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Mild Solutions of Time Fractional Navier-Stokes Equations Driven by Finite Delayed External Forces

*Md Mansur Alam and Shruti Dubey**

Department of Mathematics, IIT Madras, Chennai- 600 036, Tamilnadu, India

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Abstract: In this work, we consider time-fractional Navier-Stokes equations (NSE) with the external forces involving finite delay. Equations are considered on a bounded domain $\Omega \subset \mathbb{R}^3$ having sufficiently smooth boundary. We transform the system of equations (NSE) to an abstract Cauchy problem and then investigate local existence and uniqueness of the mild solutions for the initial datum $\phi \in C\big([-r,0]; D(A^{\frac{1}{2}})\big)$, where $r > 0$ and A is the Stokes operator. With some suitable condition on initial datum we establish the global continuation and regularity of the mild solutions. We use semigroup theory, tools of fractional calculus and Banach contraction mapping principle to establish our results.

Keywords: Fractional calculus, Navier-Stokes equations, delay differential equations, analytic semigroup, mild solutions, fractional power of operators.

1 Introduction

The Navier-Stokes equations (NSE) are the prime system of equations in the study of fluid dynamics which represent the motion of a viscous fluid passing through a region. One may consider the situation when the fluid passes through such a medium that the fluid motion behaves anomalously. To control such system one may consider the external forces having some hereditary features which depends not only on the present state of the system but also on the past history of the system. Therefore, from the last two decades, the study of NSE with force term consisting of such delay received lot of attention. For instance see [1–5] and references therein. On the other hand, the study of time fractional functional differential equations has gained a huge attention from the researchers, not only due to its novel applications in the field of science and engineering study but also due to the non-local nature of fractional derivatives [6–8]. In particular, generalized model of a diffusion phenomena in a porous media behaves much better than the classical model of that diffusion phenomena. So, it is significant to consider time fractional NSE with delay model which reads as follows:

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$.

$$\begin{cases} {}^c\partial_t^\alpha u - \Delta u + (u\cdot\nabla)u = -\nabla p + f(t,u_t),\ t>0,\ x\in\Omega \\ \nabla\cdot u = 0,\ t>0,\ x\in\Omega \\ u|_{\partial\Omega} = 0,\ t>0, \\ u(x,t) = \phi(x,t), \quad -r\le t\le 0, \quad x\in\Omega, \end{cases} \tag{1}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ represents the velocity of the fluid, $p = p(x,t)$ is the associated pressure, $[-r,0]$ is the finite delayed interval, $u_t(\theta) = u(t+\theta)$, $-r \le \theta \le 0$, f is an external force which is given in terms of the past history of the velocity, ϕ is the initial datum corresponding to delayed interval and ${}^c\partial_t^\alpha u$ is the Caputo fractional order derivative of order $\alpha \in (0,1)$ with respect to t.

J. Leray [9] was the first who has initially contributed to the mathematical study of NSE. After that Kato-Fujita [10,11] has proved the existence, uniqueness and regularity of the mild solutions in space-time variable of the classical NSE by

* Corresponding author e-mail: sdubey@iitm.ac.in

transforming the system into an abstract initial value problem and using semigroup theory. From last few decades, there has been lot of work on the study of classical NSE, for instance, see [12–15] and references therein. Caraballo et al. [1] was the first who considered integer order NSE with finite delay over a bounded domain and proved the existence of weak solution. For similar investigation of these problems on unbounded domain and unbounded delay, one may refer [2,3]. In contrast to this, M. El-Shahed et al. [16], was the first who considered time fractional Navier-Stokes equation and studied the analytical solutions by using Laplace, Fourier and Hankel transformation technique. After that, few more works have been reported on the study of analytical solutions of the similar problem in [17–19]. In 2015, Carvalho-Neto et al. [20] have studied about mild solutions to the time-fractional Navier-Stokes equations on $\mathbb{R}^N$. Yong Zhou et al. [21] have studied existence, uniqueness and regularity of mild solution for the time fractional NSE without delay on a half-space in $\mathbb{R}^n$, $n \geq 3$.

Recently, Yejuan Wang et al. [22] considered fractional Navier-Stokes inclusions and proved the global existence, regularity and decay of mild solution, when the initial velocity belongs to $C([-r,0];D(A^{\varepsilon}))$, where $0 < \varepsilon < \frac{1}{2}$. The results are obtained utilizing the techniques of measure of noncompactness in L^p-framework, where $2 \leq n < p < \infty$ (n is the dimension of consideration of the problem). In contrast to this, we present the analysis of fractional order Navier-Stokes inclusion with delay for the case $1 \leq p < n$ which is studied first time. Indeed, this work accomplishes the deficiency for a case $n = 3,\ p = 2$. In particular, we investigate the existence, uniqueness and regularity of mild solution for the fractional order NSE (1), driven by finite delayed forces in L^2. We obtain the existence result by using the Banach contraction mapping principle and present some examples for the delay functions to illustrate the results.

The paper is organized as follows. In section 2, we recall some definitions, preliminary results on estimation of analytic solution operators and the nonlinear term $Fu = -P(u \cdot \nabla)u$. In section 3, we present our main results concerning local existence of mild solution of the problem (1). To illustrate our existence result, we consider examples of some delayed force functions in this section. In section 4, we study about the maximality of interval of existence and blow up of the mild solution. Regularity of mild solution is given in section 5.

2 Preliminaries

This section recalls basic definitions, notations and preliminary results which will be used throughout the paper. We use the standard notations, $\mathbb{R}$, $\mathbb{N}$ for denoting the set of real numbers and natural numbers respectively. Let X be a Banach space with the norm $\|\cdot\|_X$. For two Banach spaces X and Y, $\mathscr{B}(X;Y)$ denotes the space of all bounded linear map from X to Y. For $X = Y$, we write $\mathscr{B}(X;X)$ as $\mathscr{B}(X)$. Let $1 \leq p < \infty$, then for any interval I in $\mathbb{R}$, $L^p(I;X)$ denotes the set of all X-valued measurable functions f on I such that $\int_I \|f(t)\|_X^p dt < \infty$, and it is a Banach space endowed with the norm $\|f\|_{I,p} = \left(\int_I \|f(t)\|_X^p\right)^{1/p}$.
For $k \in \mathbb{N}$, $W^{k,p}(I;X) = \{f \in L^p(I;X) : f \text{ has weak derivatives } f^{(j)} \text{ and } f^{(j)} \in L^p(I;X) \ \text{ for all } 1 \leq j \leq k, \text{ where } j \in \mathbb{N}\}$ is known as Sobolev spaces of order k. It is a Banach space with respect to the norm $\|f\|_{1,p} = \sum_{j=0}^{k} \|f^{(j)}\|_{I,p}$.
Let $\Omega \subset \mathbb{R}^3$ be any domain. For $k \in \mathbb{N}$,

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \text{ the weak derivatives } \partial^{\gamma} u \in L^p(\Omega),\ \forall \text{ multi-index } \gamma \text{ such that } |\gamma| \leq k\}$$

are standard real valued Sobolev spaces. For $p = 2$, $W^{k,p}(\Omega) = H^k(\Omega)$ are Hilbert spaces. Let $C_0^{\infty}(\Omega)$ be the set of all real valued infinitely differentiable function with compact support and $H_0^1(\Omega)$ be the closure of $C_0^{\infty}(\Omega)$ in $H^1(\Omega)$.
We denote $C^k(I;X)$ as the set of all X-valued continuously differentiable function upto order $k \in \mathbb{N}$ on I. $C^{\theta}(I;X)$ denotes the set of all X-valued Hölder continuous function with Hölder exponent $\theta \in (0,1)$.

Definition 1. *Let $0 < \alpha < 1$, $a, b \in \mathbb{R}$ and $f \in L^1([a,b];X)$. The Riemann Liouville integral of order α is defined by*

$$J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds \quad a.e\ t \in [a,b].$$

For $\alpha > 0$, we consider

$$g_{\alpha}(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Definition 2. *Let $0 < \alpha < 1$ and $f \in L^1([a,b];X)$ be such that $g_{1-\alpha} * f \in W^{1,1}([a,b];X)$. Then the Caputo fractional derivative of order α is defined by*

$${}^{c}D_t^{\alpha} f(t) = \frac{d}{dt} J_t^{1-\alpha}(f(t) - f(a)) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha}(f(s) - f(a)) ds \quad a.e\ t \in [a,b].$$

Note that if $f \in C^1([a,b];X)$, then ${}^cD_t^{\alpha} f(t) = J_t^{1-\alpha} f'(t)$ for all $t \in [a,b]$.

