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Differential Graded Algebras and Derived E_∞-Algebras

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Abstract: In this paper, we introduce the derived E-infinity and the homology of differential graded algebra and seen as an E-infinity algebra.We display two hypotheses of the homology theory of differential graded algebra A over commutative ground ring k. Furthermore, exhibit that it is the minimal derived E_∞-algebra. Finally, we give satisfying fitting relations in the class E-infinity algebra.

Keywords: Differential algebra, E-infinity algebras, homological algebras

1 Introduction

An operad L comprises k-modules $L(j)$, $j \geq 0$, with unit map $\eta : k \to L(1)$, which is the correct action by symmetric group Σ_j on $L(j)$, $\forall j$, and maps $\gamma : L(k) \otimes L(j_1) \otimes ... L(j_k) \longrightarrow L(j), for k \geq 1$ and $j_s \geq 0$.
An E_∞-algebra is a k-module A with maps $m_j : L(j) \otimes A^{(j)} \longrightarrow A$ since $A(j)$ is the notation of the j-fold tensor power of A. These maps must fulfill a few properties, such as: associativity,equivariance relations,and unity. We can define Operads of dg k-modules by replacing Cartesian products of spaces with tensor products for k-modules. An operad L of k-modules is an E_∞ operad if $L(j)$ is the free $k[\Sigma_j]$ resolution of k for each j; a chain operad of E_∞-operad of spaces is an example. Let L mean the chain operad gotten from our topological E_∞-operad L. We characterize an E-infinity k-algebra to be k-module A together with maps $\theta_j : L(j) \otimes A^{(j)} \longrightarrow A$ since $A(j)$ indicates the j-fold tensor power of A. These maps must fulfill properties, such as unity, equivariance and associativity relations exactly like those of definition of E-infinity ring spectra. Modules over algebras are characterized likewise regarding maps $\lambda_j : L(j) \otimes (A^{\otimes(j-1)}) \longrightarrow A$ see ([1], [2] and [3]). Without a field k with characteristic zero,one cannot plan to supplant the subsequent E_∞ algebras by quasi-isomorphic genuine $DGA\grave{}s$.
If k is a field, Kadeishvili asserts that each $k-dga$ E admits a quasi-isomorphism for A_∞-algebras may be minimal A_∞-algebra, where A_∞-algebra is minimal if $m_1 = 0$. Clearly,A graded k-algebra of minimal model for E is characterized to be homology algebra $H_*(E)$. That quasi-isomorphism type of A might be acquired from its minimal model.
Therefore, minimal A_∞-algebras provide an elective portrayal to quasi-isomorphism classes of dga and the minimal A_∞-structure on $H_*(A)$ specifies the extra majority of the data required to recreate those quasi-isomorphism sort of dga from its homology. Let k be a commutative ring, then Kadeishvilis theorem does not hold in allthough the homology $H_*(A)$ is not k-projective, for example, a k-linear cycle selection morphism may not exist.
The present paper aims to aggregate up the considered the E_∞-algebra to be a connection to which a dga A concedes the minimal model without restrictions on k or $H_*(E)$. Furthermore, gatherings give an alternate characteristic of the quasi-isomorphism type A. Baues [4], the most recent cohomology principle may be characterized as a Hochschild cohomology to a k-projective DGA resolving algebra.
First,we take a k-projective resolution of $H_*(E)$ in the direction of new grading, then we look for the analog to the minimal E_∞-structure on this bigraded k-module. To consider the extra reviewing, we present the idea of derived E_∞-algebra (dE_∞-algebra for short). A dE_∞-algebra is (Z,Z)-bigraded k-module A equipped with k-linear maps $m^A_{ij} : L(j) \otimes A^{\otimes j} \longrightarrow A$ of bidegree $(-i, 2, -i-j)$ for each $j \geq 0, j \geq 1$ satisfying appropriate relations. It is minimal if $m^A_{01} = 0$. We can study both of dgas and E-infinity algebras as dE-algebras concentrated with degree 0 of the Z-grading. The idea of equivalence

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appropriate for our motivations is that of an E_2-equivalence of dE-infinity algebras, which is identified on the iterated homology concerning m^A_{11} and m^A_{01}.
In this section, we present some facts as well as the main results in Theorems (1-20), (1-22) and (1-23).

