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Stability and Ultimate Boundedness for Non-Autonomous System with Variable Delay

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Abstract: In this paper, we investigate proper sufficient conditions for the uniform stability (US) of the zero solution, and for the uniform boundedness (UB) as well as uniform ultimate boundedness (UUB) of all solutions of a certain system of nonlinear non-autonomous third-order differential equation (DE) with variable delay.
In the proofs, the method of Lyapunov functional (LF) approach is employed as a main tool and two examples are presented in the last section to show feasibility of the established results which improve the results of the previous pieces of literature.

Keywords: (US), (UUB), (LF), third-order vector (DE), variable delay

1 Introduction

Over the last few decades, the qualitative behaviour (QB) of solutions for ordinary scalar and vector nonlinear (DEs) has been extensively investigated and several results have been obtained. Stability (S) and boundedness (B) of solutions play a key role in characterizing the behaviour of nonlinear (DEs). For a comprehensive investigation of this subject, we refer the reader to the books by Burton [1], Reissig *et al.* [2], Yoshizawa [3] and the references cited in these books. To verify the results of the above-mentioned books, Lyapunov's second method [4] has been used.

However, the response of the system, in many applications, can be delayed, or be established on the past history of the system. Dynamical systems, which respond in this way, are called delay differential equations (DDEs). The Lyapunov's second method has been developed to deal with (DDEs).

Many results have been obtained on the (S) and (B) of solutions for various second and third-order scalar vector (DEs) without delay, see for example, [5,6,7,8,9,10,11,12,13], etc. Since (S) and (B) are much more complicated for (DDEs), it is worth-while to continue to investigate the (S) and (B) of solutions for vector (DDEs).

On the other hand, for certain third-order scalar (DDEs), the (S) and the (B) results have been investigated by many researchers, see, for example, [14,15,16,17,18,19,20,21,22,23,24]. In this regard, these cited papers present outstanding results on the (QB) of solutions for the considered (DEs).

However, by this time, no attention was given to the investigation of the (S) and (B) in the nonlinear vector (DEs) of third-order with constant and variable delays, except the works of Tunç and Mohammed [25], Omeike [26], Mahmoud and Tunç [27], Tunç [28,29,30], and the references therein.

In the following, we provide some background details regarding the study of various classes of third-order vector (DEs) with variable delay.

In [26], Omeike investigated the (S) and (B) of nonlinear differential system (DS) of third-order with variable delay $r(t)$, of the following form

$$\dddot{X}+A\ddot{X}+B\dot{X}+H(X(t-r(t)))=P(t).$$

Recently, in [28], Tunç has explored the (S) and (B) of nonlinear (DS) of third-order with variable delay $\tau(t)$, as the following type

$$\dddot{X}+A\ddot{X}+G(\dot{X}(t-\tau(t)))+H(X(t-\tau(t)))=F(t,X,\dot{X},\ddot{X}).$$

In this paper, defining (LFs), we obtain proper sufficient conditions for the (S) and the (B) of solutions in the cases $P(\cdot)=0$ and $P(\cdot)\neq 0$ respectively, to the following

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nonlinear non-autonomous third-order (DS) with variable delay of the type

$$\dddot{X} + \Phi(X,\dot{X})\ddot{X} + \Psi(\dot{X}(t-r(t))) + c(t)H(X(t-r(t))) = P(\cdot), \quad (1)$$

where

$$P(\cdot) = P(t, X, X(t-r(t)), \dot{X}, \dot{X}(t-r(t)), \ddot{X}).$$

The equation (1) can be written in the following equivalent system

$$\dot{X} = Y,$$
$$\dot{Y} = Z,$$
$$\dot{Z} = -\Phi(X,Y)Z - \Psi(Y) - H(X) + \int_{t-r(t)}^{t} J_{\Psi}(Y(s))Z(s)ds + c(t)\int_{t-r(t)}^{t} J_H(X(s))Y(s)ds + P(t, X, X(t-r(t)), Y, Y(t-r(t)), Z), \quad (2)$$

where $0 \le r(t) \le \gamma$, γ is a positive constant, which will be defined later; $X \in \mathbb{R}^n$; $c : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ is a continuous function, Φ is an $n\times n$-continuous symmetric function matrix, Ψ and H are n-vector continuous functions with $\Psi(0) = H(0) = 0$ and $P(\cdot)$ is a vector continuous function in their arguments; $t \in [0,\infty)$. Moreover, it is assumed that the Jacobian matrices $J_H(X) = (\frac{\partial h_i}{\partial x_j})$ and $J_{\Psi}(Y) = (\frac{\partial \psi_i}{\partial y_j})$, $(i, j = 1, 2, \ldots, n)$, exist and are continuous.

Remark 1.1. We observed the following:

(i) If $\Phi(X,\dot{X}) = A$, $\Psi(\dot{X}(t-r(t))) = B\dot{X}$, $c(t)$ as a constant such as equal one, and $P(\cdot) = P(t)$, equation (1) reduces to nonlinear (DS) of third-order with variable delay $r(t)$ in [26].

(ii) If $\Phi(X,\dot{X}) = A$, $\Psi(\dot{X}(t-r(t))) = G(\dot{X}(t-r(t)))$, $c(t)$ as a constant such as equal one, and $P(\cdot) = F(t, X, \dot{X}, \ddot{X})$, equation (1) reduces to nonlinear (DS) of third-order with variable delay $r(t)$ in [28]. However, in this paper, we construct a new (LF) to investigate the (UB) and (UUB) of all solutions for (1).

(iii) Special cases of (1), when $n = 1$, have been investigated by several authors in the pieces of literature, see [15,16] and the references cited in these sources.

2 Preliminaries

The symbol $\langle X,Y\rangle$ corresponding to any pair X and Y of vectors in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$. The Euclidean length in $\mathbb{R}^n$ will be denoted by $\|.\|$, so that in particular $\langle X,X\rangle = \|X\|^2$ for arbitrary $X \in \mathbb{R}^n$, and $\lambda_i(M)(i = 1, 2, \ldots, n)$ are the eigenvalues of the real symmetric $n\times n$-matrix M. The matrix M is negative-definite, when $\langle MX,X\rangle < 0$, for all nonzero $X \in \mathbb{R}^n$.

The following Lemmas will be substantial for the proofs of the main Theorems.

Lemma 2.1.[5] Let M be a real symmetric positive definite $n\times n$-matrix, then for any $X \in \mathbb{R}^n$, we have

$$\alpha_M\|X\|^2 \le \langle MX,X\rangle \le \beta_M\|X\|^2,$$

where α_M, β_M are the least and the greatest eigenvalues of M, respectively.

Lemma 2.2. Assume that $\dot{X} = Y$, $\dot{Y} = Z$. Then

(1) $\frac{d}{dt}\left(\int_0^1 \langle H(\sigma X), X\rangle d\sigma\right) = \langle H(X), Y\rangle,$

(2) $\frac{d}{dt}\left(\int_0^1 \langle \Psi(\sigma Y), Y\rangle d\sigma\right) = \langle \Psi(Y), Z\rangle,$

(3) $\frac{d}{dt}\left(\int_0^1 \langle \sigma\Phi(X,\sigma Y)Y, Y\rangle d\sigma\right) \le \langle \Phi(X,Y)Y, Z\rangle.$

Proof. The proof of (2) is similar to that of (1), see [27].

$$(3)\quad \frac{d}{dt}\int_0^1 \langle \sigma\Phi(X,\sigma Y)Y,Y\rangle d\sigma = \int_0^1 \langle \sigma\Phi(X,\sigma Y)Y,Z\rangle d\sigma + \int_0^1 \langle \sigma J(\Phi(X,\sigma Y),Y|X)Y,Y\rangle d\sigma + \int_0^1 \sigma\langle \sigma J(\Phi(X,\sigma Y),Y|Y)Z,Y\rangle d\sigma$$
$$\le \int_0^1 \langle \sigma\Phi(X,\sigma Y)Y,Z\rangle d\sigma + \int_0^1 \sigma\langle \sigma J(\Phi(X,\sigma Y),Y|Y)Z,Y\rangle d\sigma.$$