To start with the problem, we transform the system of equations (1) to an abstract Cauchy problem. Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$ and $L^2_\sigma(\Omega) =$ closure of $C^\infty_{0,\sigma}(\Omega)$ in $(L^2(\Omega))^3$, where $C^\infty_{0,\sigma}(\Omega) = \{u \in (C_0^\infty(\Omega))^3 : \nabla \cdot u = 0\}$. Then $L^2_\sigma(\Omega)$, endowed with the usual inner product in $(L^2(\Omega))^3$ is a Hilbert space. We mainly denote the norm on $L^2_\sigma(\Omega)$ as $\|\cdot\|$. Also, to avoid the abundance use of notations, we use the same notation to represent the norm of an operator in $\mathscr{B}(L^2_\sigma(\Omega))$. So, the readers should use accordingly.
Let $G(\Omega) = \{f \in (L^2(\Omega))^3 : \exists p \in L^2(\Omega)$ such that $f = \nabla p\}$. Then $G(\Omega)$ is a closed subspace of $(L^2(\Omega))^3$ and the decomposition $(L^2(\Omega))^3 = L^2_\sigma(\Omega) \oplus G(\Omega)$ holds and known as *Helmholtz decomposition*. Let $P : (L^2(\Omega))^3 \to L^2_\sigma(\Omega)$ be the Projection operator.

Now, we define the bilinear form as $a(u,v) = \langle \nabla u, \nabla v\rangle$, where $u,v \in H^1_{0,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{H^1(\Omega)}$, $\langle,\rangle$ is the usual inner product on $L^2_\sigma(\Omega)$ and $A : D(A) \subset L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$ is the associated operator of the bilinear form. Following [23, Theorem 1.52], $-A$ generates analytic semigroup of contractions $\{T(t)\}_{t\geq 0}$ on $L^2_\sigma(\Omega)$. Moreover, by following [24], $A = -P\Delta$ with $D(A) = L^2_\sigma(\Omega) \cap (H^1_0(\Omega))^3 \cap (H^2(\Omega))^3$ is known as Stokes operator. Since $0 \in \rho(A)$ [24], where $\rho(A)$ is the resolvent set of A and $-A$ generates the analytic semigroup $\{T(t)\}_{t\geq 0}$, then one can define the fractional power of A as follows [25]:
For $\alpha > 0$, $A^{-\alpha}$ is defined by $A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)dt$ which is convergent in the uniform operator topology. Also, $A^{-\alpha}$ is injective, and hence define $A^\alpha := (A^{-\alpha})^{-1}$ with $D(A^\alpha) = R(A^{-\alpha})$, which is densely defined closed operator in $L^2_\sigma(\Omega)$. For $0 < \alpha < 1$, $A^{-\alpha}$ is bounded linear operator and hence $D(A^\alpha)$ with the norm $\|x\|_{D(A^\alpha)} = \|A^\alpha x\|$ $\big($ which is equivalent to the graph norm on $D(A^\alpha)$ $\big)$ is a Banach space and for $0 < \alpha < \beta < 1$, $D(A^\beta) \hookrightarrow D(A^\alpha)$. Applying projection operator on (1) and using Stokes operator, the system (1) transforms to the following evolution equation in a Banach space $L^2_\sigma(\Omega)$:

$$\left.\begin{aligned} {}^cD_t^\alpha u + Au &= Fu + Pf(t,u_t), \quad t > 0,\\ u(t) &= \phi(t), \qquad -r \leq t \leq 0, \end{aligned}\right\} \tag{2}$$

where $Fu = -P(u\cdot\nabla)u$.
Let $0 < \alpha < 1$. Now, we define following two families of operators on $L^2_\sigma(\Omega)$,

$$S_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{\rho,\eta}} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1} d\lambda, \quad t > 0, \tag{3}$$

$$T_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{\rho,\eta}} e^{\lambda t}(\lambda^\alpha + A)^{-1} d\lambda, \quad t > 0, \tag{4}$$

where $\Gamma_{\rho,\eta}$ is a suitable path in $\rho(-A)$. For more details see [26].
Using these operators and some tools of fractional calculus, we define the mild solution of (2) as follows:

Definition 3. *Let $0 < T < \infty$. A function $u : [-r,T] \to D(A^{\frac{1}{2}})$ is said to be a local mild solution of the problem (2) if $u|_{[0,T]} \in C([0,T]; D(A^{\frac{1}{2}}))$ and u satisfies the following integral equations:*

$$u(t) = \begin{cases} S_\alpha(t)\phi(0) + \int_0^t T_\alpha(t-s)Fu(s)ds + \int_0^t T_\alpha(t-s)Pf(s,u_s)ds, \ t \in (0,T],\\ \phi(t), -r \leq t \leq 0. \end{cases} \tag{5}$$

Definition 4. *Let $0 < T < \infty$. A function $u \in C([-r,T]; L^2_\sigma(\Omega))$ is said to be a classical solution of the problem (2) if it satisfies following conditions:*

(i) $u \in C([0,T]; D(A))$,
(ii) $g_{1-\alpha} * (u - u(0)) \in C^1((0,T); L^2_\sigma(\Omega))$,
(iii) u *satisfies* (2).

Lemma 1. *[26] Let $S_\alpha(t)$ be defined by (3). Then following holds:*

(i) $S_\alpha(t) \in \mathscr{B}(L^2_\sigma(\Omega))$ *for each* $t > 0$. *Moreover, there exists* $C_1 = C_1(\alpha) > 0$ *such that* $\|S_\alpha(t)\| \leq C_1$ *for all* $t > 0$.
(ii) $S_\alpha(t) \in \mathscr{B}(L^2_\sigma(\Omega); D(A))$ *for each* $t > 0$. *Moreover, there exists* $C_2 = C_2(\alpha) > 0$ *such that*

$$\|AS_\alpha(t)\| \leq C_2 t^{-\alpha} \quad \text{for all } t > 0,$$

and if $x \in D(A)$, *then* $AS_\alpha(t)x = S_\alpha(t)Ax$ *for all* $t > 0$.

(iii)The function $t \mapsto S_\alpha(t)$ belongs to $C^\infty((0,\infty);\mathscr{B}(L^2_\sigma(\Omega))$, and for every $n \in \mathbb{N}$, there exists $M_n = M_n(\alpha) > 0$ such that

$$\|S^{(n)}_\alpha(t)\| \le M_n t^{-n} \text{ for all } t > 0.$$

(iv)For each $0 < \beta < 1$, there exists $C_3 = C_3(\alpha,\beta) > 0$ such that

$$\|A^\beta S_\alpha(t)x\| \le C_3 t^{-\alpha}(t^{-\alpha(\beta-1)}+1)\|x\| \text{ for all } x \in L^2_\sigma(\Omega),\ t > 0.$$

(v)For $x \in L^2_\sigma(\Omega)$, $\lim\limits_{t\to 0}\|S_\alpha(t)x - x\| = 0$.

Lemma 2. *[26] Let $T_\alpha(t)$ be defined by (4). Then following holds:*

(i)$T_\alpha(t) \in \mathscr{B}(L^2_\sigma(\Omega))$ for each $t > 0$. Moreover, there exists $B_1 = B_1(\alpha) > 0$ such that

$$\|T_\alpha(t)\| \le B_1 t^{\alpha-1} \text{ for all } t > 0.$$

(ii)$T_\alpha(t) \in \mathscr{B}(L^2_\sigma(\Omega);D(A))$ for each $t > 0$. Moreover, there exists $B_2 = B_2(\alpha) > 0$ such that

$$\|AT_\alpha(t)\| \le B_2 t^{-1} \text{ for all } t > 0,$$

and if $x \in D(A)$, then $AT_\alpha(t)x = T_\alpha(t)Ax$ for all $t > 0$.

(iii)The function $t \mapsto T_\alpha(t)$ belongs to $C^\infty((0,\infty);\mathscr{B}(L^2_\sigma(\Omega))$, and for every $n \in \mathbb{N}$, there exists $N_n = N_n(\alpha) > 0$ such that

$$\|T^{(n)}_\alpha(t)\| \le N_n t^{\alpha-1-n} \text{ for all } t > 0.$$

(iv)For each $0 < \beta < 1$, there exists $B_3 = B_3(\alpha,\beta) > 0$ such that

$$\|A^\beta T_\alpha(t)x\| \le B_3 t^{\alpha(1-\beta)-1}\|x\| \text{ for all } x \in L^2_\sigma(\Omega),\ t > 0.$$

(v)For $x \in L^2_\sigma(\Omega)$ and $t > 0$, $\frac{d}{dt}(S_\alpha(t)x) = AT_\alpha(t)x$.