Theorem 1. *Let A be an E-infinity algebra. There exists a quasi-isomorphism $\varepsilon : BA \longrightarrow A$ and E-infinity algebra BA, such that M is an A-module. There exists BA-module BM and the quasi-isomorphism of BA-modules $\varepsilon : BM \longrightarrow M$ (natural isomorphism). If A, A' are augmented E-infinity algebras, there exist natural quasi-isomorphisms of k-modules $A \otimes BA' \longrightarrow BA \boxdot BA'$ and $BA \otimes BM \longrightarrow BA \triangleright BM$ The products $\triangleright$ and $\boxdot$ are used to obtain good algebraic properties on the areas of meaning of the multiplications on E-infinity algebras and of their actions on modules. By the theorem, we can obtain such algebraic control without changing the underlying quasi-isomorphism type. The properties of $b\boxdot$ are easily derived from associative and commutative of $\boxtimes$ together with formal arguments from the definition.*

Theorem 2. *Let A be a DGA, then we have the map $\alpha : U(A) \cong Fk \longrightarrow A$ which is the map of DGAs. It induces equivalence of categories between the derived category D_A and the E_∞-derived category $D_{U(A)}$.*

Proof. We can get the first statement as an immediate verification since C acts on A through the augmentation, $C \longrightarrow N$. Then α is a quasi-isomorphism. Now, we present a few definitions and theorems on derived E_∞algebras. A commutative ring k, (Z,Z) -bigraded k-modules we define $E = \bigoplus_{p,q\in Z} E^q_p$.

Definition 1. *A derived E-infinity algebra (dE-infinity algebra for short) is a (Z,Z)-bigraded k-module A equipped with k-linear maps; $m_{pq} : L(j) \otimes A^{\otimes q} \longrightarrow A[-p, 2, -p - q]$ of bi-degree, $\eta_A : k \longrightarrow A$ for each $p \geq 0, q \geq 1$*

$$\sum_{\substack{i+p=v,\, j+q=u+1,\\ r+1+t=q,\\ j,q\geq 1,\, i,r,p,t\geq 0}} (-1)^{rq+t+pj} m^A_{ij}(1^{\otimes r} \otimes m^A_{pq} \otimes 1^{\otimes t}) = 0 \tag{1}$$

For all $u \geq 1, v \geq 0$ and the unit condition:

$$m^A_{01}\eta_A = 0, \qquad m^A_{02}(\eta_A \otimes 1) = 1 = m^A_{01}(1 \otimes \eta_A)$$

and

$$m^A_{pq}(1^{\otimes r-1} \otimes \eta_A \otimes 1^{q-r}) = 0 \quad if \quad p+q \geq 3, \quad with \ 1 \leq r \leq q \tag{2}$$

Example 1. An important example of an E_∞-algebra given as any strict commutative and associative algebra over the ground ring.

Definition 2. *The map of dE-infinity algebras from E to F comprises a group of k-module maps $G_{ij} : L(j) \otimes A^{\otimes j} \longrightarrow F[i, 1-(i+j)]$ with $i \geq 0, j \geq 1$ satisfying;*

$$\sum_{\substack{i+p=v,\\ j+q=u+1,\\ r+1+t=q,\\ j,q\geq 1,\\ i,r,p,t\geq 0}} (-1)^{rj+t+iq} G_{pq}(1^{\otimes r} \otimes m^A_{ij} \otimes 1^{\otimes t}) = \sum_{\substack{p+i_1+\ldots+i_q=v,\\ j+q=u+1,\\ r+1+t=q,\\ 1\leq q\leq u,\\ i_s\geq o,\, j_s\geq 1}} (-1)^{\delta} m^G_{pq}(g_{i_1 j_1} \otimes \ldots \otimes g_{i_p j_q})$$

