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A Multi Server Markovian Working Vacation queue with Breakdown, N-policy and with Server State Dependent Rates

R. Kalyanaraman[1] *and A. Sundaramoorthy*[2,*]

[1]Department of Mathematics, Annamalai University, Annamalainagar, India
[2]Department of Mathematics, CK College of Engineering and Technology, Cuddalore, India

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Abstract: A multi-server Markovian queueing system with alternatives between regular busy state, repair state and working vacation state has been considered. When the system is busy, it functions as a multi-server Markovian queue. When it is on vacation, again it functions as a multi-server Markovian queue but with different arrival and service rates. The vacation policy is multiple vacation policy and the vacation period follows a negative exponential distribution. Also, the servers return to the queue only if there are N customers in the queue at a vacation completion point. Besides, during service the server may break down, the repair of the server starts immediately. The repair period follows a negative exponential distribution. The steady-state probability vector of the number of customers in the queue and the stability condition are obtained using the Matrix-Geometric method. Some performance measures and some illustrative examples are also provided.

Keywords: Working vacation, State dependent rates, Breakdown, $N-$policy, Matrix-Geometric method.

1 Introduction

In many practical queueing situations, we can see the following one or more features : (i) The queueing system has more than one server (ii) The arrival rate to the system varies (iii) The service rate to the customer varies (iv) The system may fail while in working state and (v) The vacation completion may depend on the number of customers in the queue.With all these points in mind, we propose a queuueing model in this article.

In real life, while working a system may breakdown due to some kind of service interruption, which is beyond the control of the management.Some earlier notable works in this direction are [1,5,21,26,27,31,35,36,37].

In many real-life queueing situations, it can be seen that the server works during his rest period, if necessity occurs, called the working vacation period. Recently there have been an increasing interest in queueing systems with server working vacations, due to their applications in telecommunication systems, manufacturing systems and computer systems. But, in such a queue, the server works with a variable service rate, in particular a reduced service rate, rather than completely stops service during the vacation period. [32] have first analyzed an $M/M/1$ queue with multiple working vacation, in which the vacation times are exponentially distributed. [41] extend the work for $M/G/1$ queue. [12] analyzed the queue length distribution of the $M/G/1$ queue with working vacations. [24], examined the stochastic decomposition structure of the queue length and waiting time in an $M/M/1$ working vacation queue. [42] extended the $M/M/1$ working vacation queue to an $M^{[x]}/M/1$ working vacation queue. [22] used the matrix analytic method to analyze an $M/G/1$ queue with exponential working vacation under a specific assumption. [23] consider a multi-server queue with a single working vacation. [6] investigated a single working vacation model with a server breaks down. [9] have given a short survey on vacation models in recent years.

The $C-$server Markovian queue with exponentially distributed vacation was first studied by [20]. The same model has been studied by [40] using the matrix geometric method. [2] investigated a multi-server model and provided an algorithm for finding the stationary distribution and performance measures. [38,39]

* Corresponding author e-mail: a.sundarmaths92@gmail.com

established stochastic decomposition results for a multi-server Markovian queue with vacation.

[44] have considered an unreliable single-server exponential queueing model with the arrival state depending on the operational state or breakdown state of the server. [4] analyzed the same model with the assumption that any arrival finding the server busy is lost, and they obtained the steady-state proportion of customers lost. [34] has dealt with a single server queueing model with an arrival rate depending on the server state. [33] has analyzed a single server Poisson queue with arrival rate depending on the state of the server. [7] analyzed a general bulk service queue with arrival rate depending on server breakdowns. [38] discussed the queueing system with a variable arrival rate. The authors studied the model by using the principle of quasi-birth and death process(QBD) and matrix-geometric method. Furthermore, they calculated some performance measures. Matrix-geometric method approach is a useful tool for solving the more complex queueing problems. The matrix-geometric method has been applied by many researchers to solve various queueing problems in different frameworks. [29] explained various matrix geometric solutions of stochastic models. The Matrix-geometric approach is utilized to develop the computable explicit formula for the probability distributions of the queue length and other system characteristics.

For optimal design and control of queueing system one can use the concept of N-policy. The policy states that the server remains idle until the queue size becomes N. The queueing model with N-policy has applications in flexible manufacturing systems, service systems, computer and telecommunication systems. Multiple vacation queue with N-policy was first introduced by [43]. Latter queue with N-policy was investigated by [17, 10]. Some other notable works are [3, 13, 16, 18, 19]. [8] analysed a single server Markovian queue with unreliable server, N-policy and with working vacation.

In this paper, we consider a $M/M/C$ queues with multiple working vacation. Besides the system may breakdown (Unreliable system). For this model the arrival rate and service rate depend on the server states. The model has been analyzed using the matrix geometric method. The rest of this paper is organized as follows: In section 2, we give the model description, establish its quasi-birth-death process. In section 3, we present the steady-state solution using the matrix geometric method and we derive some performance measures. In section 4, we derived the stationary waiting time distribution in queue. Section 5 gives some numerical examples related to the model discussed in this paper. The last section ends with a conclusion.

