Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/080201

Hadamard Inequality for $(k-r)$ Riemann-Liouville Fractional Integral Operator via Convexity

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Received: 3 Apr. 2020, Revised: 2 Jul. 2020, Accepted: 15 Jul. 2020
Published online: 1 Apr. 2022

Abstract: Recently, many researchers have published work on the Hermite-Hadamard inequalities, due to their immense importance in the fields of numerical analysis, statistics, optimization and convexity theory. In this paper, certain new Hermite-Hadamard type integral inequalities have been established using the $(k-r)$ Riemann-Liouville fractional integral operator. We present various inequalities based on different types of the convex functions such as quasi-convex, l-convex, η-convex in the second sense and (β,l)-convex functions. Also, we derive Hermite-Hadamard type inequalities for the product of two l-convex functions and two (β,l)-convex functions using $(k-r)$ Riemann-Liouville fractional integral operator. The results obtained in our work will be helpful in the further study of the convex functions and in the evaluation of the certain mathematical problems.

Keywords: $(k-r)$ Riemann-Liouville fractional integral, Hermite-Hadamard inequality, convex functions, quasi-convex functions, l-convex functions, η-convex in the second sense, (β,l)-convex functions.

1 Introduction and preliminaries

In the recent years, importance of the fractional calculus has increased extensively in the different fields such as inequality theory, applied mathematics, sciences and engineering. Fractional integrals are used for the description of the various properties of different physical processes like seepage flow in fluid dynamic traffic model and non-linear oscillations of earthquake.
Various integral inequalities have been obtained for the fractional integrals. These type of integrals can be used to generalize the important and known inequalities. Hermite-Hadamard inequality is one such type of inequality. It is extensively used in literature and provides necessary and sufficient conditions for a function to be convex. Hermite-Hadamard inequalities via Riemann-Liouville fractional integrals were generalized by Sarikaya [1]. Further, Sarikaya's results were extended to Hermite-Hadamard-Fejer type inequalities for the fractional integrals by Iscan [2]. Various authors have worked on different generalizations of convexity (see, e.g. [3,4,5,6]). Many generalized forms of the inequalities via fractional integrals have been discussed by various researchers (see, e.g. [7,8]). Inequalities for the different classes of the functions such as non-decreasing functions and convex functions have also been obtained [1,9].
In this paper, we use $(k-r)$ Riemann-Liouville fractional integral operator to establish certain new Hermite-Hadamard type integral inequalities for the different types of convex functions such as quasi-convex, l-convex, η-convex in the second sense and (β,l)-convex functions. Also, we derive Hermite-Hadamard type inequalities for the product of two l-convex functions and two (β,l)-convex functions.

We mention now, some definitions and basic elements useful for our study:

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Definition 1(Convex function [10]). *A function $h: D \to \mathbb{R}$, where D is an interval in $\mathbb{R}$ is said to be convex if the following inequality holds true for all $x, y \in D$*

$$h(\zeta x + (1-\zeta)y) \leq \zeta h(x) + (1-\zeta)h(y). \tag{1}$$

If the inequality in the equation (1) becomes strict inequality for all different points in D and $\zeta \in [0,1]$, then the function h is called strictly convex. Also, h is concave if $(-h)$ is convex.

Definition 2(Hermite-Hadamard inequality [11]). *Let $h: D \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $x, y \in D$, with $x < y$, then the inequality*

$$h\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y h(\zeta)d\zeta \leq \frac{h(x)+h(y)}{2}, \tag{2}$$

holds, and is known as Hermite-Hadamard integral inequality for the convex functions.

Definition 3(Quasi-convex function [12]). *The function $h: D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex function on D, if the inequality*

$$h(\zeta x + (1-\zeta)y) \leq sup[h(x), h(y)] \tag{3}$$

holds true for all $x, y \in D$ and $\zeta \in [0,1]$.

Definition 4(l-convex function [13]). *The function $h: [0,y] \to \mathbb{R}$ is said to be l-convex function, $l \in (0,1]$, if for $x_1, y_1 \in [0,y]$ and $\zeta \in (0,1)$, the inequality given below holds true*

$$h(\zeta x_1 + l(1-\zeta)y_1) \leq \zeta h(x_1) + l(1-\zeta)h(y_1). \tag{4}$$

If $l = 1$, l-convex functions coincide with the convex functions.

Definition 5(η -convex in the second sense [14]). *The function $h: \mathbb{R}^+ \to \mathbb{R}$ is called η-convex in the second sense if the inequality given below holds true for some fixed $\eta \in [0,1]$*

$$h(\Phi x + \Theta y) \leq \Phi^\eta h(x) + \Theta^\eta h(y), \tag{5}$$

where, $x, y \in [0,\infty)$, $\Phi, \Theta \geq 0$ and $\Phi + \Theta = 1$.

If $\eta = 1$, then η-convex functions reduce to the convex functions.

Definition 6((β, l)-convex function [15]). *The function $h: [0,y] \to \mathbb{R}$ is said to be (β, l)-convex function, $(\beta, l) \in (0,1] \times (0,1]$, if for $x_1, y_1 \in [0,y]$ and $\zeta \in (0,1)$, the following inequality holds true*

$$h(\zeta x_1 + l(1-\zeta)y_1) \leq \zeta^\beta h(x_1) + l(1-\zeta^\beta)h(y_1). \tag{6}$$

If $\beta = l = 1$, then (β, l)-convex functions coincide with convex functions.

Definition 7([16]). *A space of continuous real valued functions $h(\zeta)$ on $[x,y]$, denoted by $L_{r,s}[x,y]$ is given by*

$$\left(\int_x^y |h(\zeta)|^r \zeta^s d\zeta\right)^{\frac{1}{r}} < \infty, \tag{7}$$

where, $1 \leq r < \infty, s \geq 0$. Also, $L_{r,0}[x,y] = L_r[x,y]$ and $L_{1,0}[x,y] = L_1[x,y]$.