Let $J(\Phi(X,Y)Y|X)$ be a negative-definite and $J(\Phi(X,Y)Y|Y)$ be a symmetric. It follows that

$$\frac{d}{dt}\int_0^1 \langle \sigma\Phi(X,\sigma Y)Y,Y\rangle d\sigma \le \int_0^1 \langle \sigma\Phi(X,\sigma Y)Y,Z\rangle d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \Phi(X,\sigma Y)\sigma Y,Z\rangle d\sigma = \sigma\langle \Phi(X,\sigma Y),Z\rangle|_0^1 = \langle \Phi(X,Y)Y,Z\rangle.$$

Lemma 2.3. [27,28] Let $H(X)$ be a continuous vector function with $H(0) = 0$. Then,

$$(1)\ \langle H(X), H(X)\rangle = 2\int_0^1\int_0^1 \sigma_1\langle J_H(\sigma_1 X)J_H(\sigma_1\sigma_2 X)X,X\rangle d\sigma_2 d\sigma_1.$$

$$(2)\ \langle H(X), X\rangle = \int_0^1 \langle J_H(\sigma X)X,X\rangle d\sigma.$$

Lemma 2.4. Let $H(X)$ be a continuous vector function and that $H(0) = 0$. Then,

$$\alpha_H\|X\|^2 \le \int_0^1 \langle H(\sigma X,X)\rangle d\sigma \le \beta_H\|X\|^2,$$

where α_H, β_H are the least and the greatest eigenvalues of $J_H(X)$, respectively.

3 Stability

Now, we consider the (S) criteria for the general non-autonomous delay differential system:

$$\dot{\bar{x}}(t) = \bar{f}(t,\bar{x}_t),\ \bar{x}_t(s) = \bar{x}(t+s),\ -h \le s \le 0,\ t \ge 0, \quad (3)$$

where $\bar{f} : [0,\infty)\times C_H \to \mathbb{R}^n$ is a continuous mapping, $\bar{f}(t,0) = \bar{0}$, $C_H := \{\vartheta \in C([-h,0],\mathbb{R}^n) : \|\vartheta\| \le H\}$ and for $H_1 < H$, there exists $L(H_1) > 0$, with $|\bar{f}(t,\vartheta)| \le L(H_1)$ when $\|\vartheta\| \le H_1$.

Theorem 3.1. *[1]* Let $V(t,\vartheta) : C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition, $V(0) = 0$ and the functions W_i $(i = 1,2)$ are wedges, such that:

(a) $W_1(|\vartheta(0)|) \le V(t,\vartheta) \le W_2(\|\vartheta\|)$,
(b) $\dot{V}_{(3)}(t,\vartheta) \le 0$, for $\vartheta \in C_H$.

Then, the zero solution of (3) is (US).

The main (S) result of (1) with $P(\cdot) = 0$ is the following theorem:

Theorem 3.2. In addition to the fundamental assumptions imposed on the functions Φ,Ψ,H and $c(t)$ with $P(\cdot) = 0$, let us assume that there exist positive constants $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3$ and γ, such that for $(i = 1,2,...,n)$, the following conditions hold:

(i) The matrix Φ is symmetric and $\alpha_1 \le \lambda_i(\Phi(X,Y)) \le \beta_1$, for all $X,Y \in \mathbb{R}^n$.
(ii) $\Psi(0) = 0$, $J_\Psi(Y)$ is symmetric and $\alpha_2 \le \lambda_i(J_\Psi(Y)) \le \beta_2$, for all $Y \in \mathbb{R}^n$.
(iii) $H(0) = 0$, $J_H(X)$ is symmetric and $\alpha_3 \le \lambda_i(J_H(X)) \le \beta_3$, for all $X \in \mathbb{R}^n$.
(iv) $0 < \delta_c \le \lambda_i(c(t)) \le \Delta_c \le 1$ and $\lambda_i(c') \le 0$.
(v) $\frac{1}{\alpha_1} < \frac{1}{\mu} < \frac{\alpha_2}{\beta_3}$.
(vi) $0 \le r(t) \le \gamma$ and $r'(t) \le \omega$, $0 < \omega < 1$.

Then, the zero solution of (1) with $P(\cdot) = 0$ is (US), provided that

$$\gamma < min\Bigg[\frac{(\mu\alpha_2 - \beta_3\Delta_c)(1-\omega)}{\sqrt{n}\,[\{\mu(2-\omega)+1\}\beta_3\Delta_c + \mu(1-\omega)\beta_2]},$$
$$\frac{(\alpha_1-\mu)(1-\omega)}{\sqrt{n}\,\{(1-\omega)\beta_3\Delta_c + (2+\mu-\omega)\beta_2\}}\Bigg].$$

Proof.

Let λ and δ be two positive constants, which will be defined later in the proof. For the sake of brevity, we define

$$\Delta(t) = \lambda \int_{-r(t)}^{0}\int_{t+s}^{t} \|Y(\xi)\|^2 d\xi ds + \delta \int_{-r(t)}^{0}\int_{t+s}^{t} \|Z(\xi)\|^2 d\xi ds \ge 0. \quad (4)$$

Our main tool in the proof of the Theorem 3.2 is a (LF), $V_1(\cdot) = V_1(t,X_t,Y_t,Z_t)$ defined by

$$V_1(\cdot) = \mu \int_0^1 \langle c(t)H(\sigma X),X\rangle d\sigma + \int_0^1 \langle\Psi(\sigma Y),Y\rangle d\sigma + \langle c(t)H(X),Y\rangle + \mu\int_0^1 \langle\sigma\Phi(X,\sigma Y)Y,Y\rangle d\sigma + \frac{1}{2}\langle Z,Z\rangle + \mu\langle Y,Z\rangle + \Delta(t). \quad (5)$$

Using Lemma 2.3, we observe that the above (LF) can be rewritten as follows:

$$V_1(\cdot) = \mu \int_0^1\int_0^1 \sigma_1\langle c(t)J_H(\sigma_1\sigma_2 X)X,X\rangle d\sigma_2 d\sigma_1 + \int_0^1\int_0^1 \sigma_1\langle J_\Psi(\sigma_1\sigma_2 Y)Y,Y\rangle d\sigma_2 d\sigma_1 + \langle c(t)H(X),Y\rangle + \frac{1}{2}\|Z+\mu Y\|^2 + \mu\int_0^1 \langle(\sigma\Phi(X,\sigma Y) - \frac{1}{2}\mu)Y,Y\rangle d\sigma + \Delta(t).$$

From the condition $(i)-(iii)$ of Theorem 3.2, (4) and Lemma 2.4, we get

$$V_1(\cdot) \ge \mu \int_0^1\int_0^1 \sigma_1\langle c(t)J_H(\sigma_1\sigma_2 X)X,X\rangle d\sigma_2 d\sigma_1 + \frac{\alpha_2}{2}\|Y + \frac{1}{\alpha_2}c(t)H(X)\|^2 - \frac{1}{2\alpha_2}\langle c(t)H(X),c(t)H(X)\rangle + \frac{1}{2}\mu(\alpha_1-\mu)\|Y\|^2 + \frac{1}{2}\|Z+\mu Y\|^2.$$