2 The Model Definition

In this article a C-server queueing system with the following characteristics has been analysed:
1. The system alternate between two states, up state and down state. In the up state it is either in regular state or in working vacation state. In the down state it is in the repair state.
2. Arrival process follows Poisson with parameter λ during regular state.
3. The servers serve the customers based on exponential distribution with rate μ during regular state.
4. All the server takes vacation, if there are no customer in the system at a service completion point. If there are less than C customers in the system, that is, less than C servers are busy, the remaining server waits in idle mode.
5. During vacation, the arrival follows Poisson with rate λ_1 $(\lambda_1 < \lambda)$.
6. Vacation period follows negative exponential distribution with rate θ and the vacation policy is multiple vacation policy, that is, the servers continue the vacation until the servers finds $N(> C)$ customers at a vacation completion point.
7. When the servers are in vacation, if customer arrives, one of the the server serve the customer using exponential distribution with rate μ_1 $(\mu_1 < \mu)$. As this vacation period ends before the current service completion, the server instantaneously switches over to the normal service rate μ, if there are N customers waiting for service. Upon completion of a service at a vacation period, the server will (i) Continue the current vacation if it is not finished and no customer is waiting for service; (ii) Continue the service with rate μ_1, if the vacation has not expired and if there are customers waiting for service.
8. The system may break down during a service and the break downs are assumed to occur according to a Poisson process with rate α.
9. Once the system break downs, the customer whose service is interrupted goes to the head of the queue and the repair to the system starts immediately.
10. Duration of repaired period follows negative exponential distribution with rate β.
11. During repair period no service takes place but customers arrive according to a Poisson process with rate λ_2 $(\lambda_2 < \lambda_1)$.
12. The first come first served (FCFS) service rule is followed to select the customer for service.

3 The Mathematical Description and Analysis

The model defined in the Section-2 can be studied as a Quasi birth and death(QBD) process. The following notations are necessary for the analysis:

Let $L(t)$ be the number of customers in the queue at time t and let

$$J(t) = \begin{cases} 0, \text{if the servers are on working vacation} \\ 1, \text{if the servers are busy} \\ 2, \text{if the servers are on repaired period} \end{cases}$$

be the server state at time t.

Let $X(t) = (L(t), J(t))$, then $\{(X(t)) : \ t \geq 0\}$ is a Continuous time Markov chain (CTMC) with state space $S = \{(i,j) : \ i \geq 0; j = 0,1,2\}$, where i denotes the number of customer in the queue and j denotes the server state.

Using the lexicographical sequence for the states, the rate matrix Q has been formed, is the infinitesimal generator of the Markov chain.

The model has been analyzed in this section.

$$Q = \begin{array}{cc} & \begin{matrix} 0 & 1 & 2 & 3 & \cdots & C-1 & C & C+1 & \cdots & N-2 & N-1 & N & N+1 & \cdots \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ C-1 \\ C \\ C+1 \\ \vdots \\ N-2 \\ N-1 \\ N \\ N+1 \\ \vdots \end{matrix} & \begin{pmatrix} B_0 & A_0 & & & & & & & & & & & & \\ B_{10} & B_{11} & A_0 & & & & & & & & & & & \\ & B_{21} & B_{22} & A_0 & & & & & & & & & & \\ & & B_{31} & B_{32} & & & & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & & & \\ & & & & & B_{(C-1)2} & A_0 & & & & & & & \\ & & & & & A_2 & B_{C2} & A_0 & & & & & & \\ & & & & & & A_2 & B_{C2} & & & & & & \\ & & & & & & \ddots & \ddots & \ddots & & & & & \\ & & & & & & & & & B_{C2} & A_0 & & & \\ & & & & & & & & & A_2 & B_{C2} & A_0 & & \\ & & & & & & & & & & A_2 & A_1 & A_0 & \\ & & & & & & & & & & & A_2 & A_1 & \\ & & & & & & & & & & & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

where the sub-matrices A_0, A_1, and A_2 are of order 3×3 and are appearing as

$$A_0 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -(\lambda_1 + C\mu_1 + \theta) & \theta & 0 \\ 0 & -(\lambda + C\mu + \alpha) & \alpha \\ 0 & \beta & -(\lambda_2 + \beta) \end{bmatrix}$$

$$A_2 = \begin{bmatrix} C\mu_1 & 0 & 0 \\ 0 & C\mu & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the boundary matrix is defined by

$$B_0 = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ \mu & -(\lambda + \mu + \alpha) & \alpha \\ 0 & \beta & -(\lambda_2 + \beta) \end{bmatrix}$$

$$B_{10} = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} -(\lambda_1 + \mu_1) & 0 & 0 \\ 0 & -(\lambda + \mu + \alpha) & \alpha \\ 0 & \beta & -(\lambda_2 + \beta) \end{bmatrix}$$

$$B_{i1} = \begin{bmatrix} i\mu_1 & 0 & 0 \\ 0 & i\mu & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ for } i = 2,3,4, \ldots C-1$$

$$B_{i2} = \begin{bmatrix} -(\lambda_1 + i\mu_1) & 0 & 0 \\ 0 & -(\lambda + i\mu + \alpha) & \alpha \\ 0 & \beta & -(\lambda_2 + \beta) \end{bmatrix}, \text{ for }$$