Lemma 3. *[11] Let $u,v \in D(A^{\frac{1}{2}})$, then following estimations hold:*

(i)There exists $c_1 > 0$ such that $\|A^{-\frac{1}{4}}Fu\| \le c_1\|A^{\frac{1}{2}}u\|^2$,
(ii)$\|A^{-\frac{1}{4}}(Fu - Fv)\| \le c_1\|A^{\frac{1}{2}}(u-v)\|(\|A^{\frac{1}{2}}u\| + A^{\frac{1}{2}}v\|)$.

Lemma 4. *[10] Let $u,v \in D(A^{\frac{3}{4}})$, then following estimations hold:*

(i)There exists $c_2 > 0$ such that $\|Fu\| \le c_2\|A^{\frac{1}{2}}u\|\|A^{\frac{3}{4}}u\|$,
(ii)$\|Fu - Fv\| \le c_2(\|A^{\frac{1}{2}}(u-v)\|\|A^{\frac{3}{4}}u\| + \|A^{\frac{3}{4}}(u-v)\|\|A^{\frac{1}{2}}v\|)$.

Lemma 5. *[21, p. 890] Let $0 < \beta < 1$ and $T_\alpha(t)$ is defined by (4). Then there exists $B_4 = B_4(\alpha,\beta) > 0$ such that $\|A^\beta T_\alpha(t) - A^\beta T_\alpha(s)\| \le B_4(s^{\alpha(1-\beta)} - t^{\alpha(1-\beta)})$ for all $s,t > 0$ with $t > s$. In another words, $t \mapsto A^\beta T(t)$ is continuous for $t > 0$ with respect to uniform operator topology.*

Lemma 6. *[27, p. A3] Let X be a Banach space and $A : D(A) \subset X \to X$ be a closed operator. Let I be a real interval with $\inf I = a$, $\sup I = b$, where $-\infty \le a < b \le \infty$ and $f : I \to D(A)$ be such that the functions $t \to f(t)$, $t \to Af(t)$ are integrable (Bochner sense) on I. Then*

$$\int_a^b f(t)dt \in D(A), \quad A\int_a^b f(t)dt = \int_a^b Af(t)dt.$$

Theorem 1.*(Contraction Principle)* *Let B be a closed subset of a Banach space X and $f : B \to B$ be a contraction map. Then there exists a unique fixed point of f in B.*

Proposition 1.*Let $0 < \beta < 1$ and $S_\alpha(t)$ be defined by (3) on $L^2_\sigma(\Omega)$. Then, for $x \in D(A^\beta)$,*

$$A^\beta S_\alpha(t)x = S_\alpha(t)A^\beta x \text{ for all } t > 0.$$

*Proof.*Consider the Mainardi function, $M_\alpha(t) := \frac{1}{\pi}\sum\limits_{n=1}^{\infty}\frac{(-t)^{n-1}}{(n-1)!}\Gamma(\alpha n)\sin(\pi\alpha n)$, $t \ge 0$, for more details on Mainardi function, see [28]. Then, following [6], $S_\alpha(t)$ can be written as,

$$S_\alpha(t)x = \int_0^\infty M_\alpha(s)T(st^\alpha)xds,\ x \in L^2_\sigma(\Omega).$$

Since $\int_0^\infty s^q M_\alpha(s)ds = \frac{\Gamma(q+1)}{\Gamma(\alpha q+1)}$, where $-1 < q < \infty$ and $\|A^\beta T(t)\| \le t^{-\beta}$ for all $t > 0$. Therefore, by Lemma 6 we have,

$$A^\beta S_\alpha(t)x = A^\beta\int_0^\infty M_\alpha(s)T(st^\alpha)xds = \int_0^\infty M_\alpha(s)A^\beta T(st^\alpha)xds = \int_0^\infty M_\alpha(s)T(st^\alpha)A^\beta xds = S_\alpha(t)A^\beta x.$$

3 Local existence of mild solution

In this section, we establish local existence and uniqueness of mild solution to (2).

Theorem 2.*Let $Y_{\frac{1}{2}} := C([-r,0];D(A^{\frac{1}{2}}))$ and $U \subset Y_{\frac{1}{2}}$ be open. Assume that $Pf : [0,\infty)\times U \to L^2_\sigma(\Omega)$ be such that*

(i)$\|Pf(t,\varphi)\| \le \omega(t)\|\varphi\|_{Y_{\frac{1}{2}}}$ for all $t \ge 0$, $\varphi \in U$ and for some $\omega \in L^p_{loc}[0,\infty)$, where $p > \frac{2}{\alpha}$,
(ii)$\|Pf(t,\varphi) - Pf(t,\psi)\| \le L_f\|\varphi-\psi\|_{Y_{\frac{1}{2}}}$ for all $\varphi,\psi \in U$ and for some $L_f > 0$.

Then for every $\phi \in U$, there exists a unique mild solution $u : [-r,T] \to D(A^{\frac{1}{2}})$ to (2), for some $T = T(\phi) > 0$.

*Proof.*Let $\phi \in U$ and $R > 0$ be such that $\{\psi \in Y_{\frac{1}{2}} : \|\psi-\phi\|_{Y_{\frac{1}{2}}} \le R\} \subset U$. Let $T > 0$ (will be fixed later). We define the following set,

$$Z_{\frac{1}{2}} = \{u \in C\big([-r,T]);D(A^{\frac{1}{2}})\big) : \ u_0 = \phi \text{ and } \|u_t-\phi\|_{Y_{\frac{1}{2}}} \le R \ \text{ for all } t\in[0,T]\},$$

which is a non-empty closed subset of $C\big([-r,T]);D(A^{\frac{1}{2}})\big)$, where $C\big([-r,T]);D(A^{\frac{1}{2}})\big)$ is endowed with sup-norm topology. Now, we define an operator on $Z_{\frac{1}{2}}$ as follows,

$$Ku(t) = \begin{cases} S_\alpha(t)\phi(0) + \int_0^t T_\alpha(t-s)Fu(s)ds + \int_0^t T_\alpha(t-s)Pf(s,u_s)ds, \ t\in(0,T], \\ \phi(t), \ -r\le t\le 0. \end{cases} \tag{6}$$

First we prove that $K(Z_{\frac{1}{2}}) \subset Z_{\frac{1}{2}}$.
Let $u \in Z_{\frac{1}{2}}$. We note that $\|u(t)\|_{D(A^{\frac{1}{2}})} = \|u_t(0)\|_{D(A^{\frac{1}{2}})} \le \|u_t\|_{Y_{\frac{1}{2}}} \le R + \|\phi\|_{Y_{\frac{1}{2}}}$ for all $t\in[0,T]$.
Let $t_1 > 0$ be such that $\|\phi(t+\theta)-\phi(\theta)\|_{D(A^{\frac{1}{2}})} \le \frac{R}{4}$ for all $t\in[0,t_1]$ and $\theta\in[-r,0]$ with $t+\theta\le 0$. Let $t_2 > 0$ such that $\|S_\alpha(t)\phi(0)-\phi(0)\|_{D(A^{\frac{1}{2}})} \le \frac{R}{4}$ for all $t\in[0,t_2]$. By following [7, Theorem 2.6], $J_t^{\frac{\alpha}{2}}\omega(t) \to 0$ as $t\to 0$, choose $t_3 > 0$ such that $\int_0^t (t-s)^{\frac{\alpha}{2}-1}\omega(s)ds \le \frac{R}{4B_3(\alpha,\frac{1}{2})(R+\|\phi\|_{Y_{\frac{1}{2}}})}$ for all $t\in[0,t_3]$. Also, we can choose some $t_4 > 0$ such that $\int_0^t (t-s)^{\frac{\alpha}{4}-1}ds \le \frac{R}{4B_3(\alpha,\frac{3}{4})c_1(R+\|\phi\|_{Y_{\frac{1}{2}}})^2}$ for all $t\in[0,t_4]$.
Let $T_1 = \min\{t_1,t_2,t_3,t_4\}$. Now, for all $t\in[0,T_1]$ and $\theta\in[-r,0]$ such that $0\le t+\theta\le T_1$, we have

