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Explict Expressions for Moments of $K-$TH Lower Record Values from J-Shaped Distribution and a Characterization

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Abstract: In this paper we established some explicit expressions and recurrence relations for single and product moments of $k-$th lower record values from J-shaped distribution. Further, using a recurrence relation for single moments we obtain characterization of J-shaped distribution.

Keywords: Record values; single moment; product moment; recurrence relations; J-shaped distribution; Characterization.

1 Introduction

The model of record statistics defined by Chandler [8] as a model for successive extremes in a sequence of independent and identically distributed (iid) random variables. This model takes a certain dependence structure into consideration. That is, the life-length distribution of the components in the system may change after each failure of the components. For this type of model, we consider the lower record statistics. If various voltages of equipment are considered, only the voltages less than the previous one can be recorded. These recorded voltages are the lower record value sequence.
Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them e.g. Olympic records or world records in sports. Motivated by extreme weather conditions.

Let $X_1, X_2, \ldots$ be a sequence of *iid* random variables with distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$. Suppose $Y_n = min\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. we say X_j, $j \geq 1$ is a lower record value of this sequence, if $Y_j < Y_{j-1}$ for $j > 1$. And we suppose that X_1 is a first lower record value. The indices at which the lower record values occur are given by record times $\{L(n), n \geq 1 \geq$, where $L_n = min\{j | j > L(n-1), X_j < X_{L(n-1)}\}$, $n > 1$, with $L(1) = 1$. For more details and references, see Ahsanullah [1], Arnold and Balakrishnan [2] and Arnold *et al*. [3].
For a fixed $k \geq 1$ we define the sequence $\{L_n^{(k)}, n \geq 1\}$ of $k-$th lower record times of $\{X_n, n \geq 1\}$ as follows

$$L_1^{(k)} = 1,$$

$$L_{n+1}^{(k)} = min\{j > L_n^{(k)} : X_k : L_k(n) + k - 1 > X_k : j + k - 1\}.$$

For $k = 1$ and $n = 1, 2, \ldots$ we write $L_1^{(1)} = L_n$. Then $\{L_n, n \geq 1\}$ is the sequence of record times of $\{X_n, n \geq 1\}$. The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{L_n^{(k)}}$ is called the sequence of k lower record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Y_0^{(k)} = 0$. Note that $k = 1$ we have $Y_n^{(1)} = X_{L_n}, n \geq 1$, which are record value of $\{X_n, n \geq 1\}$. Moreover $Y_1^{(k)} = min\{X_1, X_2, \ldots, X_k\} = X_{1:k}$.

Let $\{X_n^{(k)}, n \geq 1\}$ be the sequence of $k-$th lower record values then from (1.1). Then the pdf of $X_{L(n)}^{(k)}$, $n \geq 1$ is given by

$$f_{X_{L(n)}^{(k)}}(x) = \frac{k^n}{(n-1)!}[-ln(F(x))]^{n-1}[F(x)]^{k-1}f(x) \quad (1.1)$$

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and the joint pdf of $X_{L(m)}^{(k)}$ and $X_{L(n)}^{(k)}$, $1 \le m < n$, $n > 2$ is given by

$$f_{X_{L(m)}^{(k)},X_{L(n)}^{(k)}}(x,y) = \frac{k^n}{(m-1)!(n-m-1)!}[-ln(F(x))]^{m-1}$$

$$\times[-ln(F(y))+ln(F(x))]^{n-m-1}[F(y)]^{k-1}\frac{f(x)}{F(x)}f(y),\ x<y. \tag{1.2}$$

We shall denote

$$\mu_{L(n):k}^{(r)} = E((X_{L(n)}^{(k)})^r), \quad r,n = 1,2,\ldots,$$

$$\mu_{L(m,n):k}^{(r,s)} = E((X_{L(m)}^{(k)})^r,(X_{L(n)}^{(k)})^s), \quad 1 \le m \le n-1 \quad and \quad r,s = 1,2,\ldots,$$

$$\mu_{L(m,n):k}^{(r,0)} = E((X_{L(m)}^{(k)})^r) = \mu_{L(m:k)}^{(r)}, \quad 1 \le m \le n-1 \quad and \quad r = 1,2,\ldots,$$

$$\mu_{L(m,n):k}^{(0,s)} = E((X_{L(n)}^{(k)})^s) = \mu_{L(n:k)}^{(s)}, \quad 1 \le m \le n-1 \quad and \quad s = 1,2,\ldots,$$