$i = 2,3,4, \ldots C$

We define the matrix $A = A_0 + A_1 + A_2$. This matrix A is a 3×3 matrix and it can be written as

$$A = \begin{bmatrix} -\theta & \theta & 0 \\ 0 & -\alpha & \alpha \\ 0 & \beta & -\beta \end{bmatrix}$$

3.1 The Steady State Solution

Let $P = (p_0, p_1, p_2, \ldots)$ be the stationary probability vector associated with Q, such that $PQ = 0$ and $Pe = 1$, where e is a column vector of $1's$ of appropriate dimension.

Let $p_i = (p_{i0}, p_{i1}, p_{i2})$ for $i \geq 0$.

If the steady state condition is satisfied, then the sub probability vectors p_i satisfies following equations:

$$p_0 B_0 + p_1 B_{10} = 0 \tag{1}$$

$$p_0 A_0 + p_1 B_{11} + p_2 B_{21} = 0 \tag{2}$$

$$p_i A_0 + p_{i+1} B_{(i+1)2} + p_{i+2} B_{(i+2)1} = 0, \text{ for } i = 1,2,3, \ldots C-3 \tag{3}$$

$$p_{C-2} A_0 + p_{C-1} B_{(C-1)2} + p_C A_2 = 0 \tag{4}$$

$$p_i A_0 + p_{i+1} B_{C2} + p_{i+2} A_2 = 0, \ i = C-1, C, \ldots N-3 \tag{5}$$

$$p_{N-2} A_0 + p_{N-1} B_{C2} + p_N A_2 = 0 \tag{6}$$

$$p_i A_0 + p_{i+1} A_1 + p_{i+2} A_2 = 0, \ i \geq N-1 \tag{7}$$

$$p_i = p_{N-1} R^{i-(N-1)}; i \geq N \tag{8}$$

where R is the rate matrix, is the minimal non-negative solution of the matrix quadratic equation (see [29]).

$$R^2 A_2 + R A_1 + A_0 = 0, \tag{9}$$

the matrices A_0, A_1, and A_2 are upper triangular matrices of order 3.

Substituting the equation (8) in (6), we have

$$p_{N-2} A_0 + p_{N-1}(B_{C2} + R A_2) = 0 \tag{10}$$

and the normalizing condition is

$$\sum_{i=0}^{N-2} p_i e + p_{N-1}(I - R)^{-1} e = 1 \tag{11}$$

Theorem 3.1 The queueing system described in section 2 is stable if and only if $\rho < 1$, where $\rho = \dfrac{(\lambda_2 \alpha + \lambda \beta)}{C\mu\beta}$

Proof. Consider the infinitesimal generator $A = \begin{bmatrix} -\theta & \theta & 0 \\ 0 & -\alpha & \alpha \\ 0 & \beta & -\beta \end{bmatrix}$, which is a square matrix of order 3, the

row vector $\pi = (\pi_1, \pi_2, \pi_3)$ satisfying the condition $\pi A = 0$ and $\pi e = 1$.
Following [29], the system is stable if and only if $\pi A_0 e < \pi A_2 e$.
That is,
The system is stable if and only if $\frac{(\lambda_2 \alpha + \lambda \beta)}{C \mu \beta} < 1$ $\square$.

Theorem 3.2 If $\rho < 1$, the matrix equation (9) has the minimal non-negative solution $R = -A_0 A_1^{-1} - R^2 A_2 A_1^{-1}$

Proof. Since A is reducible. The analysis present in [28] is not applicable. In [25], similar reducible matrix is treated for the case when the elements are probabilities.
Equation (9), can be written as,

$$A_0 A_1^{-1} + R A_1 A_1^{-1} + R^2 A_2 A_1^{-1} = 0 A_1^{-1}$$

Since A_1 is non-singular, A_1^{-1} exists

$$R = -A_0 A_1^{-1} - R^2 A_2 A_1^{-1} \tag{12}$$

where

$$A_1^{-1} = \begin{bmatrix} \frac{-1}{S_{00}} & (\lambda_2 + \beta)\theta S_0 & \alpha \theta S_0 \\ 0 & S_0 (\lambda_2 + \beta) S_{00} & S_0 \alpha S_{00} \\ 0 & S_0 \beta S_{00} & S_0 (\lambda + C\mu + \alpha) S_{00} \end{bmatrix}$$

$$S_{00} = (\lambda_1 + C\mu_1 + \theta)$$

$$S_0 = \frac{1}{-S_{00}[(\lambda_2 + \beta)(\lambda + C\mu + \alpha) - \alpha\beta]}$$

Using [30], the matrix R is numerically computed by using the recurrence relation with $R(0) = 0$ in equation (12). $\square$.