Definition 8(Beta function [17]). *The Euler's Beta function with $\Re(\rho_1), \Re(\rho_2) > 0$ where $\Re$ is the real part of the function is given by,*

$$B(\rho_1, \rho_2) = \int_0^1 t^{\rho_1 - 1}(1-t)^{\rho_2 - 1} dt. \tag{8}$$

The Pochhammer k-symbol $(y)_{p,k}$ is defined as (see, [18, Defn.1, p. 181]),

$$(y)_{p,k} = y(y+k)(y+2k)\ldots y+(p-1)k, \qquad (p \in \mathbb{N}_0, k > 0). \tag{9}$$

The $k-gamma$ function Γ_k is defined as (see, [18, Defn.3, p. 182]),

$$\Gamma_k(y) = \lim_{p\to\infty} \frac{p!k^p(pk)^{\frac{y}{k}-1}}{(y)_{p,k}}, \qquad (k>0, y\in\mathbb{C}\setminus k\mathbb{Z}_0^-, k\mathbb{Z}_0^- = [km : m\in\mathbb{Z}_0^-]). \tag{10}$$

For $k=1$, the equations (9) and (10), reduce to the Pochhammer symbol $(y)_p$ and the gamma function (see e.g., [17]), respectively, as follows:

$$(y)_p = \prod_{r=1}^{p}(y+r-1) = \frac{\Gamma(y+p)}{\Gamma(y)}, \quad p\in\mathbb{N}, y\neq 0,-1,-2,\ldots,$$

and

$$\Gamma(\varphi) = \int_0^\infty e^{-t}t^{\varphi-1}dt, \quad \Re(\varphi)>0.$$

Fractional calculus provides a powerful tool which has been recently employed to model real life problems. The derivatives and integrals of arbitrary order are used by many researchers and scientists to study various types of problems (see, e.g. [19,20,21,22,23]).

Definition 9(**$(k-r)$ Riemann-Liouville fractional integral [24]**). *Let h be a continuous function defined on a finite real interval $[x,y]$, that is, $h\in L_1[x,y]$ and $\Re(\alpha)>0$. Then, $(k-r)$ Riemann-Liouville fractional integral operator ${}^r_k\mathfrak{J}^\alpha_c$ of order $\alpha>0$ of the function h is defined as:*

$$\left({}^r_k\mathfrak{J}^\alpha_x h\right)(t) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^t \left[t^{r+1}-\zeta^{r+1}\right]^{\frac{\alpha}{k}-1}\zeta^r h(\zeta)d\zeta, \qquad (k>0, r\in\mathbb{R}, r\neq -1). \tag{11}$$

The left and right sided (k-r) Riemann-Liouville fractional integral operator are given by

$$\left({}^r_k\mathfrak{J}^\alpha_{x+} h\right)(t) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^t \left[t^{r+1}-\zeta^{r+1}\right]^{\frac{\alpha}{k}-1}\zeta^r h(\zeta)d\zeta, \tag{12}$$

$$\left({}^r_k\mathfrak{J}^\alpha_{y-} h\right)(t) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_t^y \left[\zeta^{r+1}-t^{r+1}\right]^{\frac{\alpha}{k}-1}\zeta^r h(\zeta)d\zeta, \tag{13}$$

respectively.

Special cases:

1.If $r=0$, $(k-r)$ Riemann-Liouville fractional integral reduces to k-Riemann-Liouville fractional integral [**?**, eq.8, p. 91]

$$\left({}_k\mathfrak{J}^\alpha_x h\right)(t) = \frac{1}{k\Gamma_k(\alpha)} \int_x^t [t-\zeta]^{\frac{\alpha}{k}-1} h(\zeta)d\zeta, \qquad (k>0). \tag{14}$$

Also, the left and right sided k-Riemann-Liouville fractional integrals of $h\in L_1[x,y]$, $k>0$ and $\Re(\alpha)>0$ are

$$\left({}_k\mathfrak{J}^\alpha_{x+} h\right)(t) = \frac{1}{k\Gamma_k(\alpha)} \int_x^t [t-\zeta]^{\frac{\alpha}{k}-1} h(\zeta)d\zeta, \tag{15}$$

$$\left({}_k\mathfrak{J}^\alpha_{y-} h\right)(t) = \frac{1}{k\Gamma_k(\alpha)} \int_t^y [\zeta-t]^{\frac{\alpha}{k}-1} h(\zeta)d\zeta, \tag{16}$$

respectively.

2.If $k=1$, $(k-r)$ Riemann-Liouville fractional integral reduces to Katugampola fractional integral [**?**], given by

$$\left({}^r\mathfrak{J}^\alpha_{x+} h\right)(t) = \frac{r^{1-\alpha}}{\Gamma(\alpha)} \int_x^t (t^r-\zeta^r)^{\alpha-1}\zeta^{r-1} h(\zeta)d\zeta, \qquad t>x, \tag{17}$$

$$\left({}^r\mathfrak{J}^\alpha_{y-} h\right)(t) = \frac{r^{1-\alpha}}{\Gamma(\alpha)} \int_t^y (\zeta^r-t^r)^{\alpha-1}\zeta^{r-1} g(\zeta)d\zeta, \qquad t<y, \tag{18}$$

where, $h\in L_1[x,y]$, $r\in\mathbb{R}, r\neq -1$ and $\Re(\alpha)>0$.

3.If $k=1, r=0$, $(k-r)$ Riemann-Liouville fractional integral reduces to Riemann-Liouville fractional integral [25,26], given by

$$\left(\mathfrak{I}_{x+}^{\alpha}h\right)(t)=\frac{1}{\Gamma(\alpha)}\int_{x}^{t}(t-\zeta)^{\alpha-1}h(\zeta)d\zeta, \qquad t>x, \tag{19}$$

$$\left(\mathfrak{I}_{y-}^{\alpha}h\right)(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{y}(\zeta-t)^{\alpha-1}h(\zeta)d\zeta, \qquad t<y, \tag{20}$$

where, $h\in L_1[x,y]$ and $\Re(\alpha)>0$.

In the next section, we obtain various inequalities via $(k-r)$ Riemann-Liouville fractional integral.

2 Main results

In this section, we derive inequalities for the $(k-r)$ Riemann-Liouville fractional integral of a function using convex, quasi-convex, l-convex, η-convex in the second sense and (β,l)-convex functions.