$$\begin{aligned}
\|(Ku)_t(\theta)-\phi(\theta)\|_{D(A^{\frac{1}{2}})} \le& \|S_\alpha(t+\theta)\phi(0)-\phi(0)\|_{D(A^{\frac{1}{2}})} + \|A^{\frac{1}{2}}\int_0^{t+\theta} A^{\frac{1}{4}}T_\alpha(t+\theta-s)A^{-\frac{1}{4}}Fu(s)ds\| \\
&+ \|\phi(\theta)-\phi(0)\|_{D(A^{\frac{1}{2}})} + \|A^{\frac{1}{2}}\int_0^{t+\theta} T_\alpha(t+\theta-s)Pf(s,u_s)ds\| \\
\le& \|S_\alpha(t+\theta)\phi(0)-\phi(0)\|_{D(A^{\frac{1}{2}})} + B_3(\alpha,\frac{3}{4})\int_0^{t+\theta}(t+\theta-s)^{\frac{\alpha}{4}-1}\|A^{-\frac{1}{4}}Fu(s)\|ds \\
&+ \|\phi(\theta)-\phi(0)\|_{D(A^{\frac{1}{2}})} + B_3(\alpha,\frac{1}{2})\int_0^{t+\theta}(t+\theta-s)^{\frac{\alpha}{2}-1}\omega(s)\|u_s\|_{Y_{\frac{1}{2}}}ds \\
\le& \|S_\alpha(t+\theta)\phi(0)-\phi(0)\|_{D(A^{\frac{1}{2}})} + B_3(\alpha,\frac{3}{4})c_1\int_0^{t+\theta}(t+\theta-s)^{\frac{\alpha}{4}-1}\|u(s)\|^2_{D(A^{\frac{1}{2}})}ds \\
&+ \|\phi(\theta)-\phi(0)\|_{D(A^{\frac{1}{2}})} + B_3(\alpha,\frac{1}{2})\int_0^{t+\theta}(t+\theta-s)^{\frac{\alpha}{2}-1}\omega(s)\|u_s\|_{Y_{\frac{1}{2}}}ds \\
\le& \|S_\alpha(t+\theta)\phi(0)-\phi(0)\|_{D(A^{\frac{1}{2}})} + B_3(\alpha,\frac{3}{4})c_1\int_0^{t+\theta}(t+\theta-s)^{\frac{\alpha}{4}-1}(R+\|\phi\|_{Y_{\frac{1}{2}}})^2ds \\
&+ \|\phi(\theta)-\phi(0)\|_{D(A^{\frac{1}{2}})} + B_3(\alpha,\frac{1}{2})\int_0^{t+\theta}(t+\theta-s)^{\frac{\alpha}{2}-1}\omega(s)(R+\|\phi\|_{Y_{\frac{1}{2}}})ds \\
\le& R.
\end{aligned}$$

For case $-r \le t+\theta \le 0$, $t \in [0,T_1]$, we get $\|(Ku)_t(\theta)-\phi(\theta)\|_{D(A^{\frac{1}{2}})} = \|\phi(t+\theta)-\phi(\theta)\|_{D(A^{\frac{1}{2}})} \le \frac{R}{4} \le R$.
Hence, $\|(Ku)_t-\phi\|_{Y_{\frac{1}{2}}} \le R$ for all $t \in [0,T_1]$.

Now, we prove the continuity of $t \mapsto Ku(t)$ on $(0,T_1]$ with respect to the topology induced by $D(A^{\frac{1}{2}})$-norm.
First define $v(t) := \int_0^t T_\alpha(t-s)Fu(s)ds$ and let $t_0 \in (0,T_1]$ with $t > t_0$ and $\delta > 0$ small enough.

$$\begin{aligned}
\|A^{\frac{1}{2}}(v(t)-v(t_0))\| \le & \|A^{\frac{1}{2}} \int_0^{t_0-\delta} A^{\frac{1}{4}}[T_\alpha(t-s)-T_\alpha(t_0-s)]A^{-\frac{1}{4}}Fu(s)ds\| \\
& + \|A^{\frac{1}{2}} \int_{t_0-\delta}^{t_0} A^{\frac{1}{4}}[T_\alpha(t-s)-T_\alpha(t_0-s)]A^{-\frac{1}{4}}Fu(s)ds\| \\
& + \|A^{\frac{1}{2}} \int_{t_0}^{t} A^{\frac{1}{4}}T_\alpha(t-s)A^{-\frac{1}{4}}Fu(s)ds\| \\
:= & I_1+I_2+I_3.
\end{aligned}$$

Consider I_1. We see that,

$$\begin{aligned}
I_1 \le & c_1 \sup_{0\le s\le t_0-\delta} \|A^{\frac{3}{4}}[T_\alpha(t-s)-T_\alpha(t_0-s)]\| \int_0^{t_0-\delta} \|A^{\frac{1}{2}}u(s)\|^2 ds \\
\le & c_1 \sup_{0\le s\le t_0-\delta} \|A^{\frac{3}{4}}[T_\alpha(t-s)-T_\alpha(t_0-s)]\|(R+\|\phi\|_{Y_{\frac{1}{2}}})^2(t_0-\delta).
\end{aligned}$$

Since by Lemma 5, $t \mapsto A^{\frac{3}{4}}T(t)$ is continuous in the uniform operator topology on $[\delta,T_1]$ for every $\delta > 0$, there exists $\tilde{t} \in [0,t_0-\delta]$ such that,

$$\sup_{0\le s\le t_0-\delta} \|A^{\frac{3}{4}}[T_\alpha(t-s)-T_\alpha(t_0-s)]\| = \|A^{\frac{3}{4}}[T_\alpha(t-\tilde{t})-T_\alpha(t_0-\tilde{t})]\| \to 0, \text{ as } t \to t_0$$

and hence $I_1 \to 0$ as $t \to t_0$.
Now, consider I_2. Using Lemmas 2, 3 we have,

$$\begin{aligned}
I_2 & \le \int_{t_0-\delta}^{t_0} \|A^{\frac{3}{4}}[T_\alpha(t-s)-T_\alpha(t_0-s)]A^{-\frac{1}{4}}Fu(s)\|ds \\
& \le c_1B_3 \int_{t_0-\delta}^{t_0} [(t-s)^{\frac{\alpha}{4}-1}+(t_0-s)^{\frac{\alpha}{4}-1}]\|A^{\frac{1}{2}}u(s)\|^2 ds \\
& \le 2c_1B_3(R+\|\phi\|_{Y_{\frac{1}{2}}})^2 \int_{t_0-\delta}^{t_0} (t_0-s)^{\frac{\alpha}{4}-1}ds \to 0 \text{ as } \delta \to 0.
\end{aligned}$$

Again using Lemmas 2, 3 in I_3 we have,

$$\begin{aligned}
I_3 & \le \int_{t_0}^{t} \|A^{\frac{3}{4}}T_\alpha(t-s)A^{-\frac{1}{4}}Fu(s)\|ds \\
& \le c_1B_3 \int_{t_0}^{t} (t-s)^{\frac{\alpha}{4}-1}\|A^{\frac{1}{2}}u(s)\|^2 ds \\
& \le (R+\|\phi\|_{Y_{\frac{1}{2}}})^2 c_1B_3 \int_{t_0}^{t} (t-s)^{\frac{\alpha}{4}-1}ds \to 0 \text{ as } t \to t_0.
\end{aligned}$$

Therefore, $\|A^{\frac{1}{2}}(v(t)-v(t_0))\| \to 0$ as $t \to t_0+$. Analogously it can be proved that $\|A^{\frac{1}{2}}(v(t)-v(t_0))\| \to 0$ as $t \to t_0-$ by considering $t < t_0$. Hence, $t \mapsto v(t)$ is continuous on $(0,T_1]$ with respect to the topology induced by $D(A^{\frac{1}{2}})$-norm.
Now, define $w(t) := \int_0^t T_\alpha(t-s)Pf(s,u_s)ds$ and let $t_0 \in (0,T_1]$ with $t > t_0$ and $\delta > 0$ small enough.

$$\begin{aligned}
\|A^{\frac{1}{2}}(w(t)-w(t_0))\| \le & \|A^{\frac{1}{2}} \int_0^{t_0-\delta} [T_\alpha(t-s)-T_\alpha(t_0-s)]Pf(s,u_s)ds\| \\
& + \|A^{\frac{1}{2}} \int_{t_0-\delta}^{t_0} [T_\alpha(t-s)-T_\alpha(t_0-s)]Pf(s,u_s)ds\| \\
& + \|A^{\frac{1}{2}} \int_{t_0}^{t} T_\alpha(t-s)Pf(s,u_s)ds\| \\
:= & J_1+J_2+J_3.
\end{aligned}$$

Consider J_1. We see that,

$$J_1 \leq \sup_{0\leq s\leq t_0-\delta} \|A^{\frac{1}{2}}[T_\alpha(t-s)-T_\alpha(t_0-s)]\| \int_0^{t_0-\delta} \omega(s)\|u_s\|_{Y_{\frac{1}{2}}} ds$$
$$\leq \sup_{0\leq s\leq t_0-\delta} \|A^{\frac{1}{2}}[T_\alpha(t-s)-T_\alpha(t_0-s)]\| \int_0^{t_0-\delta} \omega(s)(R+\|\phi\|_{Y_{\frac{1}{2}}})ds.$$

