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On the Maximal Positive Definite Solution of the Nonlinear Matrix Equation $X - \sum_{i=1}^{m} A_i^* X^{-1} A_i + \sum_{j=1}^{n} B_j^* X^{-1} B_j = I$

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Abstract: In this paper, the existence and uniqueness of the maximal positive definite solution of the nonlinear matrix equation $X - \sum_{i=1}^{m} A_i^* X^{-1} A_i + \sum_{j=1}^{n} B_j^* X^{-1} B_j = I$ is studied. Our technique is based on the coupled fixed-point theorem. A sufficient condition for the existence of the unique maximal solution of the above nonlinear matrix equation is investigated. Some numerical examples are presented to show the applicability and the effectiveness of our technique.

Keywords: Nonlinear matrix equation, maximal positive solution, coupled fixed point, mixed monotone.

1 Introduction

Consider the nonlinear matrix equation

$$X + \sum_{j=1}^{n} B_j^* X^{-1} B_j - \sum_{i=1}^{m} A_i^* X^{-1} A_i = I, \qquad (1)$$

where B_j , $j = 1, 2, \ldots, n$, A_i , $i = 1, 2, \ldots, m$ are $N \times N$ nonsingular complex matrices with m and n being nonnegative integers. I is an $N \times N$ identity matrix. B_j^* and A_i^* represent the conjugate transpose of the matrices B_j and A_i, respectively. This kind of equations arises in various areas of applications, such as ladder networks [1, 2], dynamic programming [3,4] and control theory [5,6]. Many authors have studied the existence and uniqueness of a fixed point for different forms of nonlinear matrix equations (see, [7]-[12]). In [13], Bhaskar and Lakshmikantham presented the fixed-point theory in partially-ordered metric spaces and its applications. M.

Berzig et al. [14] studied the existence and uniqueness of a positive definite solution to the nonlinear matrix equation

$$X = Q - A^* X^{-1} A + B^* X^{-1} B.$$

M. Berzig [15] solved the nonlinear matrix equations of the type $X = Q + \sum_{i=1}^{m} A_i^* X A_i - \sum_{i=1}^{m} B_i^* X B_i$. J. H. Long et al. [16] studied the Hermitian positive definite solution of the nonlinear matrix equation

$$X + {A_1}^* X^{-1} A_1 + {A_2}^* X^{-1} A_2 = I.$$

Y. M. He and J. H. Long [17] considered the Hermitian positive definite solution of the nonlinear matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$. Throughout this paper, we denote $H(N)$ the set of $N \times N$ Hermitian matrices. For H_1 , $H_2 \in H(N)$, $H_1 \geq 0$ $(H_1 > 0)$, which means that H_1 is a positive semi-definite (positive definite) matrix. Also, $H_1 \geq H_2$ $(H_1 > H_2)$ denotes that $H_1 - H_2 \geq 0$ $(H_1 - H_2 > 0)$. We denote by $\| . \|$ and $\| . \|_{tr}$ the spectral norm and the trace norm, respectively, where $\| H_1 \|_{tr} = \sum_{t=1}^{m} \sigma_t(H_1)$ and $\sigma_t(H_1)$, $t = 1, 2, \ldots, m$ are the singular values of H_1 . We use X_L to denote the maximal solution of the matrix Eq. (1). We organize this paper as follows: First, in section 2, some definitions, lemmas and theorems that will be needed to develop this work are introduced. In section 3, both the existence and uniqueness of the maximal positive definite solution of the matrix equation $X - \sum_{i=1}^{m} A_i^* X^{-1} A_i + \sum_{j=1}^{n} B_j^* X^{-1} B_j = I$ is presented. In section 4, numerical examples are given to show the applicability and the effectiveness of our technique.

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2 Preliminaries

The following definitions, lemmas and theorems proclaimed below are important to develop this work.

