Combining the fact that functions f and g have the same sense of variation with the conditions 1) and 2), we obtain from theorem 10 the inequality

$$D^{\alpha}_{0t}(fg)+(1-\alpha)D^{\alpha}_{0t}(f)\cdot D^{\alpha}_{0t}(g) \leq f(t)\cdot D^{\alpha}_{0t}(g)+g(t)\cdot D^{\alpha}_{0t}(f). \tag{31}$$

On the other hand, considering that functions f, g, f', g' are non-negatives, we conclude that $D^{\alpha}_{0t}f\geq 0$, $D^{\alpha}_{0t}g\geq 0$, $D^{\alpha}_{0t}fg\geq 0$. Therefore all terms of inequality (31) are non-negatives. Then, deleting the second term of the left hand side of (31), we obtain (30).

Theorem 12. *Let $0<\alpha<1$, f' and g' are two functions that have the same sense of variation on $[0,\infty)$. Then,*

$$D^{\alpha}_{0t}f(t)\cdot D^{\alpha}_{0t}g(t)\leq\frac{1-\alpha}{\alpha^2}\cdot\left(1-e^{-\frac{\alpha}{1-\alpha}t}\right)\cdot G(f'g'). \tag{32}$$

Proof. As f', g' are two functions that have the same sense of variation on $[0,\infty)$, then for all $\tau\geq 0,\rho\geq 0$ we have

$$f'(\tau)g'(\tau)+f'(\rho)g'(\rho)\geq f'(\tau)g'(\rho)+f'(\rho)g'(\tau). \tag{33}$$

Multiplying (33) by $\frac{\alpha}{(1-\alpha)^3}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}$ and integrating with respect to τ and ρ over $(0,t)\times(0,t)$, we obtain the inequality

$$\frac{1}{1-\alpha}\int_0^t\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\cdot \cdot f'(\tau)g'(\tau)d\tau\right\}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}d\rho + \frac{1}{1-\alpha}\int_0^t\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}\cdot \cdot f'(\rho)g'(\rho)d\rho\right\}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau$$
$$\geq\frac{\alpha}{1-\alpha}\frac{1}{1-\alpha}\int_0^t\left\{\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\cdot \cdot f'(\tau)d\tau\right\}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}g'(\rho)d\rho + \frac{\alpha}{1-\alpha}\frac{1}{1-\alpha}\int_0^t\left\{\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}\cdot \cdot f'(\rho)d\rho\right\}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}g'(\tau)d\tau,$$

which is equivalent to

$$\frac{1}{1-\alpha}G(f'g')\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}d\rho\geq\frac{\alpha}{1-\alpha}D^{\alpha}_{at}f(t)D^{\alpha}_{at}g(t). \tag{34}$$

Inequality (32) follows from (34).

Theorem 13. *Let $0<\alpha<1$, f' and g are two functions that have the same sense of variation on $[0,\infty)$. Then,*

$$D^{\alpha}_{at}f(t)G(g)\leq\frac{1}{\alpha}\cdot\left(1-e^{-\frac{\alpha}{1-\alpha}(t-a)}\right)G(f'g). \tag{35}$$

Proof. Suppose that f' and g are functions that have the same sense of variation on $[0,\infty)$. Then, for all $\tau\geq 0,\rho\geq 0$, we have

$$f'(\tau)g(\tau)+f'(\rho)g(\rho)\geq f'(\tau)g(\rho)+f'(\rho)g(\tau). \tag{36}$$

Multiplying both sides of (36) by $\frac{\alpha}{(1-\alpha)^3}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}$ and integrating with respect to τ and ρ over $(0,t)\times(0,t)$, we obtain the inequality

$$\begin{aligned}
&\frac{1}{1-\alpha}\int_0^t \left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\cdot\right.\\
&\left.\cdot f'(\tau)g(\tau)d\tau\right\}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}d\rho\\
&+\frac{1}{1-\alpha}\int_0^t \left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}\cdot\right.\\
&\left.\cdot f'(\rho)g(\rho)d\rho\right\}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau\\
&\geq \frac{\alpha}{(1-\alpha)^2}\int_0^t \left\{\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\cdot\right.\\
&\left.\cdot f'(\tau)d\tau\right\}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}g(\rho)d\rho\\
&+\frac{\alpha}{(1-\alpha)^2}\int_0^t \left\{\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}\cdot\right.\\
&\left.\cdot f'(\rho)d\rho\right\}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}g(\tau)d\tau,
\end{aligned}$$

which is equivalent to

$$\frac{1}{1-\alpha}G(f'g)\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}d\rho \geq D^{\alpha}_{at}f(t)\cdot G(g). \quad (37)$$

The inequality (35) follows from (37).

6 Conclusion

In this paper, we have investigated some linear differential equations involving Caputo-Fabrizio fractional derivative. Also, some theorems to characterize the Caputo-Fabrizio derivative in certain spaces have been proven. Statements on fractional inequalities containing the Caputo-Fabrizio derivative were also presented.

Data Availability. No data were used to support this study.

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Gustavo Asumu Mboro Nchama received the B.S. and M.S. degrees from Kharkov University, Ukraine in 2011 and 2012, respectively. Now he is Pursuing Ph.D in Mathematics at Havana University, Cuba. He is Professor in Mathematics at Universidad Nacional de Guinea Ecuatorial(UNGE). His research interest is image processing with fractional partial differential equation.

Leandro Daniel Lau Alfonso Received bachelor degrees from Havana University in 2017 and he is working at Institute of Cybernetic, Mathematics and Physics (ICIMAF), Cuba. He is a researcher in applied mathematics and his research interests are signal and image processing and differential equations.

Ángela León Mecías received her Ph.D in Mathematics from the University of Havana, Cuba in 2007. She is a Full Professor at the Applied Mathematics Department, Faculty of Mathematics and Computer Science, University of Havana. Dr. León is the general coordinator of the Master Program in Mathematics of the University of Havana. Her research interests include: numerical solution of differential equations and applications, mathematical methods in image processing and wavelet transform applications.

Mariano Rodríguez Ricard, Full Professor and Consultant at University of Havana. Areas of interest: bifurcations in ODE, pattern formation in reaction diffusion PDE. Hi is member of the Academic Committee of the Doctor Degree Program in Mathematics.