Let

$$Q^* = \begin{array}{cc} & \begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & \ldots & C-2 & C-1 & C & C+1 & \ldots & N-2 & N-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ C-2 \\ C-1 \\ C \\ C+1 \\ \vdots \\ N-2 \\ N-1 \end{array} & \begin{pmatrix} B_0 & A_0 & & & & & & & & & & \\ B_{10} & B_{11} & A_0 & & & & & & & & & \\ & B_{21} & B_{22} & A_0 & & & & & & & & \\ & & B_{31} & B_{32} & & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & \\ & & & & & B_{(C-2)2} & A_0 & & & & & \\ & & & & & B_{(C-1)1} & B_{(C-1)2} & A_0 & & & & \\ & & & & & & A_2 & B_{C2} & A_0 & & & \\ & & & & & & & A_2 & B_{C2} & & & \\ & & & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & & & & & B_{C2} & A_0 \\ & & & & & & & & & & A_2 & B_{C2} + RA_2 \end{pmatrix} \end{array}$$

is also irreducible and let $P^* = (p_0, p_1, p_2, \ldots, p_{N-1})$ be a solution of $P^* Q^* = 0$
Solving equations (1) and (2), we get

$$p_0 = -(p_1 B_{10} B_0^{-1}) \tag{13}$$

$$p_2 = p_1 (B_{10} B_0^{-1} - B_{11}) B_{21}^{-1} \tag{14}$$

In this way we can calculate all the $p_i's$, $0 \le i \le N-2$.
Finally we get

$$p_{N-1}[D A_0 + B_{C2} + R A_2] = 0 \tag{15}$$

where the matrix D is readily computed
p_{N-1} is the left eigen vector of the matrix $(D A_0 + B_{C2} + R A_2)$ of order 3 corresponding to the eigen value zero.
It is normalized so that,

$$\sum_{i=0}^{N-2} p_i e + p_{N-1} (I - R)^{-1} e = 1.$$

and

$$p_i = p_{N-1} R^{i-(N-1)}; \; i \ge N$$

Remark 3.1
The computation of R can be carried out using a number of well-known methods. We use Theorem 1 of [15]. The matrix R is computed by successive substitutions in the recurrence relation:

$$R(0) = 0 \tag{16}$$

$$R(n+1) = -A_0 A_1^{-1} - [R(n)]^2 A_2 A_1^{-1} \text{ for } n \ge 0 \tag{17}$$

and is the limit of the monotonically increasing sequence of matrices $\{R(n), n \ge 0\}$.

3.2 Some Performance Measures

Using straightforward calculations the following performance measures have been obtained for the Model discussed in this article:
(i) Mean queue length $E(L) = p_0 R (I - R)^{-2} e$
(ii) $E(L^2) = p_0 R (I + R)(I - R)^{-3} e$
(iii) Variance of $L = var(L)$
$= p_0 R\{(I + R) - p_0 R (I - R)^{-1} e\}(I - R)^{-3} e$
(iv) Probability that the servers are in vacation and no customer in the system $= p_0 e$
(v) Mean queue length when the servers are in vacation period $= \sum_{i=0}^{\infty} i p_{i0}$
(vi) Mean queue length when the servers are in regular busy period $= \sum_{i=0}^{\infty} i p_{i1}$
(vii) Mean queue length when the servers are in breakdown period $= \sum_{i=0}^{\infty} i p_{i2}$
(viii) Probability that the servers are in working vacation period$= pr\{J = 0\} = \sum_{i=1}^{\infty} p_{i0}$
(ix) Probability that the servers are in regular busy period$= pr\{J = 1\} = \sum_{i=1}^{\infty} p_{i1}$
(x) Probability that the servers are in breakdown period$= pr\{J = 2\} = \sum_{i=1}^{\infty} p_{i2}$

4 Stationary Waiting Time Distribution in the Queue

In this section, we derive the stationary waiting time distribution. Let $W(t)$ be the distribution function for the waiting time in the queue of an arriving (tagged) customer. Note that if there is no customer in the system, the arrival receives service immediately. If the server is not busy then also there would be no delay in getting service. Thus, the probability that the customer gets his service without waiting is $\sum_{i=0}^{N-1} p_i e$(where $e = \begin{bmatrix}1\\1\\1\end{bmatrix}$). Hence, with probability $1-(\sum_{i=0}^{N-1} p_i e)$, the customer has to wait before getting the service. The waiting time may be viewed as the time until absorption in a Markov chain with state space

