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An Investigation on Edge Ideal of a Simple Graph in the Theory of Coatomic Modules

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Abstract: We introduce local cohomology for the edge ideal of a graph, and investigate some properties of local cohomology modules for Artinian modules and coatomic modules, such that the property to be finitely generated, Artinian and coatomic. Also we present vanishing results. Moreover, we present results which involve the theory of local cohomology modules together with the edge ideal of a finite simple graph, with no isolated vertices.

Keywords: Artinian module, local cohomology, coatomic module, edge ideal of a graph

1 Introduction

Throughout this paper, R is a commutative ring with non-zero identity and I is an ideal of R.

In [1], Grothendieck introduced the definition of local cohomology. It is well-known that the local cohomology theory of Grothendieck is an important tool in commutative algebra and algebraic geometry. Let I be an ideal of R, and let M be an R-module, then

$$\mathrm{H}^i_I(M) = \varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R\left(R/I^t, M\right)$$

is called the i-th local cohomology module of M with respect to I.

For an R-module M, the I-torsion submodule of M is

$$\Gamma_I(M) = \{x \in M \mid I^n x = 0 \text{ for some integer } n \geq 1\}.$$

They are denoted by $\mathrm{H}^i_I(\bullet)$ the i-th right derived functor of the functor $\Gamma_I(\bullet)$. For more information on ordinary local cohomology modules, see [2].

The present paper aims to explore is to study some properties of local cohomology modules which involve the theory of graphs, together with the edge ideal of a graph. We present the results which involve the notion of coatomic module. An R-module M is called coatomic if every proper submodule of M is contained in a maximal submodule of M. Here, we use properties of commutative algebra and homological algebra for the development of the results (see [3] and [4]). The coatomic modules were introduced and investigated by H. Zöschinger in [5].

In Section 2, we present some definitions and prerequisites for a better understanding of the theory and results. We introduce preliminaries of the theory of graphs which involves the edge ideal of a graph G; associated to the graph G which is a monomial ideal

$$I(G) = (v_i v_j \mid v_i v_j \text{ is an edge of } G),$$

with $v_i v_j = v_j v_i$ and with $i \neq j$, in the polynomial ring $R = K[v_1, v_2, \ldots, v_s]$ over a field K, called the **edge ideal** of G. The preliminaries of the theory of graphs were introduced in Section 2 together with the concepts appropriate for the work.

In Section 3, we prove some properties of the local cohomology module, that involve the edge ideal of a graph G, which is a finite simple graph, with no isolated vertices.

In Section 4, we prove some results of vanishing about the local cohomology module. Those results involve the edge ideal of a graph G, which is a finite simple graph, with no isolated vertices.

Throughout paper, a graph G means, a finite simple graph with the vertex set $V(G)$ and with no isolated vertices.

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2 Prerequisites of the graphs theory

This section comprises the concepts of the graphs theory adopted in the present paper.

2.1 Edge ideal of a graph

This section is consistent with [6] and [7].

Let $R = K[v_1,\ldots,v_s]$ be a polynomial ring over a field K, and let $Z = \{z_1,\ldots,z_q\}$ be a finite set of monomials in R. The **monomial subring** spanned by Z is the K-subalgebra,

$$K[Z] = K[z_1,\ldots,z_q] \subset R.$$

In general, it is very difficult to certify whether $K[Z]$ has a given algebraic property - e.g., Cohen-Macaulay, normal - or to obtain a measure of its numerical invariants - e.g., Hilbert function. This arises because the number q of monomials is usually large.

Thus, consider any graph G, simple and finite without isolated vertices, with vertex set $V(G) = \{v_1,\ldots,v_s\}$

Let Z be the set of all monomials $v_iv_j = v_jv_i$, with $i \neq j$, in $R = K[v_1,\ldots,v_s]$, such that $\{v_iv_j\}$ is an edge of G, i.e. the finite simple graph G, with no isolated vertices, is such that the squarefree monomials of degree two are defining the edges of the graph G.

Definition 21A *walk* of length s in G is an alternating sequence of vertices and edges $w = \{v_1,z_1,v_2,\ldots,v_{s-1},z_h,v_s\}$, where $z_i = \{v_{i-1}v_i\}$ is the edge joining v_{i-1} and v_i.

Definition 22A walk is **closed** if $v_1 = v_s$. A walk may also be denoted by $\{v_1,\ldots,v_s\}$, and the edges become evident by context. A **cycle** of length s is a closed walk, in which the points $v_1,\ldots,v_s$ are distinct.