Lemma 1. *[18] Let $H_1 \geq 0$ and $H_2 \geq 0$ be $N \times N$ matrices, then*

$$0 \leq tr(H_1 H_2) \leq \| H_1 \| \, tr(H_2).$$

Lemma 2. *[13] Let $H_1 \in H(N)$ satisfy $-I < H_1 < I$, then $\| H_1 \| < 1$.*

Lemma 3. *[19] If $0 < \theta \leq 1$,H_1 and H_2 are positive definite matrices of the same order where $H_1, H_2 \geq aI > 0$, then we have $\| H_1{}^{\theta} - H_2{}^{\theta} \|_{tr} \leq \theta a^{\theta-1} \| H_1 - H_2 \|_{tr}$ and $\| H_1{}^{-\theta} - H_2{}^{-\theta} \|_{tr} \leq \theta a^{-(\theta+1)} \| H_1 - H_2 \|_{tr}$.*

Definition 1. *[13] Let Y be a nonempty set and $G : Y \times Y \to Y$ be a given map, we call an element $(y, z) \in Y \times Y$ a coupled fixed point of G if $y = G(y, z)$ and $z = G(z, y)$.*

Definition 2. *[13] Let $(Y, \leq)$ be a partially-ordered set and $G : Y \times Y \to Y$ be a given map, we say that G has the mixed monotone property if for all $y, z \in Y$,*

$y_1, y_2 \in Y$, $y_1 \leq y_2$ yields that, $G(y_1, z) \leq G(y_2, z)$

$z_1, z_2 \in Y$, $z_1 \leq z_2$ yields that $G(y, z_1) \geq G(y, z_2)$

Theorem 1. *[13] Let $(Y, \leq)$ be a partially-ordered set granted with a metric space d such that (Y, d) is complete. Let $G : Y \times Y \to Y$ be a continuous mapping with the mixed monotone property on Y.*

If there exists $\varepsilon \in [0, 1)$, where $d(G(y, z), G(v, w)) \leq \frac{\varepsilon}{2} [d(y, v) + d(z, w)]$ for all $(y, z), (v, w) \in Y \times Y$ where $y \geq v$ and $z \leq w$. Moreover, there exists $y_0, z_0 \in Y$ such that $y_0 \leq G(y_0, z_0)$and $z_0 \geq G(z_0, y_0)$. Then,

- (a) G has a coupled fixed point $(\tilde{y}, \tilde{z}) \in Y \times Y$;
- (b) The sequences $\{y_k\}$ and $\{z_k\}$ are defined by $y_{k+1} = G(y_k, z_k)$ and $z_{k+1} = G(z_k, y_k)$ converge to $\tilde{y}$ and $\tilde{z}$, respectively;

In addition, suppose that every pair of elements has a lower bound and an upper bound, then

- (c) G has a unique coupled fixed point $(\tilde{y}, \tilde{z}) \in Y \times Y$;
- (d) $\tilde{y} = \tilde{z}$; and
- (e) We have the following estimate:

$$\max\{d(y_k, \tilde{y}), d(z_k, \tilde{y})\} \leq \frac{\varepsilon^k}{2(1-\varepsilon)} [d(G(y_0, z_0), y_0) + d(G(z_0, y_0), z_0].$$

Theorem 2.*(Theorem of Schauder Fixed Point) [20]*

Every continuous function $g : T \to T$ mapping T into itself has a fixed point, where T is a nonempty compact convex subset of a normed vector space.

3 Main Outcomes

3.1 On the Existence and Uniqueness of the Maximal Solution of Eq. (1).

Suppose that the set of matrices Ψ is defined by $\Psi = \{X \in H(N) : X \geq \frac{1}{2} I\}$.

let the mapping $G : \Psi \times \Psi \to \Psi$ associated with Eq. (1) be defined by

$$G(X,Y) = I - \sum_{j=1}^{n} B_j^* X^{-1} B_j + \sum_{i=1}^{m} A_i^* Y^{-1} A_i. \tag{2}$$

We prove the mixed monotone property of G in the following theorem.

Theorem 3.*If $\sum_{j=1}^{n} \|B_j\|^2 < \frac{1}{4^2}$ and $\sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{4^2}$, then the mapping G which is defined by (2) including the mixed monotone property with respect to the partial order$\leq$.*

*Proof.*Consider the mapping $G : \Psi \times \Psi \to \Psi$ defined by (2) for all X , $Y \in \Psi$, that is,$X \geq \frac{1}{2} I$, $Y \geq \frac{1}{2} I$. Let $X, Y, V, W \in \Psi$ such that $X \geq V$ and$Y \leq W$.