Since

$$\frac{\alpha_2}{2}\|Y + \frac{1}{\alpha_2}c(t)H(X)\|^2 \ge 0.$$

Then

$$V_1(\cdot) \ge \int_0^1\int_0^1 \sigma_1\left\langle\left[\mu c(t) - \frac{c^2(t)}{\alpha_2}J_H(\sigma_1 X)\right]J_H(\sigma_1\sigma_2 X)X,X\right\rangle d\sigma_2 d\sigma_1 + \frac{1}{2}\|Z+\mu Y\|^2 + \frac{1}{2}\mu(\alpha_1-\mu)\|Y\|^2$$
$$\ge \frac{\kappa}{2}\|X\|^2 + \frac{1}{2}\|Z+\mu Y\|^2 + \frac{1}{2}\mu(\alpha_1-\mu)\|Y\|^2, \quad (6)$$

where

$$\kappa = \mu\alpha_3\delta_c\left(1 - \frac{\beta_3}{\mu\alpha_2}\right) > \mu\alpha_3\delta_c\left(1-\frac{\mu}{\mu}\right) = 0, \text{ and } \alpha_1 - \mu > 0,$$

by the conditions $(iii)-(v)$.
Thus, we can find a positive constant D_1, small enough such that

$$V_1(\cdot) \ge D_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2). \quad (7)$$

Using the hypotheses of Theorem 3.2, we obtain $\|\Phi(X,Y)\| \le \sqrt{n}\beta_1$; by (i), then

$$\mu\int_0^1 \langle\sigma\Phi(X,\sigma Y)Y,Y\rangle d\sigma \le \mu\int_0^1 \sqrt{n}\beta_1\langle\sigma Y,Y\rangle d\sigma = \frac{\mu}{2}\sqrt{n}\beta_1\|Y\|^2.$$

Since $\frac{\partial H(\sigma X)}{\partial\sigma} = J_H(\sigma X)X$ and $H(0) = 0$, we find from (iii) that

$$\|H(X)\| \le \int_0^1 \|J_H(\sigma X)\|\|X\|d\sigma \le \sqrt{n}\beta_3\|X\|.$$

Also, since $\frac{\partial\Psi(\sigma Y)}{\partial\sigma} = J_\Psi(\sigma Y)Y$ and $\Psi(0) = 0$, from (ii) we find

$$\|\Psi(Y)\| \le \int_0^1 \|J_\Psi(\sigma Y)\|\|Y\|d\sigma \le \sqrt{n}\beta_2\|Y\|.$$

Using the Cauchy-Schwarz inequality $|\langle m,n\rangle| \le \|m\|\|n\| \le \frac{1}{2}(\|m\|^2+\|n\|^2)$, we have

$$\begin{aligned}\langle c(t)H(X),Y\rangle &\le \|c(t)\|\|H(X)\|\|Y\|\\ &\le \sqrt{n}\beta_3\Delta_c\|X\|\|Y\|\\ &\le \frac{1}{2}\sqrt{n}\beta_3\Delta_c(\|X\|^2+\|Y\|^2).\end{aligned}$$

From (vi), we obtain

$$\begin{aligned}\int_{-r(t)}^{0}\int_{t+s}^{t}\|Y(\xi)\|^2d\xi ds &= \int_{t-r(t)}^{t}\{\xi-t+r(t)\}\|Y(\xi)\|^2d\xi\\ &\le \|Y\|^2\int_{t-r(t)}^{t}\{\xi-t+r(t)\}d\xi\\ &= \frac{1}{2}r^2(t)\|Y\|^2\\ &= \frac{1}{2}r(t)\int_{t-r(t)}^{t}\|Y\|^2d\xi\\ &\le \frac{1}{2}\gamma^2\|Y\|^2. \end{aligned} \tag{8}$$

Similarly, we find

$$\begin{aligned}\int_{-r(t)}^{0}\int_{t+s}^{t}\|Z(\xi)\|^2d\xi ds &\le \frac{1}{2}r^2(t)\|Z\|^2\\ &= \frac{1}{2}r(t)\int_{t-r(t)}^{t}\|Z\|^2d\xi\\ &\le \frac{1}{2}\gamma^2\|Z\|^2. \end{aligned} \tag{9}$$

Thus, we obtain

$$\begin{aligned}V_1(\cdot) \le{}& \left(\mu+\frac{1}{2}\right)\sqrt{n}\beta_3\Delta_c\|X\|^2+\frac{1}{2}\left(\mu+1+\delta\gamma^2\right)\|Z\|^2\\ &+\frac{1}{2}\left(\sqrt{n}\beta_3\Delta_c+\mu+\mu\sqrt{n}\beta_1+4\sqrt{n}\beta_2+\lambda\gamma^2\right)\|Y\|^2. \end{aligned} \tag{10}$$

Hence, we have a positive constant D_2 satisfying

$$V_1(\cdot) \le D_2(\|X\|^2+\|Y\|^2+\|Z\|^2). \tag{11}$$

Now, since

$$\begin{aligned}&\frac{d}{dt}\int_{-r(t)}^{0}\int_{t+s}^{t}\|Y(\xi)\|^2d\xi ds\\ &= (1-r'(t))\int_{t-r(t)}^{t}\|Y(\xi)\|^2d\xi+r(t)\|Y(t)\|^2. \end{aligned} \tag{12}$$

Similarly, we find

$$\begin{aligned}&\frac{d}{dt}\int_{-r(t)}^{0}\int_{t+s}^{t}\|Z(\xi)\|^2d\xi ds\\ &= (1-r'(t))\int_{t-r(t)}^{t}\|Z(\xi)\|^2d\xi+r(t)\|Z(t)\|^2. \end{aligned} \tag{13}$$

Then, from (2), (5), (12), (13) and Lemma 2.2, we have

$$\begin{aligned}\frac{d}{dt}V_1(\cdot) \le{}& \langle c(t)J_H(X)Y,Y\rangle+\mu\langle Z,Z\rangle-\mu\langle Y,\Psi(Y)\rangle\\ &-\langle\Phi(X,Y)Z,Z\rangle\\ &+\mu\int_0^1\langle c'(t)H(\sigma X),X\rangle d\sigma+\langle c'(t)H(X),Y\rangle\\ &+\langle\mu Y+Z,\int_{t-r(t)}^{t}J_\Psi(Y(s))Z(s)ds\rangle\\ &+\langle\mu Y+Z,\int_{t-r(t)}^{t}c(t)J_H(X(s))Y(s)ds\rangle\\ &-\lambda(1-r'(t))\int_{t-r(t)}^{t}\|Y(\xi)\|^2d\xi+\lambda r(t)\|Y(t)\|^2\\ &-\delta(1-r'(t))\int_{t-r(t)}^{t}\|Z(\xi)\|^2d\xi+\delta r(t)\|Z(t)\|^2. \end{aligned} \tag{14}$$

Now, consider the term

$$\begin{aligned}\Omega_1 &= \mu\int_0^1\langle c'(t)H(\sigma X),X\rangle d\sigma+\langle c'(t)H(X),Y\rangle\\ &\le \mu\int_0^1\int_0^1\sigma_1\langle c'(t)J_H(\sigma_1\sigma_2X)X,X\rangle d\sigma_2d\sigma_1\\ &\le \frac{\mu\beta_3}{2}\langle c'(t)X,X\rangle \le 0, \text{ by } (iv).\end{aligned}$$

From conditions $(i)-(iv)$ of Theorem 3.2 and Lemma 2.1, we can write (14) as follows:

$$\begin{aligned}\frac{d}{dt}V_1(\cdot) \le{}& -(\mu\alpha_2-\beta_3\Delta_c)\|Y\|^2-(\alpha_1-\mu)\|Z\|^2\\ &+\langle\mu Y+Z,\int_{t-r(t)}^{t}J_\Psi(Y(s))Z(s)ds\rangle\\ &+\langle\mu Y+Z,\int_{t-r(t)}^{t}c(t)J_H(X(s))Y(s)ds\rangle\\ &-\lambda(1-r'(t))\int_{t-r(t)}^{t}\|Y(\xi)\|^2d\xi+\lambda r(t)\|Y(t)\|^2\\ &-\delta(1-r'(t))\int_{t-r(t)}^{t}\|Z(\xi)\|^2d\xi+\delta r(t)\|Z(t)\|^2.\end{aligned}$$