For $u \geq 1, v \geq 0$ and the unit condition $g_{01}\eta_A = \eta_G$ and

$$g_{ij}(1^{\otimes r-1} \otimes \eta_A \otimes 1^{\otimes j-r}) = 0 \ \ if \ \ i+j \geq 2 \ \ with \ \ 1 \leq r \leq j \tag{3}$$

The δ governs the sign as

$$\delta = v + \sum_{w=1}^{q-1} (qi_w + w(j_j - w - i_w) + i_q - w(\sum_{s=q-w+1}^{q} (i_s + j_s))$$

Example 2. Every ordinary commutative associative algebra is an E_∞-algebra.

Definition 3. *An algebra bidga is a dE_∞-algebra with $m^B_{pq} = 0$ if $p+q \geq 3$. The maps of bidgas have $g_{pq} = 0$ for $p+q \geq 2$.*

Definition 4. *A twisted dga is defined as a dE-infinity algebra A for m^A_{02} and m^A_{P1} with $P \geq 0$ may be non-zero. The map $G : A \longrightarrow B$ of twisted dga is map of dE-infinity algebras with $g_{pq} = 0$ if $q \geq 2$.*

Definition 5. *A twisted chain complex C is an (Z,Z)-bigraded k-module with k-linear maps $d^C_i : C \longrightarrow C$ for $i \geq 0$, satisfying*

$$\sum_{i+p=v} (-1)^i d_i d_p = 0 \quad for \quad i \geq 0 \tag{4}$$

Definition 6. *A map of twisted chain complexes $C \longrightarrow D$ is the class of maps $f_i : C \longrightarrow D$ for $i \geq 0$, of bi-degree$(-i, -i)$ satisfying;*

$$\sum_{i+p=u} (-1)^i f_i d^C_p = \sum_{i+p=u} d^D_i f_p \tag{5}$$

a composition of $f : E \longrightarrow F$ and $g : F \longrightarrow G$ is;

$$(gf)_u = \sum_{i+p=u} g_i f_p$$

Lemma 1. *For any functor $Tot : tCh_k \longrightarrow Ch_k$ called strong monoidal, where ρ is strong monoidal transformation. So, if C is tdga, then $TotC$ is dga,where the map $\rho_E : C \longrightarrow TotC$ is intdgas. A total complex which has a filtration defined by $F^p(TotX)_n = \otimes_{s \leq p} X_{s,n-s}$. The E_2-term of the spectral takes the form $E^2_{pq} \cong H^h_p H^v_q(X)$. We write the notation $H^v_*(E)$ for the 'vertical' homology of E concerning the differentiald_0. From(5)for $u = 2, d_1$ induces a differential on $H^v_*(E)$, and the notation $H^h_*(H^v_*(E))$ defines a resulting 'horizontal' homology group. Maps which induce maps of spectral sequences: given $g : A \longrightarrow G$ in Ch_k, f_0 is a d_0-chain map on $A \longrightarrow G$, and from the formula (6)for $u = 1$, ensure that $H^v_*(f_0)$ induces the chain map with esteem to the d_1-differential $H^v_*(d_1)$. Hence,we obtain $H^h_*(H^v_*(f_0))$ on the E_2-term.*

Definition 7. *A map $f : C \longrightarrow D$ of dE-infinity is an E_2-equivalence if $H^h_s(H^v_t(f_0))$ is an isomorphism for $s,t \in Z$.*

Lemma 2. *Let the twisted chain complex X, with E_2-homology concentrated in horizontal of degree 0, then $\rho_X : X \longrightarrow TotX$ is an E_2-equivalence. The basic idea is to introduce degree with projective resolutions for an E_∞algebra that are appropriate for the E_∞structure. A dga (or DGA)can be observed as a$bi-dga$ concentrated in horizontal of degree 0 by resolving dga.*

Definition 8. *Let A be graded algebra.A term wise k-projective resolution of A is the term wise k-projective bidga E with $m_{01} = 0$ together with E_2-equivalence $E \longrightarrow A$ of bidgas.*