Various development on record values in found to be Kumar and Singh [13], Kumar and Khan [15], Kumar [14] and Kumar and Kulshrestha [16]. Explicit expression and recurrence relations for single and product moments of $k-$th lower record values from exponentiated log-logistic distribution are derived by Kumar [12]. Recurrence relations for single and product moments of record values from generalized Pareto, lomax, exponential and generalized extreme value distribution are derived by Balakrishnan and Ahsanullah [4], [5] and [6] and Balakrishnan *et al*. [7] respectively. Pawlas and Szynal [18], [19] and Saran and Singh [20] have established recurrence relations for single and product moments of $k-$th record values from Weibull, Gumbel and linear exponential distributions. Kamps [11] investigated the importance of recurrence relations of order statistics in characterization.
In the present study, we established some explicit expressions and recurrence relations satisfied by the single and product moments of lower $k-$ record values from the J-shaped distribution. A characterization of this distribution has also been obtained on using a recurrence relation for single moments.

A random variable X is said to have J-shaped distribution if its pdf is given by

$$f(x) = \frac{2\alpha}{\beta}\left(1-\frac{x}{\beta}\right)\left[\frac{x}{\beta}\left(2-\frac{x}{\beta}\right)\right]^{\alpha-1},\ 0<x<\beta,\ 0<\alpha<1.$$

and the corresponding df is

$$F(x) = \left[\frac{x}{\beta}\left(2-\frac{x}{\beta}\right)\right]^{\alpha},\ 0<x<\beta,\ 0<\alpha<1.$$

The distribution is called J-shaped because $f(x) > 0$, $f'(x) < 0$ and $f''(x) > 0$ for all $0 < x < \beta$.
Topp and Leone [21] introduced this distribution and derived its first four moments, they gave no motivation to the distribution expect referring to its suitability to model failure data.
We assume through this study, with out loss of generality, that $\beta = 1$, in which case the pdf and df are respectively reduced to

$$f(x) = 2\alpha(1-x)[x(2-x)]^{\alpha-1},\ 0<x<1,\ 0<\alpha<1 \tag{1.3}$$

and the corresponding df is

$$F(x) = [x(2-x)]^{\alpha},\ 0<x<1,\ 0<\alpha<1. \tag{1.4}$$

For more details on this distribution and its application one may refer to Topp and Leone [21] Nadarajah and Kotz [17].

2 Relations for Single moment

Note that for J-shaped distribution defined in (1.5)

$$x(2-x)f(x) = 2\alpha(1-x)F(x). \tag{2.1}$$

The relation in (2.1) will be exploited in this paper to derive recurrence relations for the moments of record values from the J-shaped distribution.
We shall first establish the explicit expression for the single moment of $k-$th lower record values $\mu_{L(n):k}^{(r)}$. Using (1.1), we have

$$\mu_{L(n:k)}^{(r)} = \frac{k^n}{(n-1)!}\int_0^1 x^r[F(x)]^{k-1}[-ln(F(x))]^{n-1}f(x)dx. \tag{2.2}$$

By setting $t = F(x)$ in (2.2), since $F^{-1}(x) = \left(1-\sqrt{1-x^{1/\alpha}}\right)$, we get

$$\mu_{L(n):k}^{(r)} = \frac{k^n}{(n-1)!}\int_0^1 [F^{-1}(t)]^r t^{k-1}[-lnt]^{n-1}dt. \tag{2.3}$$

A power series expansion for $\left(1+\sqrt{1+\lambda}\right)^u$ is given by (see p. 21 of Gradshteyn and Ryzhik [9])

$$\left(1+\sqrt{1+\lambda}\right)^u = 2^u\left[1+\frac{u}{1!}\left(\frac{\lambda}{2}\right)+\right.$$

$$\left.\frac{u(u-3)}{2!}\left(\frac{\lambda}{2}\right)^2+\frac{u(u-4)(u-5)}{3!}\left(\frac{\lambda}{2}\right)^3+\cdots\right], \tag{2.4}$$

for any real number u.
Applying (2.4) for $\left(1-\sqrt{1-t^{1/\alpha}}\right)^j$, we obtain

$$\left(1-\sqrt{1-t^{1/\alpha}}\right)^j = \sum_{v=0}^{\infty} p_v t^{(j+v)/\alpha},$$

where
$p_0 = 2^{-j}, p_1 = j2^{-j}$ and $p_u = 2^{-(j+2v)}\prod_{b=1}^{v-1} j(j+v+$

b) for $v = 2, 3, \ldots$.
Thus (2.3) becomes

$$\mu^{(r)}_{L(n):k} = \frac{k^n}{(n-1)!} \sum_{p=0}^{\infty} a_p \int_0^1 t^{((r+p)/\alpha)+k-1} [-lnt]^{n-1} dt.$$

Again by putting, $z = -lnt$, we obtain

$$\mu^{(r)}_{L(n):k} = \frac{k^n}{(n-1)!} \sum_{p=0}^{\infty} a_p \int_0^1 e^{-[((r+p)/\alpha)+k]t} \; t^n dt$$

$$= (\alpha k)^n \sum_{p=0}^{\infty} \frac{a_p}{[\alpha k + j + p]^n}. \qquad (2.5)$$

Remark 2.1: For $k = 1$ in (2.5) we deduce the explicit expression for single moments of lower record values from the J-shaped distribution.