Since $\|J_H(X)\| \le \sqrt{n}\beta_3$ and by condition (iv) of Theorem 3.2, then using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\left|\langle\mu Y+Z,\int_{t-r(t)}^{t}c(t)J_H(X(s))Y(s)ds\rangle\right|\\ &\le \|\mu Y+Z\|\left\|\int_{t-r(t)}^{t}c(t)J_H(X(s))Y(s)ds\right\|\\ &\le (\mu\|Y\|+\|Z\|)\int_{t-r(t)}^{t}\sqrt{n}\beta_3\Delta_c\|Y(s)\|ds\\ &\le \frac{\mu\sqrt{n}\beta_3\Delta_c}{2}\left(\|Y\|^2r(t)+\int_{t-r(t)}^{t}\|Y(s)\|^2ds\right)\\ &\quad+\frac{\sqrt{n}\beta_3\Delta_c}{2}\left(\|Z\|^2r(t)+\int_{t-r(t)}^{t}\|Y(s)\|^2ds\right).\end{aligned}$$

Since $\|J_\Psi(Y)\| \leq \sqrt{n}\beta_2$ by condition (ii) of Theorem 3.2, so using the Cauchy-Schwarz inequality, we get

$$\left|\langle \mu Y+Z, \int_{t-r(t)}^{t} J_\Psi(Y(s))Z(s)ds\rangle\right|$$
$$\leq \frac{\mu\sqrt{n}\beta_2}{2}\left(\|Y\|^2 r(t)+\int_{t-r(t)}^{t}\|Z(s)\|^2 ds\right)$$
$$+\frac{\sqrt{n}\beta_2}{2}\left(\|Z\|^2 r(t)+\int_{t-r(t)}^{t}\|Z(s)\|^2 ds\right).$$

Since $0\leq r(t)\leq\gamma$ and $r'(t)\leq\omega$ by condition (vi) of Theorem 3.2, it follows that

$$\frac{d}{dt}V_1(\cdot) = -\left\{\mu\alpha_2-\beta_3\Delta_c-\frac{\mu\sqrt{n}}{2}(\beta_3\Delta_c+\beta_2)\gamma-\lambda\gamma\right\}\|Y\|^2$$
$$-\left(\alpha_1-\mu-\frac{\sqrt{n}\beta_3\Delta_c}{2}\gamma-\frac{\sqrt{n}\beta_2}{2}\gamma-\delta\gamma\right)\|Z\|^2$$
$$+\left\{\frac{\mu\sqrt{n}\beta_3\Delta_c}{2}+\frac{\sqrt{n}\beta_3\Delta_c}{2}-\lambda(1-\omega)\right\}\int_{t-r(t)}^{t}\|Y(\xi)\|^2 d\xi$$
$$+\left\{\frac{\mu\sqrt{n}\beta_2}{2}+\frac{\sqrt{n}\beta_2}{2}-\delta(1-\omega)\right\}\int_{t-r(t)}^{t}\|Z(\xi)\|^2 d\xi.$$

If we take $\lambda=\frac{\sqrt{n}\beta_3\Delta_c}{2(1-\omega)}(\mu+1)>0$ and $\delta=\frac{\sqrt{n}\beta_2}{2(1-\omega)}(\mu+1)>0$, then

$$\frac{d}{dt}V_1(\cdot)\leq -\left\{\mu\alpha_2-\beta_3\Delta_c-\frac{\mu\sqrt{n}}{2}(\beta_3\Delta_c+\beta_2)\gamma\right.$$
$$\left.-\frac{\sqrt{n}\beta_3\Delta_c}{2(1-\omega)}(\mu+1)\gamma\right\}\|Y\|^2$$
$$-\left\{\alpha_1-\mu-\frac{\sqrt{n}}{2}(\beta_3\Delta_c+\beta_2)\gamma-\frac{\sqrt{n}\beta_2}{2(1-\omega)}(\mu+1)\gamma\right\}\|Z\|^2.$$

Therefore, if

$$\gamma<min\left[\frac{(\mu\alpha_2-\beta_3\Delta_c)(1-\omega)}{\sqrt{n}\,[\{\mu(2-\omega)+1\}\beta_3\Delta_c+\mu(1-\omega)\beta_2]},\right.$$
$$\left.\frac{(\alpha_1-\mu)(1-\omega)}{\sqrt{n}\,\{(1-\omega)\beta_3\Delta_c+(2+\mu-\omega)\beta_2\}}\right].$$

Then, it follows that

$$\frac{d}{dt}V_1(\cdot)\leq -D_3(\|Y\|^2+\|Z\|^2),\ \text{ for some } D_3>0. \tag{15}$$

From (7), (11) and (15) it can be seen that (LF), $V_1(\cdot)$ satisfies all the conditions of Theorem 3.1, so the zero solution of (1) with $P(\cdot)=0$ is (US).
Hence, the proof of Theorem 3.2 is now complete.

4 Boundedness

Now, we consider a system of (DDEs)

$$\dot{\bar{x}}=\bar{F}(t,\bar{x}_t),\ \bar{x}_t=\bar{x}(t+\theta),\ -r\leq\theta\leq 0, \tag{16}$$

where $\bar{F}:\mathbb{R}\times C\to\mathbb{R}^n$ is a continuous mapping and takes bounded sets into bounded sets.

The following theorem is a well-known result obtained by Burton [1].

Theorem 4.1. Let $V(t,\vartheta):\mathbb{R}\times C\to\mathbb{R}^n$ be a continuous functional and is locally Lipschitz in ϑ. If

(i) $W(|\bar{x}(t)|)\leq V(t,\bar{x}_t)\leq W_1(|\bar{x}(t)|)+W_2\left(\int_{t-r}^{t}W_3(|\bar{x}(s)|)ds\right)$
and
(ii) $\dot{V}_{(16)}(t,\bar{x}_t)\leq -W_3(|\bar{x}(t)|)+N$, for some $N>0$, where W and W_i $(i=1,2,3)$ are wedges,

then the solutions of (16) are (UB) and (UUB) for a bound B.

The following theorem is the (B) main result of (1).

Theorem 4.2. Suppose further to the conditions of Theorem 3.2, that there exists a constant $m>0$, such that $\|P(\cdot)\|\leq m$. Then, the solutions of (1) are (UB) and (UUB), provided that:

$$\gamma<min\left[\frac{(\sqrt{n}-1)\alpha_2+2\sqrt{n}\beta_3\delta_c}{2\sqrt{n}(\beta_2+\beta_3\Delta_c)},\right.$$
$$\frac{2(\mu\alpha_2-\beta_3\Delta_c)+\alpha_2(\sqrt{n}-1)(\alpha_1\alpha_2-\beta_3+\alpha_1+2\alpha_1^2)}{2\sqrt{n}\{(\mu+\alpha_1^2)(\beta_2+\beta_3)+\beta_3\mathscr{M}\}}$$
$$+\frac{(1-\Delta_c)(\beta_3+2\alpha_1\alpha_3)}{2\sqrt{n}\{(\mu+\alpha_1^2)(\beta_2+\beta_3)+\beta_3\mathscr{M}\}},$$
$$\left.\frac{2(\alpha_1-\mu)+(\sqrt{n}-1)\alpha_1\alpha_2+(1-\Delta_c)\beta_3}{2\sqrt{n}\{(\beta_2+\beta_3\Delta_c)(1+\alpha_1)+\beta_2\mathscr{M}\}}\right],$$

where

$$\mathscr{M}=\frac{\mu+1+\alpha_1\alpha_2-\beta_3+\alpha_1+\alpha_1^2}{1-\omega}.$$

Proof.

Now, we consider the (B) of the solutions of (1). We assume that $P(\cdot)$ is bounded with a bound m and the conditions of Theorem 3.2 hold.