Since for any $b>0$, $\omega \in L^p[0,b]$, therefore $\omega \in L^1[0,b]$. Also, by Lemma 5, $t \mapsto A^{\frac{1}{2}}T(t)$ is continuous in the uniform operator topology on $[\delta, T_1]$ for every $\delta > 0$, therefore there exists $\tau \in [0, t_0-\delta]$ such that

$$\sup_{0\leq s\leq t_0-\delta} \|A^{\frac{1}{2}}[T_\alpha(t-s)-T_\alpha(t_0-s)]\| = \|A^{\frac{1}{2}}[T_\alpha(t-\tau)-T_\alpha(t_0-\tau)]\| \to 0 \text{ as } t\to t_0,$$

and hence $J_1 \to 0$ as $t \to t_0$.
Similarly, considering J_2 and using Lemma 2 we have,

$$J_2 \leq \int_{t_0-\delta}^{t_0} \|A^{\frac{1}{2}}[T_\alpha(t-s)-T_\alpha(t_0-s)]\|\omega(s)\|u_s\|_{Y_{\frac{1}{2}}} ds$$
$$\leq B_3 \int_{t_0-\delta}^{t_0} [(t-s)^{\frac{\alpha}{2}-1}+(t_0-s)^{\frac{\alpha}{2}-1}]\omega(s)(R+\|\phi\|_{Y_{\frac{1}{2}}})ds$$
$$\leq 2B_3(R+\|\phi\|_{Y_{\frac{1}{2}}}) \int_{t_0-\delta}^{t_0} (t_0-s)^{\frac{\alpha}{2}-1}\omega(s)ds. \tag{7}$$

Since for any $b>0$, $\omega \in L^p[0,b]$ with $p > \frac{2}{\alpha}$, therefore R.H.S of (7) $\to 0$ as $\delta \to 0$.
Again by using Lemma 2 in J_3 we have,

$$J_3 \leq \int_{t_0}^{t} \|A^{\frac{1}{2}}T_\alpha(t-s)\omega(s)\|u_s\|_{Y_{\frac{1}{2}}} ds$$
$$\leq B_3 \int_{t_0}^{t} (t-s)^{\frac{\alpha}{2}-1}\omega(s)(R+\|\phi\|_{Y_{\frac{1}{2}}})ds$$
$$\leq B_3(R+\|\phi\|_{Y_{\frac{1}{2}}}) \int_{t_0}^{t} (t-s)^{\frac{\alpha}{2}-1}\omega(s)ds \to 0 \text{ as } t\to t_0. \tag{8}$$

Again, since for any $b>0$, $\omega \in L^p[0,b]$ with $p > \frac{2}{\alpha}$, therefore R.H.S of (8) $\to 0$ as $t \to t_0$.
Therefore, $\|A^{\frac{1}{2}}(w(t)-w(t_0))\| \to 0$ as $t \to t_0+$. Analogously it can be proved that $\|A^{\frac{1}{2}}(w(t)-w(t_0))\| \to 0$ as $t \to t_0-$ by considering $t < t_0$. Hence, $t \mapsto w(t)$ is continuous on $(0, T_1]$ with respect to the topology induced by $D(A^{\frac{1}{2}})$-norm.
Since $u(0) = \phi(0) \in D(A^{\frac{1}{2}})$, therefore by Proposition 1, Lemma 1, we can say that $\|A^{\frac{1}{2}}(S_\alpha(t)\phi(0) - S_\alpha(t_0)\phi(0))\| = \|S_\alpha(t)A^{\frac{1}{2}}\phi(0) - S_\alpha(t_0)A^{\frac{1}{2}}\phi(0)\| \to 0$ as $t \to t_0$.
The continuity of Ku is easily followed at $t = 0$.
Thus, we proved that $t \mapsto Ku(t)$ is continuous on $[-r, T_1]$ with respect the topology induced by $D(A^{\frac{1}{2}})$-norm, and hence $K(Z_{\frac{1}{2}}) \subset Z_{\frac{1}{2}}$.

Now, let $u,\, v \in Z_{\frac{1}{2}},\, t \in [0,T_1]$. Then using Lemmas 2, 3 we get,

$$\begin{aligned}
\|A^{\frac{1}{2}}\{Ku(t)-Kv(t)\}\| &\leq \|A^{\frac{1}{2}}\int_0^t A^{\frac{1}{4}}T_\alpha(t-s)A^{-\frac{1}{4}}(Fu(s)-Fv(s))ds\| \\
&\quad + \|A^{\frac{1}{2}}\int_0^t T_\alpha(t-s)\{Pf(s,u_s)-Pf(s,v_s)\}ds\| \\
&\leq B_3\int_0^t (t-s)^{\frac{\alpha}{4}-1}\|A^{-\frac{1}{4}}(Fu(s)-Fv(s))\|ds \\
&\quad + B_3\int_0^t (t-s)^{\frac{\alpha}{2}-1}\|Pf(s,u_s)-Pf(s,v_s)\|ds \\
&\leq c_1B_3\int_0^t (t-s)^{\frac{\alpha}{4}-1}\{\|A^{\frac{1}{2}}(u(s)-v(s))\|\}\{\|A^{\frac{1}{2}}u(s)\|+\|A^{\frac{1}{2}}v(s)\|\}ds \\
&\quad + B_3L_f\int_0^t (t-s)^{\frac{\alpha}{2}-1}\|u_s-v_s\|_{Y_{\frac{1}{2}}}ds \\
&\leq c_1B_3\int_0^t (t-s)^{\frac{\alpha}{4}-1}\|u-v\|_{Z_{\frac{1}{2}}}2(R+\|\phi\|_{Y_{\frac{1}{2}}})ds \\
&\quad + B_3L_f\int_0^t (t-s)^{\frac{\alpha}{2}-1}\sup_{0\leq r\leq s}\|A^{\frac{1}{2}}(u(r)-v(r))\|ds \\
&\leq 2(R+\|\phi\|_{Y_{\frac{1}{2}}})c_1B_3\|u-v\|_{Z_{\frac{1}{2}}}\int_0^t (t-s)^{\frac{\alpha}{4}-1}ds \\
&\quad + B_3L_f\|u-v\|_{Z_{\frac{1}{2}}}\int_0^t (t-s)^{\frac{\alpha}{2}-1}ds \\
&= \Big\{2(R+\|\phi\|_{Y_{\frac{1}{2}}})c_1B_3\int_0^t (t-s)^{\frac{\alpha}{4}-1}ds + B_3L_f\int_0^t (t-s)^{\frac{\alpha}{2}-1}ds\Big\}\|u-v\|_{Z_{\frac{1}{2}}}. \qquad (9)
\end{aligned}$$

Since both the integrals in R.H.S of (9) tend to zero as $t \to 0$, we can choose a small positive $T(\leq T_1)$ such that following holds:

$$\|A^{\frac{1}{2}}\{Ku(t)-Kv(t)\}\| \leq M\|u-v\|_{Z_{\frac{1}{2}}} \quad \text{for all} \quad t \in [0,T] \text{ and some } 0<M<1. \qquad (10)$$

This implies that $\|Ku-Kv\|_{Z_{\frac{1}{2}}} \leq M\|u-v\|_{Z_{\frac{1}{2}}}$ for some $0<M<1$.
Therefore, $K : Z_{\frac{1}{2}} \to Z_{\frac{1}{2}}$ is a contraction map. Consequently, by contraction mapping principle in Theorem 1, K has a unique fixed point $u \in Z_{\frac{1}{2}}$ which satisfies the integral equation (6). This proves the existence of uniqueness local mild solution of (2).

Some examples of delays

*Example 1.*We associate the present state of the system with the past history of the system by the delay function $\eta : [0,\infty) \to [0,r]$ which is continuous. Now consider a measurable function $G : [0,\infty)\times\mathbb{R}^3 \to \mathbb{R}^3$ satisfying,

(i)$|G(t,x)| \leq \omega(t)|x|$ for all $t \geq 0,\, x \in \mathbb{R}^3$ and for some $\omega \in L^p_{loc}[0,\infty)$, where $p > \frac{2}{\alpha}$,
(ii)$|G(t,x)-G(t,y)| \leq L|u-v|$ for all $t \geq 0,\, x,y \in \mathbb{R}^3$ and for some $L>0$.

Then for $\varphi \in C([-r,0];D(A^{\frac{1}{2}}))$, define $Pf(t,\varphi)(x) = G(t,\varphi(-\eta(t))(x)),\, t \geq 0,\, x \in \Omega$. Therefore $Pf(t,u_t)(x) = G(t,u(t-\eta(t))(x))$. It is easy to check that $(t,\varphi) \mapsto Pf(t,\varphi)$ satisfy the conditions of the Theorem 2.