Definition 9. *Let A be dga, A k-projective E_1-resolution of A is a bidga B together with E_2-equivalence $B \longrightarrow A$ of bidgas such that $H^v_{st}(B)$ is k-projective for each bi-degree.*

Theorem 3. *For any $k-dga$ A admits a k-projective of E_1-resolution $B \longrightarrow A$. Two such resolutions can be related by an E_2-equivalences $zig-zag$ between k projective E_1-resolutions. To get the proof, we use Quillen's language manner for the model categories [5],[6] and [7]. We start with briny recalling Bouseld's general setup for resolution model structures [8], or the dual version of [9]. If C and P are a pointed model category and the set of group-objects in $Ho(C)$, respectevily. A morphism $p : X \longrightarrow Y$ in $Ho(C)$ is a$p-epi$ if $p_* : [P,X]_n \longrightarrow [P,Y]_n$ is onto $\forall P \in P$ and $n \geq 0$, where $[X,Y]_n = Ho(C)(\sum^n X, Y)$. The object $A \in Ho(C)$ is P-projective if $p_* : [A,X]_n \longrightarrow [A,Y]_n$ is onto $\forall$ $P-epis$ p and $n \geq 0$. Maps in C are $P-epis$ if they exist in $Ho(C)$. Any map in C has the left lifting property with deference to all $P-epi$ fibrations is a P-projective cofibration. A map $X \longrightarrow Y$ in C is P-free if it is composition for inclusion map $X \longrightarrow X \prod P$ with P-cofibrant and P-projective as well as acyclic cofibration $X \prod P \longrightarrow Y$. Then $Ho(C)$ has enough P-projectives, since every object $X \in Ho(C)$ admits the: $P-epiY \longrightarrow X$ with P-projective. Then P is a class of projective models in $Ho(C)$.*

Definition 10. *If $g : X \longrightarrow Y$ is a map in sC, then get:*
(a) g is a P-equivalence if $g_* : [P,X]_n \longrightarrow [P,Y]_n$ is weak equivalence in simplicial groups $\forall P \in P$ and $n \geq 0$.
(b) g is a P-fibration if it is Reedy-fibration and $g_* : [P,X]_n \longrightarrow [P,Y]_n$ is a fibration of simplicial groups for $P \in P$ and $n \geq 0$.
(c) g is a P-cofibration if induced maps $X_n \prod_{L_n X} L_n Y \longrightarrow Y_n$ are P-projective co-fibrations for all $n\forall 0$. $g_* : [P,X]_n \longrightarrow [P,Y]_n$.

Lemma 3. [10] *P-cofibration in sC consider a P-projective cofibration in C.*

Lemma 4. *The category $Ho(Ch_k)$has enough Pprojective for P-projective cofibration in C. An object in $Ho(Ch_k)$ is P-projective if it has degree wise k-projective homology.*

Proof. For any arbitrary object in co-domain of a $P-epi$ mapping out of a sum of objects in P, there exist enough P-projectives. Applying this step to a P-projective object X, we find that X is a retract of the sum of objects in P since X has degree wise k-projective homology. The object, which is k-projective homology, is a retract of the sum of objects of P, and therefore P-projective.

Proposition 1 *The functor $U : sdga_k \longrightarrow sCh_k$ makes a co-fibrantly produced model structure on $sdga_k$ in which a map g is a P-fibration or a P-equivalence if $U(g)$ is co-fibrant objects of $sdga_k$ which are term wise P-projective co-fibrant in Ch_k. The unit map in co-fibrant replacement in $sdga_k$ splits in homology.*

Definition 11. *Let a maps; $f : X \longrightarrow Y$ and $g : X' \longrightarrow Y'$ are in Ch_k or sCh_k. We introduce the pushout product of the maps by a map $(f \otimes g)$ to be induced map, $(Y \otimes X') \prod_{X \otimes x'} (X \otimes Y' \longrightarrow Y \otimes Y')$.*

Lemma 5. *(a) The functor C is map from P-equivalences of simplicial dgas to E_2-equivalences of bidgas.*
(b) For a $k-bidgaB$, we have a standard E_2-equivalence $C(\Gamma(B)) \longrightarrow B$ of bidgas which is natural map in B.