Consider the (LF) as

$$V(t,X_t,Y_t,Z_t)=V_1(t,X_t,Y_t,Z_t)+V_2(t,X_t,Y_t,Z_t), \tag{17}$$

where $V_1(\cdot)$ is defined as (5) and $V_2(\cdot)=V_2(t,X_t,Y_t,Z_t)$ is defined as

$$V_2(\cdot)=\alpha_1^2\int_0^1\langle c(t)H(\sigma X),X\rangle d\sigma+\frac{1}{2}\alpha_2(\alpha_1\alpha_2-\beta_3)\langle X,X\rangle$$
$$+\alpha_1\langle c(t)H(X),Y\rangle+(\alpha_1\alpha_2-\beta_3)\langle X,Z+\alpha_1 Y\rangle$$
$$+\frac{1}{2}\beta_3\langle c(t)Y,Y\rangle+\frac{1}{2}\alpha_1\langle Z+\alpha_1 Y,Z+\alpha_1 Y\rangle. \tag{18}$$

Since $\lambda_i(c(t))\geq\delta_c$, then we can obtain

$$V_2(\cdot)\geq S_1+\frac{\delta_c}{2\beta_3}\|\beta_3 Y+\alpha_1 H(X)\|^2$$
$$+\frac{\alpha_1\alpha_2-\beta_3}{2\alpha_2}\|\alpha_2 X+(Z+\alpha_1 Y)\|^2+\frac{\beta_3}{2\alpha_2}\|Z+\alpha_1 Y\|^2,$$

where

$$S_1:=\alpha_1^2\int_0^1\langle c(t)H(\sigma X),X\rangle d\sigma-\frac{\alpha_1^2}{2\beta_3}\langle c(t)H(X),H(X)\rangle.$$

Since

$$\frac{\partial}{\partial\sigma_1}\langle H(\sigma_1 X),H(\sigma_1 X)\rangle=2\langle J_H(\sigma_1 X)X,H(\sigma_1 X)\rangle,$$

Integrating both sides from $\sigma_1 = 0$ to $\sigma_1 = 1$ and because of $H(0) = 0$, we get

$$\langle c(t)H(X), H(X)\rangle = 2\int_0^1 \langle c(t)J_H(\sigma_1 X)X, H(\sigma_1 X)\rangle d\sigma_1.$$

Since $\lambda_i(J_H(X)) \le \beta_3$ by condition (iii) of Theorem 3.2, which implies that

$$S_1 = \alpha_1^2\int_0^1 \langle c(t)H(\sigma X), X\rangle d\sigma - \frac{\alpha_1^2}{\beta_3}\int_0^1 \langle c(t)J_H(\sigma_1 X)X, H(\sigma_1 X)\rangle d\sigma_1$$
$$= \alpha_1^2\int_0^1 \left\langle c(t)H(\sigma_1 X), \{I - \frac{1}{\beta_3}J_H(\sigma_1 X)\}X\right\rangle d\sigma_1 \ge 0.$$

Thus, we obtain

$$V_2(\cdot) \ge \frac{\delta_c}{2\beta_3}\|\beta_3 Y + \alpha_1 H(X)\|^2 + \frac{\beta_3}{2\alpha_2}\|Z + \alpha_1 Y\|^2 + \frac{\alpha_1\alpha_2 - \beta_3}{2\alpha_2}\|\alpha_2 X + (Z + \alpha_1 Y)\|^2. \quad (19)$$

Then, from (6), (17) and (19), we have

$$V(\cdot) \ge \frac{\kappa}{2}\|X\|^2 + \frac{1}{2}\|Z + \mu Y\|^2 + \frac{1}{2}\mu(\alpha_1 - \mu)\|Y\|^2 + \frac{\beta_3}{2\alpha_2}\|Z + \alpha_1 Y\|^2 + \frac{\delta_c}{2\beta_3}\|\beta_3 Y + \alpha_1 H(X)\|^2 + \frac{\alpha_1\alpha_2 - \beta_3}{2\alpha_2}\|\alpha_2 X + (Z + \alpha_1 Y)\|^2. \quad (20)$$

From conditions $(i) - (iii)$ of Theorem 3.2, it follows that using the Cauchy-Schwarz inequality

$$V_2(\cdot) \le \left\{\alpha_1\beta_3\Delta_c\sqrt{n}(\alpha_1 + \frac{1}{2}) + \frac{(\alpha_1\alpha_2 - \beta_3)(\alpha_1 + \alpha_2 + 1)}{2}\right\}\|X\|^2 + \left\{\frac{\beta_3\Delta_c(\alpha_1\sqrt{n} + 1)}{2} + \frac{\alpha_1(\alpha_1\alpha_2 - \beta_3)}{2} + \frac{\alpha_1^2(\alpha_1 + 1)}{2}\right\}\|Y\|^2 + \left\{\frac{1}{2}(\alpha_1\alpha_2 - \beta_3) + \frac{\alpha_1}{2}(\alpha_1 + 1)\right\}\|Z\|^2. \quad (21)$$

Thus, from (8), (9), (10), (17) and (21), we have

$$V(\cdot) \le \frac{1}{2}\Big\{\beta_3\Delta_c\sqrt{n}(2\mu + 1 + \alpha_1 + 2\alpha_1^2) + (\alpha_1\alpha_2 - \beta_3)(\alpha_1 + \alpha_2 + 1)\Big\}\|X\|^2 + \frac{1}{2}\Big\{\beta_3\Delta_c\sqrt{n}(\alpha_1 + 1) + \beta_3\Delta_c + \mu(\beta_1\sqrt{n} + 1) + 2\beta_2\sqrt{n} + \alpha_1(\alpha_1\alpha_2 - \beta_3) + \alpha_1^2(\alpha_1 + 1)\Big\}\|Y\|^2 + \frac{1}{2}\Big\{\alpha_1\alpha_2 - \beta_3 + \alpha_1(\alpha_1 + 1) + \mu + 1\Big\}\|Z\|^2 + \frac{\eta\gamma}{\alpha}\left\{\int_{t-r(t)}^t \frac{\alpha}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2)ds\right\}, \quad (22)$$

where $\eta > max\{\lambda, \delta\}$.
From (5), (2) and using the conditions of Theorem 3.2, we find

$$\frac{dV_1}{dt} \le -\left(\mu\alpha_2 - \beta_3\Delta_c - \frac{\mu\beta_3\Delta_c\sqrt{n}}{2}\gamma - \frac{\mu\beta_2\sqrt{n}}{2}\gamma - \lambda\gamma\right)\|Y\|^2 - \left(\alpha_1 - \mu - \frac{\beta_3\Delta_c\sqrt{n}}{2}\gamma - \frac{\beta_2\sqrt{n}}{2}\gamma - \delta\gamma\right)\|Z\|^2 + \left\{\frac{\mu\beta_3\Delta_c\sqrt{n}}{2} + \frac{\beta_3\Delta_c\sqrt{n}}{2} - \lambda(1-\omega)\right\}\int_{t-r(t)}^t \|Y(s)\|^2 ds + \left\{\frac{\mu\beta_2\sqrt{n}}{2} + \frac{\beta_2\sqrt{n}}{2} - \delta(1-\omega)\right\}\int_{t-r(t)}^t \|Z(s)\|^2 ds + \langle\mu Y + Z, P(\cdot)\rangle.$$

In view of the condition $\|P(\cdot)\| \le m$, we get

$$\frac{dV_1}{dt} \le -\left(\mu\alpha_2 - \beta_3\Delta_c - \frac{\mu\beta_3\Delta_c\sqrt{n}}{2}\gamma - \frac{\mu\beta_2\sqrt{n}}{2}\gamma - \lambda\gamma\right)\|Y\|^2 - \left(\alpha_1 - \mu - \frac{\beta_3\Delta_c\sqrt{n}}{2}\gamma - \frac{\beta_2\sqrt{n}}{2}\gamma - \delta\gamma\right)\|Z\|^2 + \left\{\frac{\mu\beta_3\Delta_c\sqrt{n}}{2} + \frac{\beta_3\Delta_c\sqrt{n}}{2} - \lambda(1-\omega)\right\}\int_{t-r(t)}^t \|Y(s)\|^2 ds + \left\{\frac{\mu\beta_2\sqrt{n}}{2} + \frac{\beta_2\sqrt{n}}{2} - \delta(1-\omega)\right\}\int_{t-r(t)}^t \|Z(s)\|^2 ds + \mu m\|Y\| + m\|Z\|. \quad (23)$$