*Example 2.*Consider a measurable function $k : [0,\infty)\times[-r,0] \to \mathbb{R}$ satisfying $|k(t,s)| \leq (-s)^{-\nu}$ for all $t \geq 0,\, 0<s\leq -r$ and for some $\nu < 1$. Now for $\varphi \in C([-r,0];D(A^{\frac{1}{2}}))$, we define the following integral,

$$Pf(t,\phi) = \int_{-r}^0 k(t,s)\varphi(s)ds, \quad t \geq 0.$$

Then $Pf(t,u_t) = \int_{-r}^0 k(t,s)u(t+s)ds$. Now one can easily check that $(t,\varphi) \mapsto Pf(t,\varphi)$ satisfy the conditions of the Theorem 2 with $\omega(t) = 1$ for all $t \geq 0$.

4 Continuation of mild solution

Theorem 3. *Assume that all the conditions of the Theorem (2) hold for $U = Y_{\frac{1}{2}}$. Then for every $\phi \in Y_{\frac{1}{2}}$, the problem (2) has a unique mild solution on a maximal interval of existence $[-r, t_{max})$. Moreover, if $t_{max} < \infty$, then $\overline{\lim}_{t\to t_{max}} \|u(t)\|_{D(A^{\frac{1}{2}})} = \infty$.*

Proof. From the previous result, we know that the mild solution of (2) exists in the interval $[-r, T]$. Now we prove that this solution can be extended to the interval $[-r, T+\delta]$ for some $\delta > 0$.

Let u be the mild solution of (2) on $[-r, T]$. Define $v(t) = u(t+T)$ where $v(t)$ is a mild solution of

$$\left.\begin{aligned} {}^cD_t^{\alpha} v + Av(t) &= Fv(t) + Pf(t+T, v_t), \quad t > 0, \\ v_0 &= u_T. \end{aligned}\right\} \tag{11}$$

Since $u \in C\big([-r,T]; D(A^{\frac{1}{2}})\big)$, therefore $v_0 = u_T \in Y_{\frac{1}{2}}$. Hence, the existence of the mild solution of (11) on some interval $[-r, \delta]$, where $\delta > 0$, is assured by the Theorem 2. But the extended part of u on $[T, T+\delta]$ given by $u(t) = v(t-T)$ does not satisfy the integral equation (5), i.e., u is not mild solution of (2) on $[T, T+\delta]$. Therefore, the classical method to show the continuation fails for fractional case.

Now, we fix $R > 0$. Consider the following set,

$$Z^*_{\frac{1}{2}} = \Big\{ w \in C\big([-r, T+\delta]); D(A^{\frac{1}{2}})\big) : \ w(t) = u(t), \ \forall t \in [-r, T], \ \sup_{0 \le t \le T} \|w_t - \phi\|_{Y_{\frac{1}{2}}} \le R, \ \sup_{T \le t \le T+\delta} \|w(t) - u(T)\|_{D(A^{\frac{1}{2}})} \le R \Big\}.$$

Then, by following the same procedure as in Theorem 2, one can prove that there exists a $\delta > 0$ such that the problem (2) has a unique mild solution in $Z^*_{\frac{1}{2}}$, which is continuation of the mild solution u in $[-r, T+\delta]$.

Consequently, let $[-r, t_{max})$ be the maximal interval of existence of mild solution of (2).

If $t_{max} = \infty$, then the mild solution is global. If $t_{max} < \infty$, we prove that $\overline{\lim}_{t\to t_{max}} \|u(t)\|_{D(A^{\frac{1}{2}})} = \infty$.

Let us assume that $\overline{\lim}_{t\to t_{max}} \|u(t)\|_{D(A^{\frac{1}{2}})} < \infty$. Consequently, $\overline{\lim}_{t\to t_{max}} \|u_t\|_{Y_{\frac{1}{2}}} < \infty$. Then, there exists $N > 0$ such that such that $\|u_t\|_{Y_{\frac{1}{2}}} \le N$ for all $t \in [0, t_{max})$. This implies $\|u(t)\|_{D(A^{\frac{1}{2}})} \le N$ and $\|Pf(t, u_t)\| \le N\omega(t)$ for all $t \in [0, t_{max})$.

Let $0 < t < \tau < t_{max}$ and $\delta > 0$ be sufficiently small. Then we have

$$\begin{aligned}
\|u(t) - u(\tau)\|_{D(A^{\frac{1}{2}})} &= \|S_\alpha(t)\phi(0) - S_\alpha(\tau)\phi(0)\|_{D(A^{\frac{1}{2}})} + \Big\| \int_t^\tau T_\alpha(\tau - s) Pf(s, u_s) ds \Big\|_{D(A^{\frac{1}{2}})} \\
&\quad + \Big\| \int_0^{t-\delta} [T_\alpha(t-s) - T_\alpha(\tau - s)] Pf(s, u_s) ds \Big\|_{D(A^{\frac{1}{2}})} \\
&\quad + \Big\| \int_{t-\delta}^{t} [T_\alpha(t-s) - T_\alpha(\tau - s)] Pf(s, u_s) ds \Big\|_{D(A^{\frac{1}{2}})} \\
&\quad + \Big\| \int_0^{t-\delta} [T_\alpha(t-s) - T_\alpha(\tau - s)] Fu(s) ds \Big\|_{D(A^{\frac{1}{2}})} \\
&\quad + \Big\| \int_{t-\delta}^{t} [T_\alpha(t-s) - T_\alpha(\tau - s)] Fu(s) ds \Big\|_{D(A^{\frac{1}{2}})} + \Big\| \int_t^\tau T_\alpha(\tau - s) Fu(s) ds \Big\|_{D(A^{\frac{1}{2}})} \\
&\le \|S_\alpha(t)\phi(0) - S_\alpha(\tau)\phi(0)\|_{D(A^{\frac{1}{2}})} + NB_3 \int_t^\tau (\tau - s)^{\frac{\alpha}{2} - 1} \omega(s) ds \\
&\quad + N \sup_{0 \le s \le t-\delta} \|A^{\frac{1}{2}} [T_\alpha(t-s) - T_\alpha(\tau - s)]\| \int_0^{t-\delta} \omega(s) ds \\
&\quad + 2N \int_{t-\delta}^{t} (t-s)^{\frac{\alpha}{2} - 1} \omega(s) ds + N^2\, t_{max} \sup_{0 \le s \le t-\delta} \|A^{\frac{3}{4}} [T_\alpha(t-s) - T_\alpha(\tau - s)]\| \\
&\quad + 2N \int_{t-\delta}^{t} (t-s)^{\frac{\alpha}{4} - 1} ds + N^2 \int_t^\tau (\tau - s)^{\frac{\alpha}{4} - 1} ds.
\end{aligned}$$

Since $\omega \in L_p[0, t_{max})$ for $p > \frac{2}{\alpha}$, therefore by applying Hölder's inequality in 2nd and 4th integrals of the above inequality and using the fact (5), it is easy to check that R.H.S of the above inequality can be made arbitrarily small by choosing $|t - \tau|$ sufficiently small. Hence $t \mapsto u(t)$ is uniformly continuous on $(0, t_{max})$ with respect to the topology induced by $D(A^{\frac{1}{2}})$-norm. This implies that $\lim_{t \to t_{max}} u(t) = u(t_{\max})$ exists, which contradicts the maximality of the interval of existence. So our assumption is wrong. Hence the theorem is proved.

5 Regularity result

In this section, we prove the regularity of the mild solution of the problem (2). If we prove that the function $t \mapsto Fu(t) + Pf(t,u_t)$ is Hölder continuous on the interval $[0,T]$ in a Banach space $L^2_\sigma(\Omega)$, then the mild solution of (2) is classical one [29]. But we found that for the mild solution $u : [-r,T] \to D(A^{\frac{1}{2}})$ of (2), the Hölder continuity of $t \mapsto Fu(t)$ can not be proved in $L^2_\sigma(\Omega)$. To overcome this difficulty, we choose initial datum ϕ such that it belongs to the space U which is an open subset of $Y_{\frac{3}{4}} := C([-r,0]; D(A^{\frac{3}{4}}))$. Further, we consider the following assumptions:

(I) $\|Pf(t,\varphi)\| \le \omega(t)\|\varphi\|_{Y_{\frac{3}{4}}}$ for all $t \ge 0$, $\varphi \in U$ and for some $\omega \in L^p_{loc}[0,\infty)$, where $p > \frac{4}{\alpha}$,

(II) $\|Pf(t,\varphi) - Pf(t,\psi)\| \le L_f\|\varphi - \psi\|_{Y_{\frac{3}{4}}}$ for all $\varphi, \psi \in U$ and for some $L_f > 0$.