A **path** is a walk with all the points distinct. A **tree** is a connected graph without cycles and a graph is **bipartite** if all its cycles are even. A vertex of degree one will be called an **end point**.

Definition 23A subgraph $G' \subseteq G$ is called **induced** if $v_iv_j = v_jv_i$, with $i \neq j$, is an edge of G' whenever v_i and v_j are vertices of G' and v_iv_j is an edge of G.

The **complement** of a graph G, for which we write G^c, is the graph on the same vertex set in which $v_iv_j = v_jv_i$, with $j \neq i$, is an edge of G^c if and only if it is not an edge of G. Finally, let C_k denote the cycle on k vertices; a **chord** is an edge which is not in the edge set of C_k. A cycle is called **minimal** if it has no a chord.

If G is a finite simple graph without isolated vertices, then let R denote the polynomial ring on the vertices of G over some fixed field K.

Definition 24([6]) According to the previous context, the **edge ideal** of a finite simple graph G, with no isolated vertices, is defined by

$$I(G) = (v_iv_j \mid v_iv_j \text{ is an edge of } G),$$

with $v_iv_j = v_jv_i$, and with $i \neq j$.

3 Local cohomology module of the edge ideal of a graph simple

In this section, present some results on the local cohomology module which involve the theory of graphs together with the edge ideal of a graph G, which is simple and finite and has no isolated vertices.

Here, we take K a fixed field and we consider $K[v_1,v_2\ldots,v_s]$ the ring polynomial over the field K. Since K is a field, we have that K is a Noetherian ring and then $K[v_1,\ldots,v_s]$ is also a Noetherian ring (Theorem of the Hilbert Basis).

Remark 31 Based on previous context, $R = K[v_1,v_2\ldots,v_s]$ is a Noetherian ring. Thus, the edge ideal $I(G)$ is an R-module, so we can get characterizations for this module under certain hypothesis.

Definition 32 Let I be an ideal of the ring R, and let M be an R-module. The i-th local cohomology module $\mathrm{H}^i_I(M)$ of M with respect to I is defined by

$$\mathrm{H}^i_I(M) = \varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R\left(R/I^t, M\right),$$

for all $i \geq 0$.

Let I be an ideal of $R = K[v_1,\ldots,v_s]$. We have the definition of $\mathrm{V}(I)$:

$$\mathrm{V}(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\}.$$

The R-module $I(G)$ is called I-torsion if $I(G) = \Gamma_I(I(G))$.

Theorem 33*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a coatomic R-module. The following statements are equivalent:*

(1)*$I(G)$ is I-torsion;*
$(2)$$\mathrm{H}^i_I(I(G)) = 0$ *for all $i > 0$.*

Proof.$(1) \Rightarrow (2)$. It follows from [8, 1.13(1)], where we put $J = 0$.
$(2) \Rightarrow (1)$. By [5, 2.4], there exists an integer t such that

$$I(G)/\left(0 :_{I(G)} (v_1,\ldots,v_s)^t\right)$$

is finitely generated. If $I(G) = (0 :_{I(G)} (v_1,\ldots,v_s)^t)$, $I(G)$ is an $(v_1,\ldots,v_s)$-torsion R-module. Since $(v_1,\ldots,v_s)$ is maximal in R, we see that $I(G)$ is I-torsion. Now, assume that $I(G)/\left(0:_{I(G)}(v_1,\ldots,v_s)^t\right)\neq 0$. This implies that $(v_1,\ldots,v_s)\in \operatorname{Supp}\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^t)\right)$. From the short exact sequence

$$0\to(0:_{I(G)}(v_1,\ldots,v_s)^t)\to I(G)\to I(G)/\left(0:_{I(G)}(v_1,\ldots,v_s)^t\right)\to 0$$

we have

$$\operatorname{Supp}(I(G))=\operatorname{Supp}\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^t)\right)\bigcup \operatorname{Supp}(0:_{I(G)}(v_1,\ldots,v_s)^t)$$

and then

$$\operatorname{Supp}(I(G))=\operatorname{Supp}\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^t)\right)$$

and

$$\mathrm{H}^i_I(I(G))\cong \mathrm{H}^i_I\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^t)\right),$$

for all $i>0$. By the assumption, $\mathrm{H}^i_I\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^t)\right)=0$, for all $i>0$. Therefore, we can conclude that

$$\operatorname{Supp}\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^t)\right)\subseteq \mathrm{V}(I),$$

by [8, 4.2] where we put $J=0$, and the proof is complete.

Definition 34 An R-module M is minimax if there exists a finitely generated submodule N of M such that M/N is Artinian.