We have

$$\| G(X, Y) - G(V, W) \|_{tr} = \left\| \begin{array}{l} -\sum_{j=1}^{n} B_j^* X^{-1} B_j + \sum_{i=1}^{m} A_i^* Y^{-1} A_i \\ +\sum_{j=1}^{n} B_j^* V^{-1} B_j - \sum_{i=1}^{m} A_i^* W^{-1} A_i \end{array} \right\|_{tr}$$

$$= \left\| \begin{array}{l} \sum_{j=1}^{n} B_j^* (V^{-1} - X^{-1}) B_j \\ +\sum_{i=1}^{m} A_i^* (Y^{-1} - W^{-1}) A_i \end{array} \right\|_{tr}$$

$$\leq \left\| \sum_{j=1}^{n} B_j^* (V^{-1} - X^{-1}) B_j \right\|_{tr} + \left\| \sum_{i=1}^{m} A_i^* (Y^{-1} - W^{-1}) A_i \right\|_{tr}$$

$$= tr\left(\sum_{j=1}^{n} B_j^* (V^{-1} - X^{-1}) B_j \right) + tr\left(\sum_{i=1}^{m} A_i^* (Y^{-1} - W^{-1}) A_i \right)$$

$$= tr\left(\sum_{j=1}^{n} B_j^* B_j (V^{-1} - X^{-1}) \right)$$

$$+ tr\left(\sum_{i=1}^{m} A_i^* A_i (Y^{-1} - W^{-1}) \right) \tag{3}$$

Since $V^{-1} - X^{-1} \geq 0$ and $Y^{-1} - W^{-1} \geq 0$.

Using Lemma 1, we get

$$\| G(X, Y) - G(V, W) \|_{tr} \leq \left\| \sum_{j=1}^{n} B_j^* B_j \right\| tr\left(V^{-1} - X^{-1}\right) + \left\| \sum_{i=1}^{m} A_i^* A_i \right\| tr\left(Y^{-1} - W^{-1}\right) \leq \sum_{j=1}^{n} \|B_j\|^2 tr\left(V^{-1} - X^{-1}\right) + \sum_{i=1}^{m} \|A_i\|^2 tr\left(Y^{-1} - W^{-1}\right) \quad (4)$$

(4)

Moreover, since $X, Y, V, W \geq \frac{1}{2} I$, by Lemma 3 we get

$tr\left(V^{-1} - X^{-1}\right) \leq \left(\frac{1}{2}\right)^{-2} tr(V - X) = 4 tr(V - X)andtr\left(Y^{-1} - W^{-1}\right) \leq \left(\frac{1}{2}\right)^{-2} tr(Y - W) = 4 tr(Y - W)$.

So we get,

$$\| G(X, Y) - G(V W) \|_{tr} \leq 4 \sum_{j=1}^{n} \|B_j\|^2 \| V - X \|_{tr} + 4 \sum_{i=1}^{m} \|A_i\|^2 \| Y - W \|_{tr} . \tag{5}$$

Which yields that

$$\| G(X, Y) - G(V, W) \|_{tr} \leq \frac{\eta}{4} (\| V - X \|_{tr} + \| Y - W \|_{tr}) ,$$

Where $\eta = 4^2 \max \left\{ \sum_{j=1}^{n} \|B_j\|^2 , \sum_{i=1}^{m} \|A_i\|^2 \right\}$.

Since $\sum_{j=1}^{n} \|B_j\|^2 < \frac{1}{4^2}$ and $\sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{4^2}$ and Lemma 2, we can easily show that $\eta < 1$. Thus the mapping G has the mixed monotone property.

Which completes the proof of the theorem.

Now, we will prove the existence and uniqueness of the maximal positive definite solution of Eq. (1).

Theorem 4. *If the following assumptions hold*

$$\sum_{j=1}^{n} \|B_j\|^2 < \frac{1}{4^2} \ \ and \ \ \sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{4^2}, \tag{6}$$

$$6\sum_{j=1}^{n} B_j^* B_j - 2\sum_{i=1}^{m} A_i^* A_i \leq \frac{3}{2} I, \tag{7}$$

$$6\sum_{i=1}^{m} A_i^* A_i - 2\sum_{j=1}^{n} B_j^* B_j \leq \frac{3}{2} I. \tag{8}$$

Then

1. *Equation (1) has a unique maximal solution $X_L \in \Psi$.*
2. $X_L \in \begin{bmatrix} I - 2\sum_{j=1}^{n} B_j^* B_j + \frac{2}{3}\sum_{i=1}^{m} A_i^* A_i , \\ I - \frac{2}{3}\sum_{j=1}^{n} B_j^* B_j + 2\sum_{i=1}^{m} A_i^* A_i \end{bmatrix}$.