Then analogous to the proof of the Theorem 2, it can be proved that under the above assumptions (I), (II) and Lemma 4, there exists unique local mild solutions $u \in C([-r,T]; D(A^{\frac{3}{4}}))$ of (2) such that $\|u_t - \phi\|_{Y_{\frac{3}{4}}} \le R$ for all $t \in [0,T]$, for some $R > 0$.
Now, we prove the regularity of this mild solution in the following theorem.

Theorem 4. *Let $u \in C([-r,T]; D(A^{\frac{3}{4}}))$ be the local mild solution of the evolution system (2) such that $\|u_t - \phi\|_{Y_{\frac{3}{4}}} \le R$ for all $t \in [0,T]$ and for some $R > 0$. Also we assume the following hypotheses:*

(H1) $\phi \in C^\theta([-r,0]; D(A^{\frac{3}{4}}))$ such that $\phi(0) \in D(A)$, for some $\theta \in (0,1)$.
(H2) Pf be such that $\|Pf(t,\varphi) - Pf(s,\psi)\| \le L(|t-s|^\theta + \|\varphi - \psi\|_{Y_{\frac{3}{4}}})$ for all $t,s \in [0,T]$ and $\varphi, \psi \in Y_{\frac{3}{4}}$, for some $\theta \in (0,1)$.

Then the mild solution is a classical solution.

To prove the above theorem we first need to prove the following results.

Lemma 7. *Let $0 < \beta < 1$ and define $v(t) = \int_0^t T_\alpha(t-s)Pf(s,u_s)ds$, $t \ge 0$, where $u \in C([-r,T]; D(A^{\frac{3}{4}}))$. Then $v(t) \in D(A^\beta)$ for all $t \in [0,T]$. Moreover, $A^\beta v \in C^{\alpha(1-\beta)}([0,T]; L^2_\sigma(\Omega))$.*

Proof. Since $u \in C([-r,T]; D(A^{\frac{3}{4}}))$, the map $t \mapsto u_t$ is continuous on $[0,T]$ with respect to $D(A^{\frac{3}{4}})$ norm. By assumption (H2), $t \mapsto Pf(t,u_t)$ is continuous on $[0,T]$ with respect to $L^2_\sigma(\Omega)$ norm. So there exists $N > 0$ such that $\|Pf(t,u_t)\| \le N$ for all $t \in [0,T]$.
Now, by using Lemma 2, we see that $\|A^\beta T_\alpha(t-s)Pf(t,u_t)\| \le NB_3(t-s)^{\alpha(1-\beta)-1}$, which is integrable on $(0,t)$ and since A^β is closed operator, by Lemma 6

$$\|A^\beta v(t)\| \le NB_3 \int_0^t (t-s)^{\alpha(1-\beta)-1} ds = \frac{NB_3}{\alpha(1-\beta)} t^{\alpha(1-\beta)} \le \frac{NB_3}{\alpha(1-\beta)} T^{\alpha(1-\beta)} < \infty \quad \text{for all } t \in [0,T].$$

Let $t \in [0,T]$ and $h > 0$ such that $t+h \in [0,T]$. Without loss of generality, we assume $0 < h < 1$.

$$\begin{aligned} A^\beta v(t+h) - A^\beta v(t) &= A^\beta \int_0^{t+h} T_\alpha(t+h-s)Pf(s,u_s)ds - A^\beta \int_0^t T_\alpha(t-s)Pf(s,u_s)ds \\ &= A^\beta \int_0^t [T_\alpha(t+h-s) - T_\alpha(t-s)]Pf(s,u_s)ds + \int_t^{t+h} T_\alpha(t+h-s)Pf(s,u_s)ds \\ &:= v_1 + v_2. \end{aligned}$$

Now, by Lemma 5

$$\begin{aligned} \|v_1\| &\le NB_4 \int_0^t [(t-s)^{\alpha(1-\beta)-1} - (t+h-s)^{\alpha(1-\beta)-1}] ds \\ &= \frac{NB_4}{\alpha(1-\beta)} [t^{\alpha(1-\beta)} - (t+h)^{\alpha(1-\beta)} + h^{\alpha(1-\beta)}] \\ &\le \frac{NB_4}{\alpha(1-\beta)} h^{\alpha(1-\beta)}. \end{aligned}$$

Also, using Lemma 2 we have

$$\begin{aligned}\|v_2\| &\le NB_3 \int_t^{t+h} (t+h-s)^{\alpha(1-\beta)-1} ds \\ &\le \frac{NB_3}{\alpha(1-\beta)} h^{\alpha(1-\beta)}.\end{aligned}$$

Hence, $A^\beta v \in C^{\alpha(1-\beta)}([0,T];L^2_\sigma(\Omega))$.

Lemma 8.*Let $0<\beta<1$. Define $w(t) = \int_0^t T_\alpha(t-s)Fu(s)ds$, $t \ge 0$, where $u \in C([-r,T];D(A^{\frac{3}{4}}))$. Then, $w(t) \in D(A^\beta)$ for all $t \in [0,T]$. Moreover, $A^\beta w \in C^{\alpha(1-\beta)}([0,T];L^2_\sigma(\Omega))$.*

*Proof.*According to the condition $\|u_t - \phi\|_{Y_{\frac{3}{4}}} \le R$ for all $t \in [0,T]$, Lemma 4 and using the property $D(A^{\frac{3}{4}}) \hookrightarrow D(A^{\frac{1}{2}})$, it is easy to check that $t \mapsto Fu(t)$ is bounded on $[0,T]$. Then, by following the similar arguments as in the proof of Lemma 7, we can show that $A^\beta w \in C^{\alpha(1-\beta)}([0,T];L^2_\sigma(\Omega))$.

Lemma 9.*Let $0<\beta<1$ and $x \in D(A)$. Consider $\Psi(t) = S_\alpha(t)x$, $t \ge 0$. Then, the map $t \mapsto \Psi(t)$ is Hölder continuous on $[0,T]$ with respect to $D(A^\beta)$-norm.*

*Proof.*Let $t \in [0,T]$ and $h>0$ such that $t+h \in [0,T]$. Without loss of generality, we assume $0<h<1$.
By Lemma 2 we have,

$$\begin{aligned}\|A^\beta\Psi(t+h) - A^\beta\Psi(t)\| &= \|A^\beta S_\alpha(t+h)x - A^\beta S_\alpha(t)x\| \\ &= \|A^\beta \int_t^{t+h} \frac{d}{d\tau}(S_\alpha(\tau)x)d\tau\| \\ &= \|A^\beta \int_t^{t+h} AT_\alpha(\tau)xd\tau\| \\ &= \|A^\beta \int_t^{t+h} T_\alpha(\tau)Axd\tau\| \\ &\le B_3 \int_t^{t+h} \tau^{\alpha(1-\beta)-1}\|Ax\|d\tau \\ &= \frac{B_3\|Ax\|}{\alpha(1-\beta)}[(t+h)^{\alpha(1-\beta)} - t^{\alpha(1-\beta)}] \\ &\le \frac{B_3\|Ax\|}{\alpha(1-\beta)} h^{\alpha(1-\beta)}.\end{aligned}$$

Hence, $\Psi \in C^{\alpha(1-\beta)}([0,T];D(A^\beta))$.

Proof.[Proof of the Theorem 4] If $u \in C([-r,T];D(A^{\frac{3}{4}}))$ is the mild solution of the Cauchy problem (2), then $u(t) = \Psi(t) + v(t) + w(t)$ for all $t \in [0,T]$. Therefore by Lemmas 7, 8, 9, the map $t \mapsto u(t)$ is Hölder continuous on $[0,T]$ with respect to $D(A^{\frac{3}{4}})$-norm.
According to the condition $\|u_t - \phi\|_{Y_{\frac{3}{4}}} \le R$ for all $t \in [0,T]$, the estimation in Lemma 4 and using the property $D(A^{\frac{3}{4}}) \hookrightarrow D(A^{\frac{1}{2}})$, it can be proved that $t \mapsto Fu(t)$ is Hölder continuous on $[0,T]$ with respect to $L^2_\sigma(\Omega)$-norm.
Now, we prove that $t \mapsto Pf(t,u_t)$ is Hölder continuous on $[0,T]$ in $L^2_\sigma(\Omega)$.