From (18), (2) and using the conditions of Theorem 3.2, we get

$$\frac{d}{dt}V_2(\cdot) \le \alpha_2(\alpha_1\alpha_2 - \beta_3)\langle X, Y\rangle + \alpha_1\beta_3\langle Y, Y\rangle + \alpha_1\alpha_2\langle Y, Z\rangle + \alpha_1(\alpha_1\alpha_2 - \beta_3)\langle Y, Y\rangle + \alpha_1(\alpha_1\alpha_2 - \beta_3)\langle X, Z\rangle - \alpha_1(\alpha_1\alpha_2 - \beta_3)\langle X, Z\rangle - \sqrt{n}\alpha_2(\alpha_1\alpha_2 - \beta_3)\|X\|\|Y\| - \sqrt{n}\beta_3\delta_c(\alpha_1\alpha_2 - \beta_3)\|X\|^2 - \alpha_1^2\langle Z, Z\rangle - \sqrt{n}\alpha_1\alpha_2\|Y\|\|Z\| - \alpha_1^3\langle Y, Z\rangle - \sqrt{n}\alpha_1^2\alpha_2\|Y\|^2 + \alpha_1^2\langle Z, Z\rangle + \alpha_1^3\langle Y, Z\rangle + \alpha_1^2\int_0^1 \langle c'(t)H(\sigma X), X\rangle d\sigma + \alpha_1\langle c'(t)H(X), Y\rangle + \frac{1}{2}\beta_3\langle c'(t)Y, Y\rangle + (\alpha_1\alpha_2 - \beta_3)\langle X, P(\cdot)\rangle + \alpha_1\langle Z + \alpha_1 Y, P(\cdot)\rangle + (\alpha_1\alpha_2 - \beta_3)\langle X, \int_{t-r(t)}^t J_\Psi(Y(s))Z(s)ds\rangle + (\alpha_1\alpha_2 - \beta_3)\langle X, c(t)\int_{t-r(t)}^t J_H(X(s))Y(s)ds\rangle + \alpha_1\langle Z + \alpha_1 Y, \int_{t-r(t)}^t J_\Psi(Y(s))Z(s)ds\rangle + \alpha_1\langle Z + \alpha_1 Y, c(t)\int_{t-r(t)}^t J_H(X(s))Y(s)ds\rangle.$$

Since $\|J_\Psi(Y)\| \le \sqrt{n}\beta_2$ by (ii), $\alpha_3\sqrt{n}\|X\| \le \|H(X)\|$ and $\|J_H(X)\| \le \sqrt{n}\beta_3$ by (iii), we obtain

$$\begin{aligned}\frac{d}{dt}V_2(\cdot) \le& -(\sqrt{n}-1)\alpha_2(\alpha_1\alpha_2-\beta_3)\|X\|\|Y\|\\ &-\sqrt{n}\beta_3\delta_c(\alpha_1\alpha_2-\beta_3)\|X\|^2-\beta_3(1-\Delta_c)\|Y\|\|Z\|\\ &-(\sqrt{n}-1)\alpha_1\alpha_2\|Y\|\|Z\|-(\sqrt{n}-1)\alpha_1^2\alpha_2\|Y\|^2\\ &-\alpha_1\beta_3(1-\Delta_c)\|Y\|^2+\Omega_2\\ &+(\alpha_1\alpha_2-\beta_3)\langle X,\int_{t-r(t)}^t J_\Psi(Y(s))Z(s)ds\rangle\\ &+(\alpha_1\alpha_2-\beta_3)\langle X,\int_{t-r(t)}^t J_H(X(s))Y(s)ds+P(\cdot)\rangle\\ &+\alpha_1\langle Z+\alpha_1Y,\int_{t-r(t)}^t J_\Psi(Y(s))Z(s)ds\rangle\\ &+\alpha_1\langle Z+\alpha_1Y,\int_{t-r(t)}^t J_H(X(s))Y(s)ds+P(\cdot)\rangle, \quad (24)\end{aligned}$$

where

$$\begin{aligned}\Omega_2 \le& \alpha_1^2\int_0^1\langle c'(t)H(\sigma X),X\rangle d\sigma+\alpha_1\langle c'(t)H(X),Y\rangle\\ &+\frac{\beta_3}{2}\langle c'(t)Y,Y\rangle\\ \le& \alpha_1^2\int_0^1\int_0^1\sigma_1\langle c'(t)J_H(\sigma_1\sigma_2X)X,X\rangle d\sigma_2 d\sigma_1\\ \le& \frac{\alpha_1^2\beta_3}{2}\langle c'(t)X,X\rangle \le 0, \text{ since } \lambda_i(c')\le 0.\end{aligned}$$

Since $\|P(\cdot)\| \le m$ and using the Cauchy-Schwarz inequality, we can rewrite (24) as

$$\begin{aligned}\frac{dV_2}{dt} \le& -(\sqrt{n}-1)\alpha_2(\alpha_1\alpha_2-\beta_3)\|X\|\|Y\|\\ &-\sqrt{n}\beta_3\delta_c(\alpha_1\alpha_2-\beta_3)\|X\|^2-(\sqrt{n}-1)\alpha_1\alpha_2\|Y\|\|Z\|\\ &-\beta_3(1-\Delta_c)\|Y\|\|Z\|-(\sqrt{n}-1)\alpha_1^2\alpha_2\|Y\|^2\\ &-\alpha_1\beta_3(1-\Delta_c)\|Y\|^2+m\{(\alpha_1\alpha_2-\beta_3)\|X\|+\alpha_1^2\|Y\|+\alpha_1\|Z\|\}\\ &+\frac{\sqrt{n}\beta_2(\alpha_1\alpha_2-\beta_3)}{2}\left(\|X\|^2\gamma+\int_{t-r(t)}^t\|Z(s)\|^2ds\right)\\ &+\frac{\sqrt{n}\alpha_1^2\beta_2}{2}\left(\|Y\|^2\gamma+\int_{t-r(t)}^t\|Z(s)\|^2ds\right)\\ &+\frac{\sqrt{n}\alpha_1\beta_2}{2}\left(\|Z\|^2\gamma+\int_{t-r(t)}^t\|Z(s)\|^2ds\right)\\ &+\frac{\sqrt{n}\beta_3\Delta_c(\alpha_1\alpha_2-\beta_3)}{2}\left(\|X\|^2\gamma+\int_{t-r(t)}^t\|Y(s)\|^2ds\right)\\ &+\frac{\sqrt{n}\alpha_1^2\beta_3\Delta_c}{2}\left(\|Y\|^2\gamma+\int_{t-r(t)}^t\|Y(s)\|^2ds\right)\\ &+\frac{\sqrt{n}\alpha_1\beta_3\Delta_c}{2}\left(\|Z\|^2\gamma+\int_{t-r(t)}^t\|Y(s)\|^2ds\right). \quad (25)\end{aligned}$$