Minimax modules were first introduced and investigated by H. Zöschinger in [9].

Theorem 35*Let $R=K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a coatomic R-module with $d=\dim(I(G))>0$ or a minimax R-module with $d=\dim(I(G))>1$. Then, $\mathrm{H}^d_I(I(G))$ is Artinian and*

$$\operatorname{Att}\left(\mathrm{H}^d_I(I(G))\right)=\{\mathfrak{p}\in\operatorname{Supp}(I(G))\mid \operatorname{cd}(I,R/\mathfrak{p})=d\},$$

where $\operatorname{cd}(I,I(G))=\sup\{n\mid \mathrm{H}^n_I(I(G))\neq 0\}$.

*Proof.*First we assume that $I(G)$ is a coatomic R-module. Then, there exists an integer $k\geq 1$ such that $I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k)$ is finitely generated by [5, 2.4] and

$$\mathrm{H}^i_I(I(G))\cong \mathrm{H}^i_I\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k)\right),$$

for all $i>0$. Since $\dim(I(G))>0$, we can conclude that

$$\operatorname{Supp}(I(G))=\operatorname{Supp}\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k)\right)$$

and

$$\dim(I(G))=\dim\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k)\right).$$

From [10, 2.1] where we put $J=0$, we have that

$$\mathrm{H}^d_I\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k)\right)$$

is an R-module Artinian and then $\mathrm{H}^d_I(I(G))$ is also an Artinian R-module. Now, we have by [11, 2.1] where we put $J=0$, that

$$\operatorname{Att}(\mathrm{H}^d_I(I(G)))=\operatorname{Att}\left(\mathrm{H}^d_I(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k))\right)$$

and then

$$\operatorname{Att}(\mathrm{H}^d_I(I(G)))=\left\{\mathfrak{p}\in\operatorname{Supp}(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k))\mid \operatorname{cd}(I,R/\mathfrak{p})=d\right\}.$$

Thus, we have that

$$\operatorname{Att}(\mathrm{H}^d_I(I(G)))=\{\mathfrak{p}\in\operatorname{Supp}(I(G))\mid \operatorname{cd}(I,R/\mathfrak{p})=d\}.$$

In the case that $I(G)$ is a minimax R-module, we have that there exists a short exact sequence

$$0\to N\to I(G)\to A\to 0,$$

where N is finitely generated and A is Artinian. Since $\dim(I(G))>0$ and A is Artinian, we have $\operatorname{Supp}(I(G))=\operatorname{Supp}(N)$ and $\dim(I(G))=\dim(N)$. Applying the functor $\Gamma_I(\bullet)$ to the above exact sequence, we obtain an exact sequence

$$0\to\mathrm{H}^0_I(N)\to\mathrm{H}^0_I(I(G))\to\mathrm{H}^0_I(A)\to\mathrm{H}^1_I(N)\to \mathrm{H}^1_I(I(G))\to 0$$

and $\mathrm{H}^i_I(N)\cong\mathrm{H}^i_I(I(G))$, for all $i\geq 2$. Since N is a finitely generated R-module, we have by [10, 2.1], where we put $J=0$, that $\mathrm{H}^d_I(N)$ is Artinian check meaning $\mathrm{H}^d_I(I(G))$. Using [11, 2.1], where we put $J=0$ again, we have

$$\operatorname{Att}\left(\mathrm{H}^d_I(I(G))\right)=\operatorname{Att}\left(\mathrm{H}^d_I(N)\right)=\{\mathfrak{p}\in\operatorname{Supp}(N)\mid\operatorname{cd}(I,R/\mathfrak{p})=d\},$$

and then

$$\operatorname{Att}\left(\mathrm{H}^d_I(I(G))\right)=\{\mathfrak{p}\in\operatorname{Supp}(M)\mid\operatorname{cd}(I,R/\mathfrak{p})=d\},$$

and the proof is complete.

Remark 36 Note, if $I(G)$ is a minimax R-module with $\dim(I(G))=1$, then we observe that

$$\operatorname{Att}(\mathrm{H}^1_I(I(G)))\subseteq\{\mathfrak{p}\in\operatorname{Supp}(I(G))\mid\operatorname{cd}(I,R/\mathfrak{p})=1\}.$$

Proposition 37*Let $R=K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a coatomic R-module with $\dim(I(G))>0$. Assume that t is a non-negative integer such that*

$$\operatorname{Supp}(\mathrm{H}^i_I(I(G)))\subseteq\{(v_1,\ldots,v_s)\},$$

for all $i<t$. Then $\mathrm{H}^i_I(I(G))$ is Artinian for all $0<i<t$.