Let $t \in [0,T]$ and $h > 0$ be such that $t+h \in [0,T]$. Without loss of generality, we assume $0 < h < \min\{1,r\}$.

$$\begin{aligned}
\|u_{t+h} - u_t\|_{Y_{\frac{3}{4}}} &= \sup_{-r\le z\le 0} \|u(t+h+z) - u(t+z)\|_{D(A^{\frac{3}{4}})} \\
&\le \sup_{-r\le \tau\le t} \|u(h+\tau) - u(\tau)\|_{D(A^{\frac{3}{4}})} \\
&\le \sup_{-r\le \tau\le -h} \|u(h+\tau) - u(\tau)\|_{D(A^{\frac{3}{4}})} + \sup_{-h\le \tau\le 0} \|u(h+\tau) - u(\tau)\|_{D(A^{\frac{3}{4}})} \\
&\quad + \sup_{0\le \tau\le t} \|u(h+\tau) - u(\tau)\|_{D(A^{\frac{3}{4}})} \\
&\le \sup_{-r\le \tau\le -h} \|\phi(h+\tau) - \phi(\tau)\|_{D(A^{\frac{3}{4}})} + \sup_{-h\le \tau\le 0} \|u(h+\tau) - \phi(0)\|_{D(A^{\frac{3}{4}})} \\
&\quad + \sup_{-h\le \tau\le 0} \|\phi(0) - \phi(\tau)\|_{D(A^{\frac{3}{4}})} + \sup_{0\le \tau\le t} \|u(h+\tau) - u(\tau)\|_{D(A^{\frac{3}{4}})}.
\end{aligned}$$

Since $\phi \in C^{\theta}([-r,0];D(A^{\frac{3}{4}}))$ and $u \in C^{\alpha(1-\beta)}([0,T];D(A^{\frac{3}{4}}))$, therefore we have

$$\begin{aligned}
\sup_{-r\le \tau\le -h} \|\phi(h+\tau) - \phi(\tau)\|_{D(A^{\frac{3}{4}})} &\le L_1 h^{\theta}, \\
\sup_{-h\le \tau\le 0} \|u(h+\tau) - \phi(0)\|_{D(A^{\frac{3}{4}})} &\le L_2 (h+\tau)^{\alpha(1-\beta)} \le L_2 h^{\alpha(1-\beta)}, \\
\sup_{-h\le \tau\le 0} \|\phi(0) - \phi(\tau)\|_{D(A^{\frac{3}{4}})} &\le L_1 (-\tau)^{\theta} \le L_1 h^{\theta}, \\
\sup_{0\le \tau\le t} \|u(h+\tau) - u(\tau)\|_{D(A^{\frac{3}{4}})} &\le L_2 h^{\alpha(1-\beta)},
\end{aligned}$$

where, L_1, L_2, L_3, L_4 are positive constants. This shows that $t \mapsto u_t$ is Hölder continuous on $[0,T]$ in $Y_{\frac{3}{4}}$. Hence by assumption (H2), the map $t \mapsto Pf(t,u_t)$ is Hölder continuous on $[0,T]$ in a Banach space $L^2_{\sigma}(\Omega)$.
Thus, it is proved that the mild solution is a classical solution.

6 Conclusion

The existence of $D(A^{\frac{1}{2}})$-valued local mild solution has been established for a time-fractional NSE driven by finite delayed external forces by using Banach fixed point theorem when the initial datum belong to an open subset U of $D(A^{\frac{1}{2}})$. It is also proved that local mild solution can be continued globally if the initial datum curve belong to the whole space $U = D(A^{\frac{1}{2}})$. Regularity result has been demonstrated by considering more stronger initial datum curve and suitable assumption on forces.

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Conflicts of Interests
The authors declare that they have no conflicts of interests.

References

[1] T. Caraballo and J. Real, Navier-Stokes equations with delays, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457**(2014), 2441–2453 (2001).
[2] M. J. Garrido-Atienza and P. Marín-Rubio, Navier-Stokes equations with delays on unbounded domains, *Nonlinear Anal.* **64**(5), 1100–1118 (2006).
[3] S. Guzzo and G. Planas, Existence of solutions for a class of Navier-Stokes equations with infinite delay, *Appl. Anal.* **94**(4), 840–855 (2015).
[4] P. Marín Rubio, M. Márquez Durán and J. Valero Cuadra, Three dimensional system of globally modified navier-stokes equations with infinite delays, *Discr. Contin. Dyn. Syst. Ser. B.* **14**(2), 655-673 (2010).

[5] P. Marín-Rubio, J. Real and J. Valero, Pullback attractors for a two-dimensional Navier-Stokes model in an infinite delay case, *Nonlin. Anal.* **74**(5), 2012–2030 (2011).
[6] P. M. CarvalhoÑeto, *Fractional differential equations: a novel study of local and global solutions in Banach spaces*, PhD thesis, Universidade de São Paulo, 2013.
[7] K. Diethelm, *The analysis of fractional differential equations : An application-oriented exposition using differential operators of Caputo type*, Lecture Notes in Mathematics, Vol. **2004**, Springer-Verlag, Berlin, 2010.
[8] M. Sharma and S. Dubey, Controllability of Sobolev type nonlinear nonlocal fractional functional integro differential equations, *Prog. Fract. Differ. Appl.* **1**(4), 281–293 (2015).
[9] J. Leray, Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pures Appl.* **12**, 1–82 (1933).
[10] T. Kato and H. Fujita, On the nonstationary Navier-Stokes system, *Rend. Semin. Mat. Univ. Padova* **32**, 243–260 (1962).
[11] H. Fujita and T. Kato, On the Navier-Stokes initial value problem. I, *Arch. Rational Mech. Anal.* **16**, 269–315 (1964).
[12] Y. Giga, Weak and strong solutions of the navier-stokes initial value problem, *Publ. Res. Inst. Math. Sci.* **19**(3), 887–910 (1983).
[13] Y. Giga and T. Miyakawa, Solutions in L_r of the navier-stokes initial value problem, *Arch. Ration. Mech. Anal.* **89**(3), 267–281 (1985).
[14] T. Miyakawa, On the initial value problem for the Navier-Stokes equations in L^p spaces, *Hiroshima Math. J.* **11**(1), 9–20 (1981).
[15] F. B. Weissler, The Navier-Stokes initial value problem in L^p, *Arch. Ration. Mech. Anal.* **74**, 219–230 (1980).
[16] M. El-Shahed and A. Salem, On the generalized Navier–Stokes equations, *Appl. Math. Comput.* **156**(1), 287–293 (2004).
[17] S. Momani and Z. Odibat, Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method, *Appl. Math. Comput.* **177**(2), 488–494 (2006).
[18] Z. Odibat and S. Momani, Analytical spherically symmetric solution for the time-fractional Navier-Stokes equation, *Adv. Theor. Appl. Math.* **1** (2-3), 97–107 (2006).
[19] Z. Z. Ganji, D. D. Ganji, A. D. Ganji and M. Rostamian, Analytical solution of time-fractional Navier–Stokes equation in polar coordinate by homotopy perturbation method, *Numer. Meth. Part. Differ. Equ.* **26**(1), 117–124 (2010).
[20] P. M. Carvalho Neto, G. Planas, Mild solutions to the time fractional Navier–Stokes equations in $\mathbb{R}^N$, *J. Differ. Equ.* **259**(7), 2948–2980 (2015).
[21] Y. Zhou and L. Peng, On the time-fractional Navier-Stokes equations, *Comput. Math. Appl.* **73**(6), 874–891 (2017).
[22] Y. Wang and T. Liang, Mild solutions to the time fractional Navier-Stokes delay differential inclusions, *Discr. Contin. Dyn. Syst. Ser. B.* **24**(8), 3713-3740 (2019).
[23] E. Ouhabaz, *Analysis of heat equations on domains*, London Mathematical Society Monographs Series, Vol. **31**, Princeton Univ. Press. Princeton NJ. 2005.
[24] H. Sohr, *The Navier-Stokes equations: An elementary functional analytic approach*, Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2001.
[25] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, *Appl. Math. Sci.* **44**, Springer-Verlag, New York, 1983.
[26] B. Hendriya Guswanto, On the properties of solution operators of fractional evolution equations, *J. Fract. Calc. Appl.* **6**(1), 131–159 (2015).
[27] G. M. Lorenzi, A. Lunardi and D. Pallara, Analytic semigroups and reaction–diffusion equations, *Internate seminar 2004-2005*, http://www.math.unipr.it/~lunardi/LectureNotes/I-Sem2005.pdf.
[28] F. Mainardi, A. Mura and G. Pagnini, The m -wright function in time-fractional diffusion processes: A tutorial survey, *Int. J. Differ. Equ.* **2010**, Special Issue:29 pages (2010).
[29] C. Li and M. Li, Hölder regularity for abstract fractional cauchy problems with order in $(0,1)$, *J. Appl. Math. Phys.* **6**, 310–319 (2018).