Lemma 1. *Let $h:D\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on D, where, $x,y\in D$ with $\zeta\in[0,1]$. If $h\in L_1[x,y]$, then, for all $x\le x_1<y_1\le y$, $k>0, r\in\mathbb{R}, r\neq -1$ and $\alpha>0$, we have,*

$$\begin{aligned}&\frac{h(x_1^r)+h(y_1^r)}{r(y_1^r-x_1^r)}-\frac{\alpha\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}+1}}\left[{}_k^r\mathfrak{I}_{x_1^+}^{\alpha}h(y_1^r)+{}_k^r\mathfrak{I}_{y_1^-}^{\alpha}h(x_1^r)\right]\\&=\int_0^1\zeta^{r\left(\frac{\alpha}{k}+1\right)-1}h'\left((1-\zeta^r)x_1^r+\zeta^r y_1^r\right)d\zeta-\int_0^1\zeta^{r\left(\frac{\alpha}{k}+1\right)-1}h'\left(\zeta^r x_1^r+(1-\zeta^r)y_1^r\right)d\zeta.\end{aligned} \tag{21}$$

Proof. Consider the integral,

$$I=\int_0^1\zeta^{\frac{\alpha r}{k}-1}h\left(\zeta^r x_1^r+(1-\zeta^r)y_1^r\right)d\zeta+\int_0^1\zeta^{\frac{\alpha r}{k}-1}h\left((1-\zeta^r)x_1^r+\zeta^r y_1^r\right)d\zeta. \tag{22}$$

Let $\zeta^r x_1^r+(1-\zeta^r)y_1^r=\mu^r$ and $(1-\zeta^r)x_1^r+\zeta^r y_1^r=\nu^r$, then equation (22) becomes,

$$\begin{aligned}I&=\frac{1}{(y_1^r-x_1^r)^{\frac{\alpha}{k}}}\int_{x_1}^{y_1}(y_1^r-\mu^r)^{\frac{\alpha}{k}-1}h(\mu^r)\mu^{r-1}d\mu+\frac{1}{(y_1^r-x_1^r)^{\frac{\alpha}{k}}}\int_{x_1}^{y_1}(\nu^r-x_1^r)^{\frac{\alpha}{k}-1}h(\nu^r)\nu^{r-1}d\nu\\&=\frac{k\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}}}{}_k^r\mathfrak{I}_{x_1^+}^{\alpha}h(y_1^r)+\frac{k\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}}}{}_k^r\mathfrak{I}_{y_1^-}^{\alpha}h(x_1^r).\end{aligned}$$

Therefore, we have,

$$\begin{aligned}&\int_0^1\zeta^{\frac{\alpha r}{k}-1}h\left(\zeta^r x_1^r+(1-\zeta^r)y_1^r\right)d\zeta+\int_0^1\zeta^{\frac{\alpha r}{k}-1}h\left((1-\zeta^r)x_1^r+\zeta^r y_1^r\right)d\zeta\\&=\frac{k\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}}}\left[{}_k^r\mathfrak{I}_{x_1^+}^{\alpha}h(y_1^r)+{}_k^r\mathfrak{I}_{y_1^-}^{\alpha}h(x_1^r)\right].\end{aligned} \tag{23}$$

Using integration by parts on the LHS of the equation (23), we get,

$$\begin{aligned}LHS=&\frac{k\left[h(x_1^r)+h(y_1^r)\right]}{\alpha r}\\&+\frac{k\left(y_1^r-x_1^r\right)}{\alpha}\left[\int_0^1\zeta^{r\left(\frac{\alpha}{k}+1\right)-1}\left(h'\left(\zeta^r x_1^r+(1-\zeta^r)y_1^r\right)-h'\left((1-\zeta^r)x_1^r+\zeta^r y_1^r\right)\right)d\zeta\right].\end{aligned}$$

Substituting the value of LHS in the equation (23) and simplifying, we get,

$$\begin{aligned}&\frac{[h(x_1^r)+h(y_1^r)]}{r}-\frac{\alpha\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}}}\left[{}_k^r\mathfrak{I}_{x_1^+}^{\alpha}h(y_1^r)+{}_k^r\mathfrak{I}_{y_1^-}^{\alpha}h(x_1^r)\right]\\&=(y_1^r-x_1^r)\int_0^1\zeta^{r\left(\frac{\alpha}{k}+1\right)-1}h'\left((1-\zeta^r)x_1^r+\zeta^r y_1^r\right)d\zeta-\int_0^1\zeta^{r\left(\frac{\alpha}{k}+1\right)-1}h'\left(\zeta^r x_1^r+(1-\zeta^r)y_1^r\right)d\zeta.\end{aligned}$$

$$\Longrightarrow \frac{h(x_1^r)+h(y_1^r)}{r(y_1^r-x_1^r)}-\frac{\alpha\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}+1}}\left[{}_k^r\mathfrak{J}_{x_1^+}^{\alpha}h(y_1^r)+{}_k^r\mathfrak{J}_{y_1^-}^{\alpha}h(x_1^r)\right]$$
$$=\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}h'((1-\zeta^r)x_1^r+\zeta^r y_1^r)\,d\zeta-\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}h'(\zeta^r x_1^r+(1-\zeta^r)y_1^r)\,d\zeta.$$

Thus, we get our required result (21).

The following theorem is based on $\eta-convexity$ of the function h' in the second sense and establishes the relationship between $(k-r)$ Riemann fractional integral and Beta function $B(\rho_1,\rho_2)$.

Theorem 1. *Let $h: D\subseteq\mathbb{R}\to\mathbb{R}$ be a positive function which is differentiable on D, with $x,y\in D$ and $\zeta\in[0,1]$. Let $h'\in L_1[x_1,y_1]$, then for all $x\le x_1<y_1\le y$, $\alpha,k>0$ and $r\in\mathbb{R}, r\neq-1$. If h' is $\eta-convex$ in the second sense on $[x_1,y_1]$ and $\eta\in[0,1]$, then the inequality given below holds true*

$$\begin{aligned}&\frac{h(x_1^r)+h(y_1^r)}{r(y_1^r-x_1^r)}-\frac{\alpha\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}+1}}\left[{}_k^r\mathfrak{J}_{x_1^+}^{\alpha}h(y_1^r)+{}_k^r\mathfrak{J}_{y_1^-}^{\alpha}h(x_1^r)\right]\\&\le\left[h'(x_1^r)+h'(y_1^r)\right]\left[\frac{1}{r}B(\frac{\alpha}{k}+1,\eta+1)+\frac{k}{r[k(\eta+1)+\alpha]}\right].\end{aligned}\tag{24}$$