*Proof.*It follows from the proof of the Theorem 35 that, there exists an integer $k\geq 1$ such that $I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k)$ is finitely generated and

$$\operatorname{Supp}(I(G))=\operatorname{Supp}\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k)\right).$$

By the hypothesis we see that

$$\mathrm{Supp}\left(\mathrm{H}_I^i(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k))\right) \subseteq \{(v_1,\ldots,v_s)\},$$

for all $i<t$. Since finitely generated modules are minimax modules, we have by [12, 2.8], where we put $J=0$, that $\mathrm{H}_I^i(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k))$ is Artinian for all $i<t$. Note that $\mathrm{H}_I^i(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k)) \cong \mathrm{H}_I^i(I(G))$ for all $i>0$ and which completes the proof.

Now, we prove the following result.

Proposition 38 *Let $R=K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a coatomic R-module. The following statements hold:*

(1) $\mathrm{H}_I^i(I(G))=0$ *for all* $i>\dim(I(G))$.
(2) *Suppose that* $\sqrt{I}=(v_1,\ldots,v_s)$. *Then,*

$$\sup\{n \mid \mathrm{H}_I^n(I(G))\neq 0\}=\dim(I(G)).$$

Proof. (1). If $\dim(I(G))=-1$, then $I(G)=0$. Now, suppose that $\dim(I(G))\geq 0$. By the assumption on $I(G)$, there exists an integer $t\geq 1$ such that $I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^t)$ is finitely generated. Let $N=(0:_{I(G)}(v_1,\ldots,v_s)^t)$. Now the short exact sequence

$$0\to N\to I(G)\to I(G)/N\to 0$$

gives rise that $\mathrm{Supp}(\mathrm{Im}(\alpha))\subseteq \mathrm{Supp}(N)\subseteq\{(v_1,\ldots,v_s)\}$. This implies that $\dim(\mathrm{Im}(\alpha))\leq 0$. If $I(G)=N$, we can easily check the claim. So, in the remainder of the proof, we may assume that N is a submodule own of $I(G)$. Now, from the short exact sequence

$$0\to \mathrm{Im}(\alpha)\to I(G)\to I(G)/N\to 0$$

we get $\dim(I(G))=\dim(I(G)/N)$. Since $I(G)/N$ is a finitely generated R-module, we have $\mathrm{H}_I^i(I(G)/N)=0$ for all $i>\dim(I(G))$, by [8, 4.3] where we put $J=0$. Now, the conclusion follows from the isomorphism

$$\mathrm{H}_I^i(I(G))\cong \mathrm{H}_I^i(I(G)/N),$$

for all $i>0$.
(2). Combining [8, 4.5], where we put $J=0$, with the isomorphism $\mathrm{H}_I^i(I(G))\cong\mathrm{H}_I^i(I(G)/N)$ for all $i>0$, we get the assertion.

4 Application of vanishing results

In this section, we continue with the same context of the previous section.

Here, we prove a result of vanishing which involves the theory of graphs.

Theorem 41 *Let $R=K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a finitely generated R-module and let t be a positive integer. The following statements are equivalent:*

(1) $\mathrm{H}_I^i(I(G))=0$ *for all* $i\geq t$;
(2) $\mathrm{H}_I^i(I(G))$ *is finitely generated for all* $i\geq t$;
(3) $\mathrm{H}_I^i(I(G))$ *is coatomic for all* $i\geq t$.

Proof. $(1)\Rightarrow(2)\Rightarrow(3)$. Trivial.
$(3)\Rightarrow(1)$. The proof is by induction on $\dim(I(G))$. Let $n=\dim(I(G))$. If $n=0$, $\mathrm{H}_I^i(I(G))=0$, for all $i>0$.

Let $n>0$, it follows from [8, 1.13] where we put $J=0$, that

$$\mathrm{H}_I^i(I(G))\cong \mathrm{H}_I^i\left(I(G)/\Gamma_I(I(G))\right),$$

for all $i>0$. Denote by $I(\bar{G})=I(G)/\Gamma_I(I(G))$, it is clear that $I(\bar{G})$ is I-torsion-free. Thus, $I(\bar{G})$ is $(v_1,\ldots,v_s)$-torsion-free and there exists an element $x\in(v_1,\ldots,v_s)$ which is regular on $I(G)$. Now, the short exact sequence

$$0\to I(\bar{G})\xrightarrow{\cdot x} I(\bar{G})\to I(\bar{G})/xI(\bar{G})\to 0$$

induces a long exact sequence

$$\ldots\to \mathrm{H}_I^i(I(\bar{G}))\xrightarrow{\cdot x}\mathrm{H}_I^i(I(\bar{G}))\to \mathrm{H}_I^i\left(I(\bar{G})/xI(\bar{G})\right)\to\ldots\ .$$