$$\Omega_1 = \{*\}\bigcup\{N, N+1, N+2\dots\}$$

Here $*$ is the absorbing state, which corresponds to taking the tagged customer into service and is obtained by lumping together the level states $0 = \{(0,0),(0,1),(0,2)\}$ and $i = \{(i,0),(i,1),(i,2)\};\ 1 \le i \le N-1$. For $i \ge N$, the level i is given by $i = \{(i,j), j = 0,\ 1,\ \text{or}\ 2\}$. The states other than the absorbing state correspond to the number of customers present in the system as the tagged customer arrives. Once the tagged customer joins the queue, the subsequent arrivals will not affect his waiting time in the queue. Hence the parameter λ does not show up in the generator matrix $\widetilde{Q}$ of this Markov process, given by

$$\widetilde{Q} = \begin{array}{c} \\ * \\ C \\ C+1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} * & N & N+1 & \dots \end{array} \\ \begin{pmatrix} & & & \\ A_2 e & D & & \\ & A_2 & D & \\ & & \ddots & \end{pmatrix} \end{array},$$

where

$$D = \begin{bmatrix} -C\mu_1-\theta & \theta & 0 \\ 0 & -C\mu-\alpha & \alpha \\ 0 & \beta & -\beta \end{bmatrix}$$

Now define the vector

$$Y(t) = (Y_*(t), Y_N(t), Y_{N+1}(t), \dots),$$

where

$$Y_i(t) = (y_{i0}(t), y_{i1}(t), y_{i2}(t)),\ \text{for}\ i \ge N.$$

The components of the $Y_i(t)$ are the corresponding probabilities in regular state and working vacation state at time t, the CTMC with generator $\widetilde{Q}$ is in the respective state of level i. Note that the scalar $Y_*(t)$ is the probability that the process is in the absorbing state at time t. By the PASTA property, we get

$$Y(0) = (p_{00}+p_{01}+p_{02}+\dots+p_{(N-1)0}+p_{(N-1)1}$$
$$+p_{(N-1)2}, p_N, p_{N+1}, \dots).$$

Clearly

$$W(t) = Y_*(t),\ \text{for}\ t \ge 0$$

The LST of $W(t)$ is given by (see [29])

$$\widetilde{W}(s) = \sum_{i=N}^{\infty} Y_i(0)\left[(sI-D)^{-1}A_2\right]^{i-N}(sI-D)^{-1}A_2 e \quad (18)$$

The mean waiting time can be obtained from $\widetilde{W}(s)$ as

$$E(W) = -\widetilde{W}'(0) = \sum_{i=1}^{\infty} p_{N+i} \sum_{j=0}^{i-1} U^j(-D)^{-1}U^{i-j}Ue$$
$$+\sum_{i=0}^{\infty} p_{N+i}U^i(-D)^{-2}A_2 e,$$

where $U = (-D)^{-1}A_2$ is a stochastic matrix. Hence, (18) can be simplified as

$$E(W) = -\widetilde{W}'(0) = \sum_{i=1}^{\infty} p_{N+i} \sum_{j=0}^{i-1} U^j(-D)^{-1}e$$
$$+\sum_{i=0}^{\infty} p_{N+i}U^i(-D)^{-1}e, \quad (19)$$

Let

$$H = \sum_{i=0}^{\infty} p_{N+i}U^i,$$

Since U is stochastic, we get

$$He = p_N(I-R)^{-1}e = 1 - p_{00} - p_{01} - p_{02} - p_{10} - p_{11} - p_{12}$$
$$- \dots - p_{(N-1)0} - p_{(N-1)1} - p_{(N-1)2}$$

This result can be used to find an approximate value of H and hence that of the second term in (19) to any desired degree of accuracy. Thus, only the first term in (19) demands serious computation. For this we make use of the ideas in ([14,29,30]).

Now consider the matrix

$$U_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

which has the property that

$$UU_2 = U_2U = U_2$$

Then we get

$$\sum_{j=0}^{i-1} U^j(I-U+U_2) = I - U^i + iU_2,\ \text{for}\ i \ge 1$$

By the classical theorem on finite Markov chains, the matrix $(I-U+U_2)$ is nonsingular (see [11]). In view of the last equation, the first term in (19) becomes

$$\left[\sum_{i=1}^{\infty} p_{N+i}(I-U^i+iU_2)\right](I-U+U_2)^{-1}(-D)^{-1}e$$

With this simplification, we get

$$E(W) = \left[p_N\left(R(I-R)^{-1} + I + R(I-R)^{-2}U_2\right) - H\right]$$

$$(I-U+U_2)^{-1}(-D)^{-1}e + H(-D)^{-1}e \qquad (20)$$

Equation (20) becomes

$$E(W) = \left[p_N\left(R(I-R)^{-1} + I + R(I-R)^{-2}U_2\right) - H\right]$$

$$(I-U+U_2)^{-1}(-D)^{-1}e + H(-D)^{-1}e$$

5 Numerical Study

In this section, some examples are given to show the effect of the parameters λ, λ_1, λ_2, μ, μ_1, θ, α, β, C and N on the performance measures mean queue length, $E(L^2)$, variance of L, probability that no customer in the queue, mean queue length when the servers are in vacation period, mean queue length when the servers are in regular busy period, probability that the servers are in working vacation period and probability that the servers are in regular busy period analyzed in this paper. The corresponding results are presented as case(1), case(2) and case(3).