Therefore, from (23) and (25), we get

$$\begin{aligned}\frac{d}{dt}V(\cdot) \le& (\alpha_1\alpha_2-\beta_3)m\|X\|+(\mu+\alpha_1^2)m\|Y\|+(\alpha_1+1)m\|Z\|\\ &-\Big\{\frac{\sqrt{n}-1}{2}\alpha_2(\alpha_1\alpha_2-\beta_3)+\sqrt{n}\beta_3\delta_c(\alpha_1\alpha_2-\beta_3)\\ &-\frac{\sqrt{n}}{2}\beta_2(\alpha_1\alpha_2-\beta_3)\gamma-\frac{\sqrt{n}}{2}\beta_3\Delta_c(\alpha_1\alpha_2-\beta_3)\gamma\Big\}\|X\|^2\\ &-\Big\{\mu\alpha_2-\beta_3\Delta_c-\frac{\mu\sqrt{n}\beta_3\Delta_c}{2}\gamma+\frac{\sqrt{n}-1}{2}\alpha_2(\alpha_1\alpha_2-\beta_3)\\ &-\frac{\mu\sqrt{n}\beta_2}{2}\gamma+\frac{\sqrt{n}-1}{2}\alpha_1\alpha_2+(\sqrt{n}-1)\alpha_1^2\alpha_2-\frac{\sqrt{n}}{2}\alpha_1^2\beta_2\gamma\\ &-\frac{\sqrt{n}}{2}\alpha_1^2\beta_3\Delta_c\gamma+\alpha_1\beta_3(1-\Delta_c)+\frac{\beta_3}{2}(1-\Delta_c)-\lambda\gamma\Big\}\|Y\|^2\\ &-\Big\{\alpha_1-\mu-\frac{\sqrt{n}\beta_3\Delta_c}{2}\gamma-\frac{\sqrt{n}\beta_2}{2}\gamma+\frac{\sqrt{n}-1}{2}\alpha_1\alpha_2\\ &-\frac{\sqrt{n}}{2}\alpha_1\beta_2\gamma-\frac{\sqrt{n}}{2}\alpha_1\beta_3\Delta_c\gamma+\frac{\beta_3}{2}(1-\Delta_c)-\delta\gamma\Big\}\|Z\|^2\\ &+\Big\{\frac{\sqrt{n}\beta_3\Delta_c}{2}(\mu+1)+\frac{\sqrt{n}\beta_3\Delta_c}{2}(\alpha_1\alpha_2-\beta_3)+\frac{\sqrt{n}}{2}\alpha_1^2\beta_{3\Delta_c}\\ &+\frac{\sqrt{n}}{2}\alpha_1\beta_3\Delta_c-\lambda(1-\omega)\Big\}\int_{t-r(t)}^t\|Y(s)\|^2ds\\ &+\Big\{\frac{\sqrt{n}\beta_2}{2}(\mu+1)+\frac{\sqrt{n}\beta_2}{2}(\alpha_1\alpha_2-\beta_3)+\frac{\sqrt{n}}{2}\alpha_1^2\beta_2\\ &+\frac{\sqrt{n}}{2}\alpha_1\beta_2-\delta(1-\omega)\Big\}\int_{t-r(t)}^t\|Z(s)\|^2ds.\end{aligned}$$

Let $\lambda=\frac{\sqrt{n}}{2}\beta_3\Delta_c\mathscr{M}$, $\delta=\frac{\sqrt{n}}{2}\beta_2\mathscr{M}$, where

$$\mathscr{M}=\frac{\mu+1+\alpha_1\alpha_2-\beta_3+\alpha_1+\alpha_1^2}{1-\omega}.$$

If

$$\begin{aligned}\gamma < min\Big[&\frac{(\sqrt{n}-1)\alpha_2+2\sqrt{n}\beta_3\delta_c}{2\sqrt{n}(\beta_2+\beta_3\Delta_c)},\\ &\frac{2(\mu\alpha_2-\beta_3\Delta_c)+\alpha_2(\sqrt{n}-1)(\alpha_1\alpha_2-\beta_3+\alpha_1+2\alpha_1^2)}{2\sqrt{n}\{(\mu+\alpha_1^2)(\beta_2+\beta_3)+\beta_3\mathscr{M}\}}\\ &+\frac{(1-\Delta_c)(\beta_3+2\alpha_1\alpha_3)}{2\sqrt{n}\{(\mu+\alpha_1^2)(\beta_2+\beta_3)+\beta_3\mathscr{M}\}},\\ &\frac{2(\alpha_1-\mu)+(\sqrt{n}-1)\alpha_1\alpha_2+(1-\Delta_c)\beta_3}{2\sqrt{n}\{(\beta_2+\beta_3\Delta_c)(1+\alpha_1)+\beta_2\mathscr{M}\}}\Big],\end{aligned}$$

then we can take

$$K=m\max\{\alpha_1\alpha_2-\beta_3,\mu+\alpha_1^2,\alpha_1+1\},$$

so

$$\frac{dV(\cdot)}{dt} \leq -\sigma\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right)+K\sigma\left(\|X\|+\|Y\|+\|Z\|\right)$$
$$= -\frac{\sigma}{2}\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right)$$
$$-\frac{\sigma}{2}\left\{(\|X\|-K)^2+(\|Y\|-K)^2+(\|Z\|-K)^2\right\}+\frac{3\sigma}{2}K^2$$
$$\leq -\frac{\sigma}{2}\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right)+\frac{3\sigma}{2}K^2, \quad (26)$$

for some $\sigma, K > 0$. Therefore, from (20), (22) and (26), the (LF) $V(\cdot)$ satisfies all the conditions of Theorem 4.1, by taking $W_3(|\bar{x}|) = \frac{\sigma}{2}(\|X\|^2+\|Y\|^2+\|Z\|^2)$ and $N = \frac{3\sigma}{2}K^2$. Then, the solutions of (1) are (UB) and (UUB) for a bound m.
Thus, the proof of Theorem 4.2 is completed.

5 Examples

In this section, we provide two examples to illustrate the application of the results we obtained in the previous sections.

Example 5.1 (An application of Theorem 3.2)

As a special case of equation (1) with $P(\cdot) = 0$, for $n = 2$, we choose

$$\Phi(X,Y) = \begin{bmatrix} 9+e^{-(x_1^2+y_1^2)} & 1 \\ 1 & 9+e^{-(x_2^2+y_2^2)} \end{bmatrix},$$

$$\Psi(Y(t-r(t))) = \begin{bmatrix} y_1(t-r(t))+tan^{-1}y_1(t-r(t)) \\ y_2(t-r(t))+tan^{-1}y_2(t-r(t)) \end{bmatrix},$$

$$H(X(t-r(t))) = \begin{bmatrix} x_1(t-r(t))+x_1(t-r(t))e^{-x_1^2(t-r(t))} \\ x_2(t-r(t))+x_2(t-r(t))e^{-x_2^2(t-r(t))} \end{bmatrix},$$

where $r(t) = \frac{1}{20}\sin^2\frac{t}{2}$. It follows that

(i) $\lambda_1(\Phi(.)) = 8+e^{-(x_1^2+y_1^2)}$, $\lambda_2(\Phi(.)) = 10+e^{(x_2^2+y_2^2)}$, then

$8 \leq \lambda_i(\Phi(.)) \leq 11,\ \alpha_1 = 8,\ \beta_1 = 11.$

(ii) $\Psi(0) = 0$,

$$J_\Psi(Y) = \begin{bmatrix} 1+\frac{1}{1+y_1^2(t-r(t))} & 0 \\ 0 & 1+\frac{1}{1+y_2^2(t-r(t))} \end{bmatrix}.$$

Therefore, we get

$$\lambda_1(J_\Psi(.)) = 1+\frac{1}{1+y_1^2(t-r(t))},$$
$$\lambda_2(J_\Psi(.)) = 1+\frac{1}{1+y_2^2(t-r(t))},$$
$$1 \leq \lambda_i(J_\Psi(.)) \leq 2,\ \alpha_2 = 1,\ \beta_2 = 2.$$

(iii) $H(0) = 0$,

$$J_H(X) = \begin{bmatrix} 1+(1-2x_1^2)e^{-x_1^2} & 0 \\ 0 & 1+(1-2x_2^2)e^{-x_2^2} \end{bmatrix}.$$

It follows that

$$\lambda_1(J_H(.)) = 1+(1-2x_1^2)e^{-x_1^2},$$
$$\lambda_2(J_H(.)) = 1+(1-2x_2^2)e^{-x_2^2},$$
$$1 \leq \lambda_i(J_H(.)) \leq 2, \alpha_3 = 1, \beta_3 = 2.$$

(iv)

$$c(t) = \begin{bmatrix} \frac{e^{-t^2}+1}{2} & 0 \\ 0 & \frac{1}{2(1+t)}+\frac{1}{2} \end{bmatrix}.$$

It follows that

$$\lambda_1(c(t)) = \frac{1}{2(1+t)}+\frac{1}{2},$$
$$\lambda_2(c(t)) = \frac{e^{-t^2}}{2}+\frac{1}{2},$$
$$\frac{1}{2} \leq \lambda_i(c(t)) \leq 1, \text{ it tends to } \delta_c = \frac{1}{2} \text{ and } \Delta_c = 1.$$

(v) Let $\mu = 5$, where $\frac{1}{8} < \frac{1}{\mu} < \frac{1}{2}$.