Proof. From Lemma 1, we have,

$$\begin{aligned}&\frac{h(x_1^r)+h(y_1^r)}{r(y_1^r-x_1^r)}-\frac{\alpha\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}+1}}\left[{}_k^r\mathfrak{J}_{x_1^+}^{\alpha}h(y_1^r)+{}_k^r\mathfrak{J}_{y_1^-}^{\alpha}h(x_1^r)\right]\\&=\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}h'((1-\zeta^r)x_1^r+\zeta^r y_1^r)\,d\zeta-\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}h'(\zeta^r x_1^r+(1-\zeta^r)y_1^r)\,d\zeta.\end{aligned}\tag{25}$$

Since h' is $\eta-convex$, from (5) we have,

$$\begin{aligned}I_1&=\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}h'((1-\zeta^r)x_1^r+\zeta^r y_1^r)\,d\zeta\\&\le\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}\left[(1-\zeta^r)^{\eta}h'(x_1^r)+\zeta^{r\eta}h'(y_1^r)\right]d\zeta\\&=h'(x_1^r)\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}(1-\zeta^r)^{\eta}d\zeta+h'(y_1^r)\int_0^1\zeta^{r(\frac{\alpha}{k}+\eta+1)-1}d\zeta\\&=\frac{h'(x_1^r)}{r}B\left(\frac{\alpha}{k}+1,\eta+1\right)+h'(y_1^r)\frac{k}{r[k(\eta+1)+\alpha]}.\end{aligned}\tag{26}$$

Similarly,

$$\begin{aligned}I_2&=-\left[\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}h'(\zeta^r x_1^r+(1-\zeta^r)y_1^r)\,d\zeta\right]\\&\le\left[\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}h'(\zeta^r x_1^r+(1-\zeta^r)y_1^r)\,d\zeta\right]\\&\le\frac{h'(y_1^r)}{r}B\left(\frac{\alpha}{k}+1,\eta+1\right)+h'(x_1^r)\frac{k}{r[k(\eta+1)+\alpha]}.\end{aligned}\tag{27}$$

Substituting equations (26) and (27) in (25), we get the desired result (24).

If the function h is quasi-convex, then we have the following result.

Theorem 2. *let $h:[x,y]\to\mathbb{R}$ be a positive function. Let $h\in L_1[x,y]$ be quasi-convex on $[x,y]$, then the inequality given below holds true for $(k-r)$ Riemann-Liouville fractional integral*

$$\frac{\alpha\Gamma_k(\alpha)r^{\frac{\alpha}{k}}}{2(y^r-x^r)^{\frac{\alpha}{k}}}\left[{}_k^r\mathfrak{J}_{x+}^{\alpha}h(y^r)+{}_k^r\mathfrak{J}_{y-}^{\alpha}h(x^r)\right]\le\sup\left[h(x^r),h(y^r)\right],\tag{28}$$

where, $\alpha>0, k>0$ and $r\in\mathbb{R}, r\neq-1$.

*Proof.*By quasi-convexity of the function h on $[x,y]$, for $\zeta \in [0,1]$, we have,

$$\frac{1}{2}\left[h\left(\zeta^r x^r + (1-\zeta^r)y^r\right) + h\left((1-\zeta^r)x^r + \zeta^r y^r\right)\right] \le \sup\left[h(x^r), h(y^r)\right]. \tag{29}$$

Multiplying both sides of the equation (29) by $\zeta^{\frac{\alpha r}{k}-1}$ and integrating with respect ζ over $[0,1]$, we get,

$$\frac{1}{2}\int_0^1 \zeta^{\frac{\alpha r}{k}-1}\left[h\left(\zeta^r x^r + (1-\zeta^r)y^r\right) + h\left((1-\zeta^r)x^r + \zeta^r y^r\right)\right]d\zeta \le \int_0^1 \zeta^{\frac{\alpha r}{k}-1}\sup\left[h(x^r), h(y^r)\right].$$

Following the procedure similar to the Lemma 1, we get,

$$\Longrightarrow \frac{1}{2}\frac{k\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y^r-x^r)^{\frac{\alpha}{k}}}\left[{}^r_k\mathfrak{I}^{\alpha}_{x^r+}h(y^r) + {}^r_k\mathfrak{I}^{\alpha}_{y-}h(x^r)\right] \le \frac{k}{\alpha r}\sup\left[h(x^r), h(y^r)\right].$$

On simplifying, we get our desired result (28).

If h is a differentiable function and $|h'|^n$ is quasi-convex, then we have the following result:

Theorem 3.*If $h:[x,y]\to\mathbb{R}$ is a differentiable mapping on (x,y) with $x<y$ such that $h'\in L_1[x,y]$. If $|h'|^n$ is quasi-convex on $[x,y]$ and $m>1$, then the inequality given below holds true:*

$$\begin{aligned}&\left|\frac{h(x_1^r)+h(y_1^r)}{r(y_1^r-x_1^r)} - \frac{\alpha\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}+1}}\left[{}^r_k\mathfrak{I}^{\alpha}_{x_1+}h(y_1^r) + {}^r_k\mathfrak{I}^{\alpha}_{y_1-}h(x_1^r)\right]\right| \\ &\le 2\left[\frac{k}{mr\alpha + k(mr-m+1)}\right]^{\frac{1}{m}}\left[\sup\left(|h'(x_1^r)|^n, |h'(y_1^r)|^n\right)\right]^{\frac{1}{n}},\end{aligned} \tag{30}$$

where, $x\le x_1<y_1\le y$, $\frac{1}{m}+\frac{1}{n}=1$, $\alpha,k>0, r\in\mathbb{R}, r\neq -1$ and $m,n\in[1,\infty)$.