Case(1): If $\lambda = 0.6$, $\lambda_1 = 0.4$, $\lambda_2 = 0.2$, $\mu = 4$, $\mu_1 = 3$, $\theta = 2.5$, $\alpha = 0.5$, $\beta = 0.9$, $C = 3$ and $N = 5$, the matrix $R = R_1$ is obtained using the equations (16) & (17)

$$R = \begin{bmatrix} 0.034515 & 0.007448 & 0.003385 \\ 0 & 0.05 & 0.022728 \\ 0 & 0.016667 & 0.189394 \end{bmatrix}$$

and the invariant probability vector is
$P = (p_0, p_1, p_2, \ldots)$ where
p_0=(0.875168979, 0.000000324, 0.000000147)
p_1=(0.116688751, 0.000000380, 0.000000199)
p_2=(0.007779024, 0.000000195, 0.000000125)
p_3=(0.000345583, 0.000000120, 0.000000077)
p_4=(0.000015208, 0.000000121, 0.000000070)
and the remaining vectors $p_i^\prime$s are evaluated using the relation
$p_i = p_4 R^{i-4}$, for $i \geq 5$
p_5=(0.000000525, 0.000000120, 0.000000067)
p_6=(0.000000018, 0.000000011, 0.000000017)
p_7=(0.000000001, 0.000000001, 0.000000004)

For the chosen parameters $p_7 \to 0$, and the sum of the steady state probabilities is found to be 0.999999225.
The performance measures are
(i) Mean queue length $E(L) = 0.134452$
(ii) $E(L^2) = 0.235557$
(iii) Variance of L=$var(L) = 0.217853$
(iv) Probability that no customer in the queue =0.875169
(v) Mean queue length when the servers are in vacation period =0.127435
(vi) Mean queue length when the servers are in regular busy period =0.008967
(vii) Probability that the servers are in working vacation period= $pr\{J = 0\} = 0.998886$
(viii) Probability that the servers are in regular busy period=$pr\{J = 1\} = 0.006874$

Case(2): If $\lambda = 0.8$, $\lambda_1 = 0.5$, $\lambda_2 = 0.3$, $\mu = 5$, $\mu_1 = 2$, $\theta = 1.8$, $\alpha = 0.4$, $\beta = 0.8$, $C = 3$ and $N = 5$, the matrix $R = R_1$ is obtained using the equations (16) & (17)

$$R = \begin{bmatrix} 0.063121 & 0.008085 & 0.002940 \\ 0 & 0.053333 & 0.019394 \\ 0 & 0.02 & 0.28 \end{bmatrix}$$

and the invariant probability vector is
$P = (p_0, p_1, p_2, \ldots)$ where
p_0=(0.778764546, 0.000004010, 0.000001458)
p_1=(0.194681108, 0.000004739, 0.000002121)
p_2=(0.024330126, 0.000002448, 0.000001469)
p_3=(0.002024168, 0.000001497, 0.000000945)
p_4=(0.000165339, 0.000001435, 0.000000780)
and the remaining vectors $p_i^\prime$s are evaluated using the relation
$p_i = p_4 R^{i-4}$, for $i \geq 5$
p_5=(0.000010436, 0.000001429, 0.000000732)
p_6=(0.000000659, 0.000000175, 0.000000263)
p_7=(0.000000042, 0.000000020, 0.000000079)
p_8=(0.000000003, 0.000000003, 0.000000023)

For the chosen parameters $p_8 \to 0$, and the sum of the steady state probabilities is found to be 0.999986172.
The performance measures are
(i) Mean queue length $E(L) = 0.206637$
(ii) $E(L^2) = 0.326744$
(iii) Variance of L=$var(L) = 0.296738$
(iv) Probability that no customer in the queue =0.778770
(v) Mean queue length when the servers are in vacation period =0.308647
(vi) Mean queue length when the servers are in regular busy period =0.007986
(vii) Probability that the servers are in working vacation period= $pr\{J = 0\} = 0.977646$
(viii) Probability that the servers are in regular busy period=$pr\{J = 1\} = 0.004743$

Case(3): If $\lambda_1 = \lambda_2 = \lambda = 0.4$, $\mu_1 = \mu = 3$, $\theta = 3.6$, $\alpha = 0.5$, $\beta = 0.8$, $C = 3$ and $N = 5$, the matrix $R = R_1$ is obtained using the equations (16) & (17)

$$R = \begin{bmatrix} 0.031454 & 0.012990 & 0.005413 \\ 0 & 0.044445 & 0.018519 \\ 0 & 0.044444 & 0.351852 \end{bmatrix}$$

and the invariant probability vector is
$P = (p_0, p_1, p_2, \ldots)$ where
p_0=(0.875167668, 0.000000591, 0.000000246)
p_1=(0.116688430, 0.000000702, 0.000000375)
p_2=(0.007778933, 0.000000367, 0.000000278)
p_3=(0.000345533, 0.000000226, 0.000000187)
p_4=(0.000015160, 0.000000215, 0.000000152)
and the remaining vectors $p_i^\prime$s are evaluated using the relation