Proof. Since C is commuting with the vertical homology, A map in (b) is inducing the co-unit $N(\Gamma(B)) \longrightarrow B$. This map is a map of *bidgas*, since the co-unit is monoidal and the shuffle map is compatible with the degeneracies. Every C or Γ commutes with taking vertical homology,so the Dold-Kan writing for simplicial k-modules and Ch^+_k devoted in each vertical degree shows that $C(\Gamma(B)) \longrightarrow B$ is an E_2-equivalence.

Lemma 6. *A map from A to A' of $k-dgas$ and k-projective E_1-resolutions $B_1 \longrightarrow A_1$ and $B'_1 \longrightarrow A'$, there exists the following commutative diagram:*

$$\begin{array}{ccccccc} B_2 & \longrightarrow & B_3 & \longrightarrow & B'_3 & \longleftarrow & B'_2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & A & \longrightarrow & A' & \longleftarrow & B'_1 \end{array}$$

of bidgas which all maps for all $iB_i \longrightarrow A$ and $B'_1 \longrightarrow A'$ are k-projective E_1-resolutions.

*Proof.*We play the cofibrant substitute functor in a P-model structure on $sdga_k$ to $\Gamma(B_1) \longrightarrow \Gamma(A) \longrightarrow \Gamma(A') \longleftarrow \Gamma(B_1^{'})$ and obtain

$$\begin{array}{ccccccc} C\Gamma(B_1)^{cof} & \longrightarrow & C\Gamma(A)^{cof} & \longrightarrow & C\Gamma(A')^{cof} & \longleftarrow & C\Gamma(B_1^{'})^{cof} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C\Gamma(B_1) & \longrightarrow & C\Gamma(A) & \longrightarrow & C\Gamma(A') & \longleftarrow & C\Gamma(B_1^{'}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & A & \longrightarrow & A' & \longleftarrow & B_1^{'} \end{array}$$

From lemma (1.9), the upper and the lower push a mass to the coveted outline.

2 Hochschild cohomology of E_∞algebras

Hochschild cohomology is an intense drive in numerous regions around variable based algebra and topology, from relations to the geometry of loop spaces to distortion hypothesis of algebras and feasibility inquiries in topology.The meaning of Simplicial cohomology of associative graded algebras can be stretched out to a meaning of Simplicial cohomology of E_∞-algebras. We can view an E_{n+1}-algebra as an E_n-algebra. Commutative algebras are E_n-algebras for every n,similarly for E_∞-algebras.
We recall some results on minimal models of E-infinity algebras. It relates differential Graded algebras to E_∞-structures on their homology. We showed that *dgas* have reasonable minimal models in the world of E_∞-algebras. The aim of this part is presenting (2.2),(23) and (2.4).

Definition 12.[3] *Let $A \longrightarrow R$ augmented commutative $R-$algebra and $\bar{A}$ its augmentation ideal. For commutative algebras there are maps:*

$$H_*^{E_1}(\bar{A}) \longrightarrow H_*^{E_2}(\bar{A}) \longrightarrow ... \longrightarrow H_*^{E_\infty}(\bar{A}).$$

Fresses depiction as far as iterated bar developments gives an immediate recognizable proof (in the commutative case over k) of $H_*^{E_n}(\bar{A})$ with $HH_{*+n}^{[n]}$ that is Pirashvilis simplicial homology in order n.

Definition 13.[11] *An E_∞-algebra is called minimal if $m_1 = 0$.*

Definition 14.*An E_∞-structure m is trivial if $m_n = 0$ for $n \geq 3$.*

Theorem 4.[12]
$H^*(E)$ has an E_∞-structure such that:
$m_1 = 0$ and the multiplication m_2 is incited by the duplication on E,
There exists morphism of E_∞-algebras $f : H^*(E) \longrightarrow E$ such that f_1 is quasi-isomorphism; for E, there exists dga over field k, and $H^*(E)$ is its homology module. This E_∞-algebra $H^*(E)$ is known as the minimal model.