*Proof.*Taking modulus on both sides of (21), and using Holder's inequality thereupon, we get,

$$\begin{aligned}&\left|\frac{h(x_1^r)+h(y_1^r)}{r(y_1^r-x_1^r)} - \frac{\alpha\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}+1}}\left[{}^r_k\mathfrak{I}^{\alpha}_{x_1+}h(y_1^r) + {}^r_k\mathfrak{I}^{\alpha}_{y_1-}h(x_1^r)\right]\right| \\ &= \left|\int_0^1 \zeta^{r\left(\frac{\alpha}{k}+1\right)-1}h'\left((1-\zeta^r)x_1^r+\zeta^r y_1^r\right)d\zeta - \int_0^1 \zeta^{r\left(\frac{\alpha}{k}+1\right)-1}h'\left(\zeta^r x_1^r+(1-\zeta^r)y_1^r\right)d\zeta\right| \\ &\le \left|\int_0^1 \zeta^{r\left(\frac{\alpha}{k}+1\right)-1}h'\left((1-\zeta^r)x_1^r+\zeta^r y_1^r\right)d\zeta\right| + \left|\int_0^1 \zeta^{r\left(\frac{\alpha}{k}+1\right)-1}h'\left(\zeta^r x_1^r+(1-\zeta^r)y_1^r\right)d\zeta\right| \\ &\le \left[\int_0^1\left(\zeta^{r\left(\frac{\alpha}{k}+1\right)-1}\right)^m d\zeta\right]^{\frac{1}{m}}\left[\int_0^1\left(h'\left((1-\zeta^r)x_1^r+\zeta^r y_1^r\right)\right)^n d\zeta\right]^{\frac{1}{n}} \\ &+ \left[\int_0^1\left(\zeta^{r\left(\frac{\alpha}{k}+1\right)-1}\right)^m d\zeta\right]^{\frac{1}{m}}\left[\int_0^1\left(h'\left(\zeta^r x_1^r+(1-\zeta^r)y_1^r\right)\right)^n d\zeta\right]^{\frac{1}{n}}.\end{aligned}$$

Since $|h'|^n$ is quasi-convex, we have,

$$\begin{aligned}&\left|\frac{h(x_1^r)+h(y_1^r)}{r(y_1^r-x_1^r)} - \frac{\alpha\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y_1^r-x_1^r)^{\frac{\alpha}{k}+1}}\left[{}^r_k\mathfrak{I}^{\alpha}_{x_1+}h(y_1^r) + {}^r_k\mathfrak{I}^{\alpha}_{y_1-}h(x_1^r)\right]\right| \\ &\le 2\left[\frac{k}{mr\alpha + k(mr-m+1)}\right]^{\frac{1}{m}}\left[\sup\left(|h'(x_1^r)|^n, |h'(y_1^r)|^n\right)\right]^{\frac{1}{n}}.\end{aligned}$$

Hence, we get our desired result (30).

We now obtain Hermite-Hadamard type inequalities for the product of two functions. When the functions f and g are $l_1-convex$ and $l_2-convex$ respectively, we have the following result.

Theorem 4. *Let $f,g:\mathbb{R}\to\mathbb{R}$, $0<x<y$ be functions such that $fg\in L_1[x,y]$. If f is $l_1-convex$ and g is $l_2-convex$ on $[x,y]$ with $l_1,l_2\in(0,1]$, then the following inequalities hold true for $\alpha,k>0, r\in\mathbb{R}, r\neq -1$:*

$$\begin{aligned}(a)\quad &\frac{k\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y^r-x^r)^{\frac{\alpha}{k}}}\,{}^r_k\mathfrak{J}^{\alpha}_{x^+}\left(f(y^r)g(y^r)\right)\\ &\leq\frac{k}{r(\alpha+2k)}\left[f(x^r)g(x^r)\right]+\frac{k^2}{r(\alpha+k)(\alpha+2k)}\left[l_2 f(x^r)g\left(\frac{y^r}{l_2}\right)+l_1 g(x^r)f\left(\frac{y^r}{l_1}\right)\right]\\ &+\frac{2l_1l_2k^3}{r\alpha(\alpha+k)(\alpha+2k)}\left[f\left(\frac{y^r}{l_1}\right)g\left(\frac{y^r}{l_2}\right)\right].\end{aligned}\tag{31}$$

$$\begin{aligned}(b)\quad &\frac{k\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y^r-x^r)^{\frac{\alpha}{k}}}\,{}^r_k\mathfrak{J}^{\alpha}_{y^-}\left(f(x^r)g(x^r)\right)\\ &\leq\frac{k}{r(\alpha+2k)}\left[f(y^r)g(y^r)\right]+\frac{k^2}{r(\alpha+k)(\alpha+2k)}\left[l_2 f(y^r)g\left(\frac{x^r}{l_2}\right)+l_1 g(y^r)f\left(\frac{x^r}{l_1}\right)\right]\\ &+\frac{2l_1l_2k^3}{r\alpha(\alpha+k)(\alpha+2k)}\left[f\left(\frac{x^r}{l_1}\right)g\left(\frac{x^r}{l_2}\right)\right].\end{aligned}\tag{32}$$

Proof.(a) Since f is $l_1-convex$ and g is $l_2-convex$, for $\zeta\in[0,1]$, we have

$$f(\zeta^r x^r+(1-\zeta^r)y^r)\leq\zeta^r f(x^r)+l_1(1-\zeta^r)f\left(\frac{y^r}{l_1}\right).\tag{33}$$

$$g(\zeta^r x^r+(1-\zeta^r)y^r)\leq\zeta^r g(x^r)+l_2(1-\zeta^r)g\left(\frac{y^r}{l_2}\right).\tag{34}$$

From the inequalities (33) and (34), we have,

$$\begin{aligned}&f(\zeta^r x^r+(1-\zeta^r)y^r)g(\zeta^r x^r+(1-\zeta^r)y^r)\\ &\leq\zeta^{2r}f(x^r)g(x^r)+l_2(1-\zeta^r)\zeta^r f(x^r)g\left(\frac{y^r}{l_2}\right)+l_1(1-\zeta^r)\zeta^r g(x^r)f\left(\frac{y^r}{l_1}\right)\\ &+l_1l_2(1-\zeta^r)^2 f\left(\frac{y^r}{l_2}\right)g\left(\frac{y^r}{l_2}\right).\end{aligned}\tag{35}$$