(vi) Since $0 \leq r(t) = \frac{1}{20}\sin^2\frac{t}{2} \leq \frac{1}{20}$, $\gamma = \frac{1}{20}$,
and since $r'(t) = \frac{1}{20}\sin\frac{t}{2}\cos\frac{t}{2} \leq \frac{1}{40}$, $\omega = \frac{1}{40}$.

Next,

$$\frac{(\mu\alpha_2-\beta_3\Delta_c)(1-\omega)}{\sqrt{n}\,[\{\mu(2-\omega)+1\}\beta_3\Delta_c+\mu(1-\omega)\beta_2]}$$
$$= \frac{3(1-\frac{1}{40})}{\sqrt{2}[2\{5(2-\frac{1}{40})+1\}+(5)(2)(1-\frac{1}{40})]} \simeq 0.06566,$$

$$\frac{(\alpha_1-\mu)(1-\omega)}{\sqrt{n}\,\{(1-\omega)\beta_3\Delta_c+(2+\mu-\omega)\beta_2\}}$$
$$= \frac{3(1-\frac{1}{40})}{\sqrt{2}\{2(1-\frac{1}{40})+2(2+5-\frac{1}{40})\}} = \frac{3(1-\frac{1}{40})}{\sqrt{2}(16-\frac{4}{40})} \simeq 0.13008,$$

Thus, all the conditions of Theorem 3.2 are satisfied, provided that

$$\gamma < min\{0.06566, 0.13008\} \simeq 0.06566.$$

Example 5.2 (An application of Theorem 4.2)

As a special case of equation (1), let us take for $n = 2$ that

$$\Phi(X,Y) = \begin{bmatrix} 11+e^{-(x_1^2+y_1^2)} & 1 \\ 1 & 11+e^{-(x_2^2+y_2^2)} \end{bmatrix},$$

$$\Psi(Y(t-r(t))) = \begin{bmatrix} y_1(t-r(t))+tan^{-1}y_1(t-r(t)) \\ y_2(t-r(t))+tan^{-1}y_2(t-r(t)) \end{bmatrix},$$

$$H(X(t-r(t))) = \begin{bmatrix} x_1(t-r(t))+x_1(t-r(t))e^{-x_1^2(t-r(t))} \\ 2x_2(t-r(t))+x_2(t-r(t))e^{-x_2^2(t-r(t))} \end{bmatrix},$$

where $r(t) = \frac{1}{80}\sin^2\frac{t}{2}$. It follows that

(i) $\lambda_1(\Phi(.)) = 10 + e^{-(x_1^2+y_1^2)}$,
$\lambda_2(\Phi(.)) = 12 + e^{-(x_2^2+y_2^2)}$, then
$10 \le \lambda_i(\Phi(.)) \le 13$, $\alpha_1 = 10$, $\beta_1 = 13$.

The calculations for condition (ii) and (iv) is the same as in Example 5.1.

(iii) $H(0) = 0$,

$$J_H(X) = \begin{bmatrix} 1+(1-2x_1^2)e^{-x_1^2} & 0 \\ 0 & 2+(1-2x_2^2)e^{-x_2^2} \end{bmatrix}.$$

Thus, we find

$$\lambda_1(J_H(.)) = 1 + (1-2x_1^2)e^{-x_1^2},$$

$$\lambda_2(J_H(.)) = 2 + (1-2x_2^2)e^{-x_2^2},$$

$$1 \le \lambda_i(J_H(.)) \le 3, \alpha_3 = 1, \beta_3 = 3.$$

(v) Let $\mu = 6.5$, where $\frac{1}{10} < \frac{1}{\mu} < \frac{1}{3}$.
(vi) Since $0 \le r(t) = \frac{1}{80}\sin^2\frac{t}{2} \le \frac{1}{80}$, $\gamma = \frac{1}{80}$ and since $r'(t) = \frac{1}{80}\sin\frac{t}{2}\cos\frac{t}{2} \le \frac{1}{160}$, $\beta = \frac{1}{160}$.

Next,

$$P(\cdot) = \begin{bmatrix} p_1(\cdot) \\ p_2(\cdot) \end{bmatrix} = \begin{bmatrix} \frac{2+2t^2+x_1y_1z_1+x_1(t-r(t))+y_1^2(t-r(t))}{1+2t^2+x_1y_1z_1+x_1(t-r(t))+y_1^2(t-r(t))} \\ \frac{2+2t^2+x_2y_2z_2+x_2(t-r(t))+y_2^2(t-r(t))}{1+2t^2+x_2y_2z_2+x_2(t-r(t))+y_2^2(t-r(t))} \end{bmatrix},$$

Hence, we get

$$|P(\cdot)| = \begin{bmatrix} 1+\frac{1}{1+2t^2+|x_1y_1z_1|+|x_1(t-r(t))|+y_1^2(t-r(t))} \\ 1+\frac{1}{1+2t^2+|x_2y_2z_2|+|x_2(t-r(t))|+y_2^2(t-r(t))} \end{bmatrix},$$

$|p_1(\cdot)| \le 2$, $|p_2(\cdot)| \le 2$, and $m = 2$.
Then, we obtain

$$\mathscr{M} = \frac{6.5+1+7+10+100}{1-\frac{1}{160}} \simeq 125.2830,$$

$$\frac{(\sqrt{n}-1)\alpha_2 + 2\sqrt{n}\beta_3\delta_c}{2\sqrt{n}(\beta_2+\beta_3\Delta_c)} = \frac{\sqrt{2}-1+3\sqrt{2}}{2\sqrt{2}(2+3)} \simeq 0.3293,$$

$$\frac{2(\mu\alpha_2-\beta_3\Delta_c)+\alpha_2(\sqrt{n}-1)(\alpha_1\alpha_2-\beta_3+\alpha_1+2\alpha_1^2)}{2\sqrt{n}\{(\mu+\alpha_1^2)(\beta_2+\beta_3)+\beta_3\mathscr{M}\}}$$

$$+\frac{(1-\Delta_c)(\beta_3+2\alpha_1\alpha_3)}{2\sqrt{n}\{(\mu+\alpha_1^2)(\beta_2+\beta_3)+\beta_3\mathscr{M}\}},$$

$$= \frac{7+(\sqrt{2}-1)7+(\sqrt{2}-1)(10+200)}{2\sqrt{2}\{(6.5+100)(2+3)+3(125.283)\}} \simeq 0.0377,$$

$$\frac{2(\alpha_1-\mu)+(\sqrt{n}-1)\alpha_1\alpha_2+(1-\Delta_c)\beta_3}{2\sqrt{n}\{(\beta_2+\beta_3\Delta_c)(1+\alpha_1)+\beta_2\mathscr{M}\}}$$

$$= \frac{7+10(\sqrt{2}-1)}{2\sqrt{2}\{(2+3)(1+10)+2(125.283)\}} \simeq 0.0129,$$

Thus, all the conditions of Theorem 4.2 are satisfied, provided that

$$\gamma < min\{0.3293, 0.0377, 0.0129\} \simeq 0.0129.$$

6 Conclusion

Using (LFs), we derived the sufficient conditions for the (US) of the zero solution, and the (UB) as well as (UUB) of all solutions of the third-order nonlinear non-autonomous vector (DDE) (1). We constructed two examples to illustrate our main results of (S) and (B).

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