$p_i = p_4 R^{i-4}$, for $i \geq 5$
p_5=$(0.000000477, 0.000000213, 0.000000140)$
p_6=$(0.000000015, 0.000000022, 0.000000056)$
p_7=$(0.000000001, 0.000000004, 0.000000020)$

For the chosen parameters $p_7 \to 0$, and the sum of the steady state probabilities is found to be 0.999999046.
The performance measures are
(i) Mean queue length $E(L) = 0.236748$
(ii) $E(L^2) = 0.495774$
(iii) Variance of L=$var(L) = 0.428754$
(iv) Probability that no customer in the queue =0.875168
(v) Mean queue length when the servers are in vacation period =0.126477
(vi) Mean queue length when the servers are in regular busy period =0.007576
(vii) Probability that the servers are in working vacation period= $pr\{J = 0\} = 0.958638$
(viii) Probability that the servers are in regular busy period=$pr\{J = 1\} = 0.008576$

6 Conclusion

An N-policy multi-server Markovian working vacation queue and with unreliable system has been analyzed in this article. Besides the arrival rates, the service rate also depends on the server state. The waiting time distribution is also derived. The model can be generalized by taking arrival time/service time to follow a general distribution.

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References

[1] B. Avi-Itzhak and P. Naor, Some queueing problems with the service station subject to breakdowns, *Oper. Res.*, **11**, 303-320 (1963).
[2] X. Chao and Y.Q. Zhao, Analysis of multi-server queues with station and server vacations, *European Journal of Operational Research*, **110(2)**, 392-406 (1998).
[3] G. Choudhury and K.C. Madan, A two stage batch arrival queueing system with a modified Bernoulli schedule vacation under $N-$Policy, *Mathematics and Computer Modelling*, **42**, 71-85 (2005).
[4] S. Fond and S. Ross, *A heterogeneous arrival and service queueing loss model*, Tech-Report ORC 77-12, Operations Research Center, University of California, Berkeley, CA, (1977).
[5] D.P. Gaver, A waiting line with intrrupted service including priorities, *Jr. R. Stat. Soc.*, **24**, 73-90 (1962).
[6] M. Jain and A. Jain, Working vacation queueing model multiple type of server breakdown, *Appl. math. modelling*, **34(1)**, 1-13 (2010).
[7] D. Jayaraman, R. Nadarajan and M.R. Sitrarasu, A general bulk service queue with arrival rate dependent on server breakdowns, *Appl. math. modelling*, **18**, 156-160 (1994).
[8] R. Kalyanaraman and A. Sundaramoorthy, A Markovian working vacation queue with server state dependent arrival rate, $N-$policy and with unreliable server, *Mathematical Sciences International Research Journal*, **7(1)**, 320-330 (2018).
[9] J.C. Ke, C.H. Wu and Z.G. Zhang, Recent development in vacation queueing models: A short surrey, *Int. Jr. of Oper. Res.*, **7(4)**, 3-8 (2010).
[10] O. Kella, The threshold policy in the $M/G/1$ queues with server vacations, *Naval Research Logistics*, **36**, 111-123 (1989).
[11] J.C. Kemeny and J.L. Snell, *Finite Markov Chains*, The University Series in Undergraduate Mathematics, D. Van Nostrand, Princeton, NJ, USA, (1960).
[12] J.D. Kim, D.W. Choi and K.C. Chae, *Analysis of queue-length distribution of the $M/G/1$ with working vacation,* $(M/G/1/WV)$. In Proceeding of the International Conference on Statistics and related, Honolulu, Hawaii, USA, (2003).
[13] G.V. Krishna Reddy, R. Nadarajan and R. Arumuganathan, Analysis of a bulk queue with $N-$Policy multiple vacations and setup times, *Computers Ops. Res.*, **25(11)**, 957-967 (1998).
[14] B. Krishna Kumar and S. Pavai Madheswari, An $M/M/2$ queueing system with heterogeneous servers and multiple vacations, *Mathematical and Computer Modelling*, **41(13)**, 1415-1429 (2005).
[15] G. Latouche and M.F. Neuts, Efficient algorithmic solutions to exponential tandem queues with blocking, *SIAM Jr. Algebraic Discrete Math.*, **1**, 93-106 (1980).
[16] H.S. Lee, Steady state probabilities for the server vacation model with group arrivals and under control operation policy, *Jr. Korean OR/MS Soc.*, **16**, 36-48 (1991).
[17] H.S. Lee and M.M. Srinivasan, Control policies for the $M^x/G1/1$ queueing system, *Management Science*, **35**, 708-729 (1989).
[18] H.W. Lee, S.S. Lee, J.O. Park and K.C. Chae, Analysis of the $M^x/G/1$ queue with $N-$Policy and multiple vacations, *Jr. Appl. Prob.*, **31**, 476-496 (1994).
[19] S.S. Lee, H.W. Lee, S.H. Yoon and K.C. Chae, Batch arrival queue with $N-$Policy and single vacation, *Oper. Res.*, **22(2)**, 173-189 (1995).
[20] Y. Levy and U. Yechiali, An $M/M/s$ queue with servers vacations, *Information Systems and Operational Research*, **14(2)**, 153-163 (1976).
[21] Q. Li, D. Shi and J. Chao, Reliability approximation of a Markovian queueing system with server breakdown and repair, *Microelectron Reliab.*, **37**, 1203-1212 (1997).
[22] J.H. Li, N. Tian, Z.G. Zhang and H.P. Luh, Analysis of the $M/G/1$ queue with exponentially working vacation- a matrix analytic approach, *Queueing systems*, **61**, 139-166 (2009).
[23] Ch.H. Lin and J.Ch. Ke, Multi-server system with single working vacation, *Applied Mathematical Modeling*, **33**, 2967-2977, 2009.