Multiplying both sides of the equation (35) by $\zeta^{\frac{\alpha r}{k}-1}$ and integrating with respect to ζ from $[0,1]$, we get,

$$\begin{aligned}&\frac{k\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y^r-x^r)^{\frac{\alpha}{k}}}\,{}^r_k\mathfrak{J}^{\alpha}_{x^+}\left(f(y^r)g(y^r)\right)\\ &\leq f(x^r)g(x^r)\int_0^1\zeta^{r(\frac{\alpha}{k}+2)-1}d\zeta+l_2f(x^r)g\left(\frac{y^r}{l_2}\right)\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}(1-\zeta^r)d\zeta\\ &+l_1g(x^r)f\left(\frac{y^r}{l_1}\right)\int_0^1\zeta^{r(\frac{\alpha}{k}+1)-1}(1-\zeta^r)d\zeta+l_1l_2f\left(\frac{y^r}{l_2}\right)g\left(\frac{y^r}{l_2}\right)\int_0^1\zeta^{\frac{\alpha r}{k}-1}(1-\zeta^r)^2d\zeta\\ &=\frac{k}{r(\alpha+2k)}\left[f(x^r)g(x^r)\right]+\frac{k^2}{r(\alpha+k)(\alpha+2k)}\left[l_2f(x^r)g\left(\frac{y^r}{l_2}\right)\right]\\ &+\frac{k^2}{r(\alpha+k)(\alpha+2k)}\left[l_1g(x^r)f\left(\frac{y^r}{l_1}\right)\right]+\frac{2l_1l_2k^3}{r\alpha(\alpha+k)(\alpha+2k)}\left[f\left(\frac{y^r}{l_1}\right)g\left(\frac{y^r}{l_2}\right)\right]\end{aligned}$$

Hence we get our desired result (31).

(b) Similarly,

$$f(\zeta^r y^r+(1-\zeta^r)x^r)\leq\zeta^r f(y^r)+l_1(1-\zeta^r)f\left(\frac{x^r}{l_1}\right).\tag{36}$$

$$g(\zeta^r y^r + (1-\zeta^r)x^r) \leq \zeta^r g(y^r) + l_2(1-\zeta^r) g\left(\frac{x^r}{l_2}\right). \tag{37}$$

From the inequalities (36) and (37), we have,

$$\begin{aligned} & f(\zeta^r y^r + (1-\zeta^r)x^r)\, g(\zeta^r y^r + (1-\zeta^r)x^r) \\ \leq & \zeta^{2r} f(y^r)g(y^r) + l_2(1-\zeta^r)\zeta^r f(y^r) g\left(\frac{x^r}{l_2}\right) + l_1(1-\zeta^r)\zeta^r g(y^r) f\left(\frac{x^r}{l_1}\right) \\ & + l_1 l_2 (1-\zeta^r)^2 f\left(\frac{x^r}{l_2}\right) g\left(\frac{x^r}{l_2}\right). \end{aligned} \tag{38}$$

Multiplying both sides of the equation (38) by $\zeta^{\frac{\alpha r}{k}-1}$ and integrating with respect to ζ from $[0,1]$, we get our desired result (32).

Corollary 1. *Let $f,g : \mathbb{R} \to \mathbb{R}$, $0 < x < y$ be convex functions such that $fg \in L_1[x,y]$. Then the inequalities given below hold true for $\alpha, k > 0, r \in \mathbb{R}, r \neq -1$:*

$$\begin{aligned} (a) \quad & \frac{k\Gamma_k(\alpha) r^{\frac{\alpha}{k}-1}}{(y^r - x^r)^{\frac{\alpha}{k}}} {}^r_k\mathfrak{I}^{\alpha}_{x^+} (f(y^r)g(y^r)) \\ \leq & \frac{k}{r(\alpha+2k)} [f(x^r)g(x^r)] + \frac{k^2}{r(\alpha+k)(\alpha+2k)} [f(x^r)g(y^r) + g(x^r)f(y^r)] \\ & + \frac{k^3}{r\alpha(\alpha+k)(\alpha+2k)} [f(y^r)g(y^r)]. \end{aligned} \tag{39}$$

$$\begin{aligned} (b) \quad & \frac{k\Gamma_k(\alpha) r^{\frac{\alpha}{k}-1}}{(y^r - x^r)^{\frac{\alpha}{k}}} {}^r_k\mathfrak{I}^{\alpha}_{y^-} (f(x^r)g(x^r)) \\ \leq & \frac{k}{r(\alpha+2k)} [f(y^r)g(y^r)] + \frac{k^2}{r(\alpha+k)(\alpha+2k)} [f(y^r)g(x^r) + g(y^r)f(x^r)] \\ & + \frac{k^3}{r\alpha(\alpha+k)(\alpha+2k)} [f(x^r)g(x^r)]. \end{aligned} \tag{40}$$

Proof. Substituting $l_1 = l_2 = 1$ in the equations (31) and (32) of the Theorem 4, we get (39) and (40), respectively.

For the (β_1, l_1)-convex and (β_2, l_2)-convex functions f and g, respectively, we have the following theorem:

Theorem 5. *Let $f,g : \mathbb{R} \to \mathbb{R}$, $0 < x < y$ be functions such that $fg \in L_1[x,y]$. If f is $(\beta_1, l_1)-$convex and g is $(\beta_2, l_2)-$convex on $[x,y]$ with $(\beta_1, l_1), (\beta_2, l_2) \in (0,1] \times (0,1]$, then the inequalities given below hold true for $\alpha, k > 0, r \in \mathbb{R}, r \neq -1$:*

$$\begin{aligned} (a) \quad & \frac{k\Gamma_k(\alpha) r^{\frac{\alpha}{k}-1}}{(y^r - x^r)^{\frac{\alpha}{k}}} {}^r_k\mathfrak{I}^{\alpha}_{x^+} (f(y^r)g(y^r)) \\ \leq & \frac{k}{r(\alpha+\beta_1 k+\beta_2 k)} [f(x^r)g(x^r)] + \frac{rk^2\beta_2}{(\alpha r + rk\beta_1)(\alpha r + rk\beta_1 + rk\beta_2)} \left[l_2 f(x^r) g\left(\frac{y^r}{l_2}\right)\right] \\ & + \frac{rk^2\beta_1}{(\alpha r + rk\beta_2)(\alpha r + rk\beta_1 + rk\beta_2)} \left[l_1 g(x^r) f\left(\frac{y^r}{l_1}\right)\right] \\ & + \left[l_1 l_2 f\left(\frac{y^r}{l_1}\right) g\left(\frac{y^r}{l_2}\right)\right] \left[\frac{k}{\alpha r} - \frac{k}{(\alpha r + kr\beta_2)} - \frac{k}{(\alpha r + kr\beta_1)} + \frac{k}{(\alpha r + kr\beta_1 + kr\beta_2)}\right]. \end{aligned} \tag{41}$$