By the assumption, $\mathrm{H}_I^i\left(I(\bar{G})/xI(\bar{G})\right)$ is coatomic for all $i\geq t$. Since

$$\dim(I(\bar{G})/xI(\bar{G}))<\dim(I(\bar{G}))\leq n,$$

and $I(\bar{G})$ is a finitely generated R-module, it follows from the inductive hypothesis that $\mathrm{H}_I^i\left(I(\bar{G})/xI(\bar{G})\right)=0$, for all $i\geq t$. Now, the long exact sequence yields

$$\mathrm{H}_I^i(I(\bar{G}))\cong x\mathrm{H}_I^i\left(I(\bar{G})\right),$$

for all $i\geq t$. Note that coatomic modules satisfy Nakayama's Lemma. Thus, we have that $\mathrm{H}_I^i(I(\bar{G}))=0$ for all $i\geq t$, and the proof is complete.

Now, we have some consequences.

Corollary 42 *Let $R=K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a coatomic R-module and let t be a positive integer. The following statements are equivalent:*

(1) $\mathrm{H}_I^i(I(G))=0$ *for all* $i\geq t$;
(2) $\mathrm{H}_I^i(I(G))$ *is finitely generated for all* $i\geq t$;
(3) $\mathrm{H}_I^i(I(G))$ *is coatomic for all* $i\geq t$.

Proof. Since $I(G)$ is a coatomic R-module, we have that there exists an integer $k\geq 1$ such that $I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k)$ is finitely generated by [5, 2.4]. Therefore, we have the isomorphisms

$$\mathrm{H}_I^i(I(G))\cong\mathrm{H}_I^i\left(I(G)/(0:_{I(G)}(v_1,\ldots,v_s)^k)\right)$$

for all $i>0$. The assertion it follows immediately from Theorem 41.

Corollary 43*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a minimax R-module and let t be a positive integer. The following statements are equivalent:*

(1)$\mathrm{H}^i_I(I(G)) = 0$ *for all* $i \geq t$;
(2)$\mathrm{H}^i_I(I(G))$ *is finitely generated for all* $i \geq t$;
(3)$\mathrm{H}^i_I(I(G))$ *is coatomic for all* $i \geq t$.

Proof.(1) $\Rightarrow$ (2) $\Rightarrow$ (3). Trivial.
(3) $\Rightarrow$ (1). Since $I(G)$ is a minimax R-module, there exists a short exact sequence

$$0 \to N \to I(G) \to A \to 0,$$

where N is finitely generated and A is Artinian. Applying the functor $\Gamma_I(\bullet)$ to the above exact sequence, we get a long exact sequence

$$0 \to \mathrm{H}^0_I(N) \to \mathrm{H}^0_I(I(G)) \to \mathrm{H}^0_I(A) \to \mathrm{H}^1_I(N) \to \mathrm{H}^1_I(I(G)) \to 0,$$

and $\mathrm{H}^i_I(N) \cong \mathrm{H}^i_I(I(G))$ for all $i \geq 2$. By the hypothesis, $\mathrm{H}^i_I(N)$ is coatomic for all $i \geq t$. It follows from Theorem 41 that $\mathrm{H}^i_I(N) = 0$ for all $i \geq t$ and which completes the proof.

Corollary 44*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a finitely generated R-module with $\mathrm{cd}(I,I(G)) > 0$. Then,*

$$\mathrm{H}^{\mathrm{cd}(I,I(G))}_I(I(G))$$

is not finitely generated.

Combining [8, 4.5], where we put $J = 0$, with Corollary 44, we have an immediate consequence.

Corollary 45*Let $R = K[v_1,\ldots,v_s]$ be the ring polynomial, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a finitely generated R-module with $\dim(I(G)) > 0$ and $\sqrt{I} = (v_1,\ldots,v_s)$. Then, $\mathrm{H}^{\dim(I(G))}_I(I(G))$ is not finitely generated.*

5 Conclusion

In this paper, we can relate the theory of graphs, with respect to the edge ideal of a finite simple graph, to the theory of local cohomology modules. The results indicated the importance of local cohomology theory as a study tool within the commutative algebra theory.

Moreover, establishing this relationship, helps us get applications for the edge ideal in the general theory of modules.

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