[24] W. Liu, X. Xu and N. Tian, Stochastic decompositions in the $M/M/1$ queue with working vacation, Oper. Res. Lett., **35**, 595-600 (2007).

[25] D.M. Lucantoni, *A $GI/M/C$ queue with a different service rate for customers who need not wait an algorithmic solution*, Technical Rep. Univ. of Delware, USA, (1979).

[26] K.C. Madan, A single channel queue with bulk service subject to interruptions, *Microelectron Reliab.*, **29(5)**, 813-818 (1989).

[27] I.L. Mitrany and B. Avi-Itzhak, A many server queue with service interruptions, *Oper. Res.*, **16**, 628-638 (1968).

[28] M.F. Neuts, Markov chains with applications in queueing theory which have a matrix-geometric invariant probability vector, *Adv. Appl. Probab.*, **10**, 185-212 (1978).

[29] M.F. Neuts, *Matrix-Geometric solution in stochastic models*, Vol 2 of John Hopkins series in the Mathematical Sciences, Johns Hopkins University press, Baltimore, md, USA, (1981).

[30] M.F. Neuts and D.M. Lucantoni, A Markovian queue with N servers subject to breakdowns and repairs, *Management Science*, **25(9)**, 849-861 (1979).

[31] B. Sengupta, A queue with service interruptions in an alternating random environment, *Oper. Res.*, **38**, 308-318 (1990).

[32] L.D. Servi and S.G. Finn, $M/M/1$ queues with working vacations $(M/M/1/WV)$, *Perform. Eval.*, **50**, 41-52 (2002).

[33] J.G. Shanthikumar, Analysis of a single server queue with time and operational dependent server failures, *Adv. in mgnt. studies*, **1**, 339-359 (1982).

[34] A.W. Shogan, A single server queue with arrival rate dependent on server breakdowns, *Naval Res. Log. Quart.*, **26**, 487-497 (1979).

[35] T. Takine and B. Sengupta, A single server queue with service interruptions, *Queue. Syst.*, **26**, 285-300 (1998).

[36] Y. Tang, A single server $M/G/1$ queueing system subject to breakdowns- some reliability and queueing problems, *Microelectron Reliab.*, **37**, 315-321 (1997).

[37] K. Thiruvengadam, Queueing with breakdowns, *Oper. Res.*, **11**, 303-320 (1963).

[38] N. Tian and D. Yue, *Quasi-birth and death process and the matrix geometric solution*, Beijing: Science Press, (2002).

[39] N. Tian, Q. Li and J. Gao, Conditional Stochastic decomposition in $M/M/C$ queue with server vacations, *Communication in Statistics. Stochastic Models*, **15**, 367-377 (1999).

[40] B. Vinod, Exponential queues with servers vacations, *Journal of the Operational Research Society*, **37(10)**, 1007-1014 (1986).

[41] D.A. Wu and H. Takagi, $M/G/1$ queue with multiple working vacations, *Perform. Eval.*, **63**, 654-681 (2006).

[42] X. Xu, Z. Zhang and N. Tian, Analysis for the $M^X/M/1$ working vacation queue, *Int. Jr. of Infor. & Manag. Sci.*, **20(3)**, 379-394 (2009).

[43] M. Yadin and P. Naor, Queueing system with a removable service station, *Operational Research Quarterly*, **14(4)**, 393-405 (1963).

[44] V. Yechiali and P. Naor, Queueing problems with heterogeneous arrivals and service, *Oprs. Res*, **19**, 722-734 (1971).

[45] Z.G. Zhang and N. Tian, Analysis of queueing systems with synchronous single vacation for some servers, *Queueing Systems*, **45**, 161-175 (2003).