$$(b)\quad \frac{k\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y^r-x^r)^{\frac{\alpha}{k}}}\,{}^r_k\mathfrak{I}^{\alpha}_{y^-}\left(f(x^r)g(x^r)\right)$$
$$\leq \frac{k}{r(\alpha+\beta_1 k+\beta_2 k)}\left[f(y^r)g(y^r)\right]+\frac{rk^2\beta_2}{(\alpha r+rk\beta_1)(\alpha r+rk\beta_1+rk\beta_2)}\left[l_2 f(y^r)g\left(\frac{x^r}{l_2}\right)\right]$$
$$+\frac{rk^2\beta_1}{(\alpha r+rk\beta_2)(\alpha r+rk\beta_1+rk\beta_2)}\left[l_1 g(y^r)f\left(\frac{x^r}{l_1}\right)\right]$$
$$+\left[l_1 l_2 f\left(\frac{x^r}{l_1}\right)g\left(\frac{x^r}{l_2}\right)\right]\left[\frac{k}{\alpha r}-\frac{k}{(\alpha r+kr\beta_2)}-\frac{k}{(\alpha r+kr\beta_1)}+\frac{k}{(\alpha r+kr\beta_1+kr\beta_2)}\right]. \quad (42)$$

Proof.(a) Since f is $(\beta_1,l_1)-convex$ and g is $(\beta_2,l_2)-convex$, for $\zeta\in[0,1]$ we have

$$f(\zeta^r x^r+(1-\zeta^r)y^r)\leq \zeta^{r\beta_1}f(x^r)+l_1(1-\zeta^{r\beta_1})f\left(\frac{y^r}{l_1}\right). \quad (43)$$

$$g(\zeta^r x^r+(1-\zeta^r)y^r)\leq \zeta^{r\beta_2}g(x^r)+l_2(1-\zeta^{r\beta_2})g\left(\frac{y^r}{l_2}\right). \quad (44)$$

Multiplying the inequalities (43) and (44), we get

$$f(\zeta^r x^r+(1-\zeta^r)y^r)\,g(\zeta^r x^r+(1-\zeta^r)y^r)$$
$$\leq \zeta^{r\beta_1+r\beta_2}f(x^r)g(x^r)+l_2(1-\zeta^{r\beta_2})\zeta^{r\beta_1}f(x^r)g\left(\frac{y^r}{l_2}\right)+l_1(1-\zeta^{r\beta_1})\zeta^{r\beta_2}g(x^r)f\left(\frac{y^r}{l_1}\right)$$
$$+l_1 l_2(1-\zeta^{r\beta_1})(1-\zeta^{r\beta_2})f\left(\frac{y^r}{l_2}\right)g\left(\frac{y^r}{l_2}\right). \quad (45)$$

Multiplying both sides of equation (45) by $\zeta^{\frac{\alpha r}{k}-1}$ and integrating with respect to ζ on $[0,1]$, we get,

$$\frac{k\Gamma_k(\alpha)r^{\frac{\alpha}{k}-1}}{(y^r-x^r)^{\frac{\alpha}{k}}}\,{}^r_k\mathfrak{I}^{\alpha}_{x^+}\left(f(y^r)g(y^r)\right)$$
$$\leq f(x^r)g(x^r)\int_0^1\zeta^{r(\frac{\alpha}{k}+\beta_1+\beta_2)-1}d\zeta+l_2 f(x^r)g\left(\frac{y^r}{l_2}\right)\int_0^1\zeta^{r(\frac{\alpha}{k}+\beta_1)-1}(1-\zeta^{r\beta_2})d\zeta$$
$$+l_1 g(x^r)f\left(\frac{y^r}{l_1}\right)\int_0^1\zeta^{r(\frac{\alpha}{k}+\beta_2)-1}(1-\zeta^{r\beta_1})d\zeta$$
$$+l_1 l_2 f\left(\frac{y^r}{l_2}\right)g\left(\frac{y^r}{l_2}\right)\int_0^1\zeta^{\frac{\alpha r}{k}-1}(1-\zeta^{r\beta_1})(1-\zeta^{r\beta_2})d\zeta.$$

On simplification, we get our desired result (41).

(b) Similarly,

$$f(\zeta^r y^r+(1-\zeta^r)x^r)\leq \zeta^{r\beta_1}f(y^r)+l_1(1-\zeta^{r\beta_1})f\left(\frac{x^r}{l_1}\right). \quad (46)$$

$$g(\zeta^r y^r+(1-\zeta^r)x^r)\leq \zeta^{r\beta_2}g(y^r)+l_2(1-\zeta^{r\beta_2})g\left(\frac{x^r}{l_2}\right). \quad (47)$$

From inequalities (46) and (47), we have,

$$f(\zeta^r y^r+(1-\zeta^r)x^r)\,g(\zeta^r y^r+(1-\zeta^r)x^r)$$
$$\leq \zeta^{r\beta_1+r\beta_2}f(y^r)g(y^r)+l_2(1-\zeta^{r\beta_2})\zeta^{r\beta_1}f(y^r)g\left(\frac{x^r}{l_2}\right)+l_1(1-\zeta^{r\beta_1})\zeta^{r\beta_2}g(y^r)f\left(\frac{x^r}{l_1}\right)$$
$$+l_1 l_2(1-\zeta^{r\beta_1})(1-\zeta^{r\beta_2})f\left(\frac{x^r}{l_2}\right)g\left(\frac{x^r}{l_2}\right). \quad (48)$$

Multiplying both sides of the equation (48) by $\zeta^{\frac{\alpha r}{k}-1}$ and integrating with respect to ζ from $[0,1]$, we get our desired result (42).