Recurrence relations for single moments of $k-$th lower record values from df (1.1) can be derived in the following theorem.
Theorem 2.1: For a positive integer $k \geq 1$ and for $n \geq 1$ and $r = 0, 1, 2, \ldots$,

$$\left(1 + \frac{2\alpha k}{r+1}\right) \mu^{(r+1)}_{L(n):k} = 2\left(1 + \frac{\alpha k}{r}\right) \mu^{(r)}_{L(n):k}$$

$$+2\alpha k \left(\frac{1}{r+1} \mu^{(r+1)}_{L(n-1):k} - \frac{1}{r} \mu^{(r)}_{L(n-1):k}\right). \qquad (2.6)$$

Proof We have

$$2\mu^{(r)}_{L(n):k} - \mu^{(r+1)}_{L(n):k} = \frac{k^n}{(n-1)!}$$

$$\int_0^1 x^r (2-x)[F(x)]^{k-1} [-ln(F(x))]^{n-1} f(x) dx. \qquad (2.7)$$

On using (2.1), then (2.7) reduced to

$$2\mu^{(r)}_{L(n):k} - \mu^{(r+1)}_{L(n):k} = \frac{2\alpha k^n}{(n-1)!}$$

$$\int_0^1 x^{r-1} (1-x)[F(x)]^{k} [-ln(F(x))]^{n-1} dx. \qquad (2.8)$$

Integrating by parts treating $x^{r-1}(1-x)$ for integration and the rest of the integrand for differentiation,then (2.8) becomes

$$2\mu^{(r)}_{L(n):k} - \mu^{(r+1)}_{L(n):k} = \frac{2\alpha k^n}{(n-2)!}$$

$$\int_0^1 \left(\frac{x^r}{r} - \frac{x^{r+1}}{r+1}\right) [F(x)]^{k-1} [-ln(F(x))]^{n-2} f(x) dx$$

$$-\frac{2\alpha k^n}{(n-1)!} \int_0^1 \left(\frac{x^r}{r} - \frac{x^{r+1}}{r+1}\right) [F(x)]^{k-1} [-ln(F(x))]^{n-1} f(x) dx$$

$$= 2\alpha k \left(\frac{1}{r} \mu^{(r)}_{L(n-1):k} - \frac{1}{r+1} \mu^{(r+1)}_{L(n-1):k}\right)$$

$$-2\alpha k \left(\frac{1}{r} \mu^{(r)}_{L(n):k} - \frac{1}{r+1} \mu^{(r+1)}_{L(n):k}\right)$$

and hence the result we obtained.
Remark 2.2 Setting $k = 1$ in (2.6) we deduce the recurrence relation for single moments of lower record values from the J-shaped distribution.

3 Relations for Product moment

Making use of (1.2), we can derive recurrence relations for product moments of $k-$th lower record values from (1.5).
Theorem 3.1: For $1 \leq m \leq n-2$ and $r, s = 1, 2, \ldots$,

$$\left(1 + \frac{2\alpha k}{s+1}\right) \mu^{(r,s+1)}_{L(m,n):k} = 2\left(1 + \frac{\alpha k}{s}\right) \mu^{(r,s)}_{L(m,n):k}$$

$$+2\alpha k \left(\frac{1}{s+1} \mu^{(r,s+1)}_{L(m,n-1):k} - \frac{1}{s} \mu^{(r,s)}_{L(m,n-1):k}\right). \qquad (3.1)$$

Proof: From equation (1.2) for $1 \leq m \leq n-2$ and $r, s = 0, 1, 2, \ldots$,

$$2\mu^{(r,s)}_{L(m,n):k} - \mu^{(r,s+1)}_{L(m,n):k}$$

$$= \frac{k^n}{(m-1)!(n-m-1)!} \int_0^{\infty} x^r [-ln(F(x))]^{m-1} \frac{f(x)}{[F(x)]} I(x) dx, \qquad (3.2)$$

where

$$I(x) = \int_0^x y^s (2-y)[-ln(F(y)) + ln(F(x))]^{n-m-1} [F(y)]^{k-1} f(y) dy. \qquad (3.3)$$