Lemma 7.*Let A be a bigraded k-module and let*

$$g_{pq}^A : L(j) \otimes A^{\otimes q} \longrightarrow F[p, 1-(p+q)]$$

be the set of k-linear maps since $p \geq 0, q \geq 1$ satisfying the unit condition (3). The accompanying is identical:
(i) The g_{pq}^A specifies a dE-infinity algebra structure on A.
(ii) $\sum_{p+i=v}(-1)^p \check{m}_p^A \check{m}_i^A = 0$ for all $v \geq 0$.
(iii) $\sum_{p+i=v}(-1)^p \check{m}_p^A \check{m}_i^A = 0$ for all $v \geq 0$.
(iv) $\bar{T}SE$ is twisted chain complex with $d_p = \check{m}_i^A$.

*Proof.*We interchange δ with $\bar{m}_{ij}^{A,1}$ as:

$$(-1)^p \bar{m}_{pq}^1(1^{\otimes r} \otimes \bar{m}_{ij}^1 \otimes 1^{\otimes t}) = (-1)^{i+t(1+i)} \delta^{-1} m_{pq} (\delta^{\otimes r} \otimes m_{ij} \delta^{\otimes j} \otimes \delta^{\otimes t})$$

$$= (-1)^{rj+t+iq} \delta^{-1} m_{pq}(1^{\otimes r} \otimes m_{ij} \otimes 1^{\otimes t}) \delta^{\otimes e+j+t}$$

In terms $\bar{m}_{pq}^{A,1}$ the unit condition (1) may be rephrased as

$$\bar{m}_{01}^{A,1}(\delta_A^{-1}\eta_A) = 0, \quad -\bar{m}_{02}^{A,1}(\delta_A^{-1}\eta_A \otimes 1) = 1 = \bar{m}_{02}^{A,1}(1 \otimes \delta_A^{-1}\eta_A)$$

and

$$\bar{m}_{pq}^{A,1}(1^{\otimes r-1} \otimes \delta_A^{-1}\eta_A \otimes 1^{q-r} = 0 \quad if \quad p+q \geq 3 \quad with \quad 1 \leq r \leq q$$

A comparable depiction applies to maps. Given bigraded k-modules A and G and k-linear maps,

$$g_{ij} : L(j) \otimes A^{\otimes j} \longrightarrow G[i, 1-(i+j)] \quad with \quad i \geq 0, j \geq 1$$

we define the map $\bar{g}_{ij}^1$ to be $\psi_j(g) : SA^(\otimes j) \longrightarrow SG[i, -i]$. The $\bar{g}_{ij}^1$ assembles to $\bar{g}_i^1 : \bar{T}SE \longrightarrow SG[i, -i]$. We define $\bar{g}_{ij}^q = \sum_{i_1+...+i_p=i, j_1+...+j_q=j} \bar{g}_{i_1 j_1}^1 \otimes ... \otimes \bar{g}_{i_p j_q}^1 : SA^(\otimes j) \longrightarrow SG^(\otimes q)[i, -i]$ and write $\bar{g}_i : \bar{T}SE \longrightarrow \bar{T}SG[i, -i], i \geq 0$,for the morphism mapping $SA^{\otimes j}$ to $SA^{\otimes q}[i, -i]$ by $\bar{g}_{ij}^q$.

Lemma 8.*Consider E and F as dA-infinity algebras and let $f_{pq} : E^{\otimes q} \longrightarrow F[p, 1-(p+q)]$. The family of k-linear maps with $p \geq 0$ and $q \geq 1$ satisfying the unit condition (3). And the following are equivalent:*
(i) The g_{pq}^A forms a dE-infinity algebra map from A tog.
(ii) $\sum_{p+i=v}(-1)^p \check{g}_p^1 \check{m}_i^{A,1} = \sum_{p+i=v} \check{m}_p^{G,1} \check{g}_i^1$ for all $v \geq 0$.
(iii) $\sum_{p+i=v}(-1)^p \check{g}_p \check{m}_i^A = \sum_{p+i=v} \check{m}_p^G \check{g}_i$ for all $v \geq 0$.
(iv) The $\check{g}_p$ form a map of twisted chain complexes.