3 Conclusion

In this paper, we begin with the review of fractional integral operators and Hermite-Hadamard inequality. We generalize Hermite-Hadamard inequality using $(k-r)$ Riemann-Liouville fractional integral operator. Our work includes several Hermite-Hadamard inequalities for convex, quasi-convex, l-convex, η-convex in the second sense and (β,l)-convex functions. Also, we have obtained the inequalities for the product of two l-convex and two (β,l)-convex functions. We observed that, under certain conditions the inequalities for the product of two l-convex and two (β,l)-convex functions reduce to the inequalities for the product of the convex functions. Also, some of the results obtained in this paper are generalizations of some already known results [12,27]. Various authors have worked on the inequalities for different types of functions and fractional operators. Rashid et al. [28] discussed Hermite-Hadamard and Ostrowski type inequalities for n-polynomials, s-type convex functions by employing k-fractional integral operators and studied the quadrature rules that are helpful in fractal theory, optimization and machine learning. Hermite-Hadamard inequalities for the differentiable exponentially convex and exponential quasi-convex functions, which were applied to numerical analysis and statistics, have also been discussed in the recent years [29]. Inequalities for the p-th order differentiation useful in the Banach Spaces are also obtained [30]. Further, Gr̈uss type inequalities for the generalized k-fractional integral operator are discussed and applied in the real world mathematical problems [31,32]. As future scope, these results can be extended using $(k-r)$ Riemann-Liouville fractional integral operator and applications can be found in the fractal theory, machine learning, numerical analysis, statistics and various others.

Conflicts of Interests
The authors declare that they have no conflicts of interests.

References

[1] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.* **57**, 2403-2407 (2013).
[2] I. Iscan, Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals, *Stud. Univ. Babes-Bolyai Math.* **60**(3), 355-366 (2015).
[3] G. Adilov and A. M. Rubinov, B-Convex sets and functions, *Numer. Funct. Anal. Optim.* **27**(3-4), 237-257 (2006).
[4] G. Adilov and G. Tnaztepe, The sharpening of some inequalities via abstract convexity, *Math. Ineq. Appl.* **12**, 33-51 (2009).
[5] G. Adilov and I. Yesilce, On generalization of the concept of convexity, *Hacet. J. Math. Stat.* **41**(5), 723-730 (2012).
[6] S. Jain, K. Mehrez, D. Baleanu and P. Agarwal, Certain Hermite–Hadamard inequalities for logarithmically convex functions with applications, *Mathematics* **7**(2), 163 (2019).
[7] P. Agarwal, Some inequalities involving Hadamard-type k-fractional integral operators, *Math. Meth. Appl. Sci.* **40**(11), 3882-3891 (2017).
[8] P. Agarwal, M. Jleli and M. Tomar, Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals, *J. Ineq. Appl.* **2017**(1), 55 (2017).
[9] J. Wang, X. Li and C. Zhu, Refinements of Hermite-Hadamard type inequalities involving fractional integrals, *Bull. Belg. Math. Soc. Simon Stevin* **20**, 655-666 (2013).
[10] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, reprint of 1952 edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988.
[11] J. Hadamard, Étude sur les propriétés des fonctions entiéres et en particulier dúne fonction considérée par Riemann, *J. Math. Pures Appl.* **9**, 171–215 (1893).
[12] M. E. Ozdemir and C. Yildiz, The Hadamard's inequality for quasi-convex functions via fractional integrals, *Ann. Univ. Craiova Math. Comput. Sci. Ser.* **40**(2), 167-173 (2013).
[13] G. H. Toader, Some generalizations of the convexity, *Proc. Colloq. Approx. Opt., Univ. Cluj-Napoca, Cluj-Napoca*, 329-338 (1985).
[14] H. Hudzik and S. Maligranda, Some remarks on s-convex functions, *Aequationes Math.* **48**, 100-111 (1994).
[15] V. G. Mihesan, *A generalization of the convexity*, Seminar on functional equations, approximation and convexity, Cluj-Napoca, Romania, 1993.
[16] S. Mubeen and S. Iqbal, Grüss type integral inequalities for generalized Riemann Liouville k-fractional integrals, *J. Ineq. Appl.* **109**, 13 (2016).
[17] E. D Rainville, *Special functions*, Chelsea Publications, Bronx, New York, 1971.
[18] R. Diaz and E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, *Divulgacion. Mat.* **15**(2), 179-192 (2007).
[19] M. P. Yadav and R. Agarwal, Numerical investigation of fractional-fractal Boussinesq equation, *Chaos* **29**(1), 013109, (2019).
[20] R. Agarwal, Kritika and S. D Purohit, A mathematical fractional model with non-singular kernel for thrombin receptor activation in calcium signalling, *Math. Meth. Appl. Sci.* **42**, 7160-7171 (2019).
[21] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley Interscience, 1993.

[22] K. S. Nisar, D. L. Suthar, R. Agarwal and S. D. Purohit, Fractional calculus operators with Appell function kernels applied to Srivastava polynomials and extended Mittag-Leffler function, *Adv. Differ. Equ.* **148**, (2020).
[23] R. Agarwal, M. P. Yadav, R. P. Agarwal and R. Goyal, Analytic solution of fractional advection dispersion equation with decay for contaminant transport in porous media, *Mat. Vesnik* **71**, 5-15 (2019).
[24] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris and F. Ahmed, (k,s)-Riemann-Liouville fractional integral and applications, *Hacet. J. Math. Stat.* **45**(1), 77-89 (2016).
[25] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.
[26] J. Liouville, Memoire: sur le calcul des differentielles a indices quelconquessur, *J. l´Ecole Polytech.* **13**(21), 71-162 (1823).
[27] F. Chen, On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, *Chin. J. Math.* **2014**, (2014).
[28] S. Rashid, İ. İşcan, D. Baleanu and Y. M. Chu, Generation of new fractional inequalities via n polynomials s-type convexity with applications, *Adv. Differ. Equ.* , Article number: 264 (2020).
[29] D. Nie, S. Rashid, A. O. Akdemir, D. Baleanu and J. B. Liu, On some new weighted inequalities for differentiable exponentially convex and exponentially quasi-convex functions with applications, *Mathematics* **7**(727), (2019).
[30] S. Rashid, H. Kalsoom, Z. Hammouch, R. Ashraf, D. Baleanu and Y. M. Chu, New multi-parametrized estimates having pth-order differentiability in fractional calculus for predominating h-convex functions in Hilbert space, *Symmetry* **12**(222), (2020).
[31] S. Rashid, F. Jarad, M. A. Noor, K.I. Noor, D. Baleanu and J. B. Liu, On Grüss inequalities within generalized K-fractional integrals, *Adv. Differ. Equ.* Article number: 203 (2020).
[32] S. Rashid, F. Jarad and M. A. Noor, Grüss-type integrals inequalities via generalized proportional fractional operators, *Rev. Real Acad. Ciencias Exactas, Físic. Natur. Serie A. Mat.* **114**(93), (2020).