On using (2.1), then (3.3) reduced to

$$I(x) = 2\alpha \int_0^x y^{s-1} (1-y)[-ln(F(y)) + ln(F(x))]^{n-m-1} [F(y)]^{k} dy.$$

Integrating $I(x)$ by parts treating $y^{s-1}(1-y)$ for integration and the rest of the integrand for differentiation, and substituting the resulting expression in (3.2), we get

$$2\mu^{(r,s)}_{L(m,n):k} - \mu^{(r,s+1)}_{L(m,n):k} = \frac{2\alpha k^n}{(m-1)!(n-m-1)!} \int_0^1 \int_0^x x^r \left(\frac{y^s}{s} - \frac{y^{s+1}}{s+1}\right)$$

$$\times [-ln(F(x))]^{m-1} [-ln(F(y)) + ln(F(x))]^{n-m-1} [F(y)]^{k-1} \frac{f(x)}{[F(x)]} f(y) dy dx$$

$$+ \frac{2\alpha(n-m-1)k^n}{(m-1)!(n-m-1)!} \int_0^1 \int_0^x x^r \left(\frac{y^s}{s} - \frac{y^{s+1}}{s+1}\right)$$

$$\times [-ln(F(x))]^{m-1} [-ln(F(y)) + ln(F(x))]^{n-m-2} [F(y)]^{k-1} \frac{f(x)}{[F(x)]} f(y) dy dx$$

and hence the result we obtain.
Remark 3.1 Setting $k = 1$ in (3.3), we deduce the recurrence relation for product moments of lower record values from the J-shaped distribution.

4 Characterization

Theorem 4.1: Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, then

$$\left(1+\frac{2\alpha k}{r+1}\right)\mu^{(r+1)}_{L(n):k}=2\left(1+\frac{\alpha k}{r}\right)\mu^{(r)}_{L(n):k}$$

$$+2\alpha k\left(\frac{1}{r+1}\mu^{(r+1)}_{L(n-1):k}-\frac{1}{r}\mu^{(r)}_{L(n-1):k}\right) \qquad (4.1)$$

if and only if

$F(x)=[x(2-x)]^{\alpha},\ 0<x<1,\ 0<\alpha<1.$

Proof: The necessary part follows immediately from equation (2.5). On the other hand if the recurrence relation in equation (4.1) is satisfied, then on using equation (1.2), we have

$$\frac{k^n}{(n-1)!}\int_0^1 x^{r+1}[F(x)]^{k-1}[-ln(F(x))]^{n-1}f(x)dx$$

$$=\frac{2k^n}{(n-1)!}\int_0^1 x^{r}[F(x)]^{k-1}[-ln(F(x))]^{n-1}f(x)dx$$

$$+\frac{2\alpha k^n}{r(n-1)!}\int_0^1 x^{r}[F(x)]^{k-1}[-ln(F(x))]^{n-1}f(x)dx$$

$$-\frac{2\alpha k^n}{(r+1)(n-1)!}\int_0^1 x^{r+1}[F(x)]^{k-1}[-ln(F(x))]^{n-1}f(x)dx$$

$$+\frac{2\alpha k^n}{(r+1)(n-2)!}\int_0^1 x^{r+1}[F(x)]^{k-1}[-ln(F(x))]^{n-2}f(x)dx$$

$$-\frac{2\alpha k^n}{r(n-2)!}\int_0^1 x^{r}[F(x)]^{k-1}[-ln(F(x))]^{n-2}f(x)dx. \qquad (4.2)$$

Integrating the last two integrals on the right hand side of equation (4.2) by parts and simplifying the resulting expression, which reduces to

$$\frac{k^n}{(n-1)!}\int_0^1 x^{r}[F(x)]^{k-1}[-ln(F(x))]^{n-1}\{(x-2)f(x)$$

$$-2\alpha F(x)+2\alpha xF(x)\}dx. \qquad (4.3)$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, [10]) to equation (4.3), we get

$$\frac{f(x)}{F(x)}=\frac{2\alpha(1-x)}{x(2-x)}$$

which proves that

$F(x)=[x(2-x)]^{\theta},\ 0<x<1,\ 0<\theta<1.$

5 Conclusion

In this study some explicit expression and recurrence relations for single and product moments of lower record values from the J-shaped distribution have been established. Further, characterization of this distribution has also been obtained on using a recurrence relation for single moments.

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