Definition 15.*For the map $f : D \longrightarrow E$ and $g : E \longrightarrow F$, the component $\widetilde{fg}_u$ of fg is $\sum_{i+p=u} \check{f}_i \check{g}_p$.*

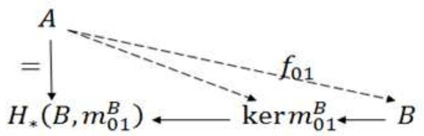
Definition 16.*A dE_∞-algebra A is minimal if $m_{01}^A = 0$.*

Proposition 2*Let B be a bidga, and A be its homology as for m_{01}^B and let d_1 and μ be the differential and the multiplication on A. If A is k-projective in every bidegree and the unit map $\eta_A : k \longrightarrow A$ induced by $\eta_B : k \longrightarrow B$ splits as a k-module map, then there exists:*
(i) A minimal dE-infinity structure on A with; $m_{11}^A = d_1, m_{02}^A = \mu$.
(ii) An E_2-equivalence of dE_∞-algebras $f : A \longrightarrow B$.

*Proof.*Before we begin to prove this proposition; we need to define

$$\tilde{Z}_{ln} = \tilde{m}^{B,1}_{02}\tilde{f}^{2}_{ln} + \tilde{m}^{B,1}_{11}\tilde{f}^{2}_{l-1,n} - \sum_{i+p=l,i+j\geq 2,j<1} (-1)^{i}\tilde{f}^{1}_{ij}\tilde{m}^{A,j}_{pn}$$

For the k-projective A, we choose a k-linear cycle homomorphism $f_{01} : A \longrightarrow B$ as

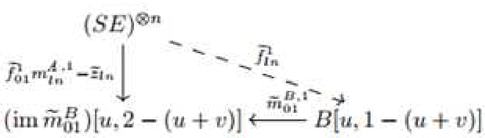


If $m^A_{01} = 0$, then

$$\sum_{\substack{i+p=v,\, j+q=u+1,\\ r+1+t=q,\\ j,q\geq 1,\\ i,r,p,t\geq 0}} (-1)^{rj+t+iq} f_{pq}(1^{\otimes r}\otimes m^A_{ij}\otimes 1^{\otimes t}) =$$

$$\sum_{\substack{p+i_1+...+i_q=v,\\ j+q=u+1,\\ r+1+t=q,\\ 1\leq q\leq u,\\ i_s\geq 0, h_s\geq 1}} (-1)^{\delta} m^f_{pq}(g_{i_1j_1}\otimes . \otimes g_{i_pj_q})(g_{i_1j_1}\otimes . \otimes g_{i_pj_q})$$

holds if $u+v=1.\tilde{z}_{ln}$ has values in $ker\ \tilde{m}^B_{01}$. Then we can define $\tilde{m}^{A,1}_{ln}$ as a composition of $\tilde{z}_{ln}$ with $ker\ \tilde{m}^B_{01} \longrightarrow SE$. Since both of $\tilde{f}^1_{01}\tilde{m}^A_{ln}$ and $\tilde{z}_{ln}$ have values in $ker\tilde{m}^B_{01}$, their difference lies in $im\tilde{m}^B_{01}$ as;



Since we can assume that f_{ln} is zero.

Corollary 1.*Minimal differential E infinity algebra models of quasi-isomorphic dgas may be related by a zig − zag of E_2-equivalences between minimal dE_∞-algebras.*

3 Conclusion

We study the homology theory of differential graded algebra over commutative ground ring and give and define the homology of E_∞-algebra. So, we study the minimal derived E_∞-algebra.

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