Proof. We demand that there exists $(X , Y) \in H(N) \times H(N)$ a solution to the system

$$X = I - \sum_{j=1}^{n} B_j^* X^{-1} B_j + \sum_{i=1}^{m} A_i^* Y^{-1} A_i,$$

$$Y = I - \sum_{j=1}^{n} B_j^* Y^{-1} B_j + \sum_{i=1}^{m} A_i^* X^{-1} A_i. \tag{9}$$

Now, taking $X_0 = \frac{1}{2} I$ and $Y_0 = \frac{3}{2} I$

From condition (7) we have

$$6\sum_{j=1}^{n} B_j^* B_j - 2\sum_{i=1}^{m} A_i^* A_i \leq \frac{3}{2} I$$

then

$$2\sum_{j=1}^{n} B_j^* B_j - \frac{2}{3}\sum_{i=1}^{m} A_i^* A_i \leq \frac{1}{2} I$$

so we have

$$G(\frac{1}{2} I , \ \frac{3}{2} I) = I - 2\sum_{j=1}^{n} B_j^* B_j + \frac{2}{3}\sum_{i=1}^{m} A_i^* A_i \geq \frac{1}{2} I,$$

that is

$$X_0 = \frac{1}{2} I \ \leq \ G(\frac{1}{2} I , \ \frac{3}{2} I).$$

Moreover, from condition (8) we get

$$6\sum_{i=1}^{m} A_i^* A_i - 2\sum_{j=1}^{n} B_j^* B_j \leq \frac{3}{2} I,$$

then

$$2\sum_{i=1}^{m} A_i^* A_i - \frac{2}{3}\sum_{j=1}^{n} B_j^* B_j \leq \frac{1}{2} I,$$

so we have

$$G(\frac{3}{2} I , \ \frac{1}{2} I) = I - \frac{2}{3}\sum_{j=1}^{n} B_j^* B_j + 2\sum_{i=1}^{m} A_i^* A_i \leq \frac{3}{2} I$$

and

$$Y_0 = \frac{3}{2} I \geq G(\frac{3}{2} I , \ \frac{1}{2} I) .$$

From Theorem 1 (a), there exists $(X , Y) \in H(N) \times H(N)$ where $G(X,Y) = X$ and $G(Y,X) = Y$ that is, (X,Y) is a solution to (9). On the other hand, for every $X,Y \in H(N)$ there is a greatest lower bound and a least upper bound. Note also that the partial order G is a continuous mapping, by Theorem 1, (X,Y) is the unique coupled fixed point of G that is

$$X = Y = X_L.$$

Thus, the unique solution of Eq. (1) is X_L. Thus, the proof of 1) is completed. To prove 2) we should used Theorem of Schauder Fixed Point, we state the mapping

$M : \left[G(\frac{1}{2} I , \ \frac{3}{2} I) , G(\frac{3}{2} I , \ \frac{1}{2} I)\right] \to \Psi$ by:

$$M(X_L) = G(X_L, X_L) = I - \sum_{j=1}^{n} B_j^* X_L^{-1} B_j + \sum_{i=1}^{m} A_i^* X_L^{-1} A_i ,$$

For all $X_L \in \left[G(\frac{1}{2} I , \ \frac{3}{2} I) , G(\frac{3}{2} I , \ \frac{1}{2} I)\right]$.

We want to prove that

$$M \ \left(\left[G(\frac{1}{2} I , \ \frac{3}{2} I) , G(\frac{3}{2} I , \ \frac{1}{2} I)\right]\right) \subseteq \left[G(\frac{1}{2} I , \ \frac{3}{2} I) , G(\frac{3}{2} I , \ \frac{1}{2} I)\right] .$$

Let $X_L \in \left[G(\frac{1}{2} I , \ \frac{3}{2} I) , G(\frac{3}{2} I , \ \frac{1}{2} I)\right]$, that is $G(\frac{1}{2} I , \ \frac{3}{2} I) \leq X_L \leq G(\frac{3}{2} I , \ \frac{1}{2} I)$.

Applying the property of mixed monotone of G yields that

$$G \ \left(G(\tfrac{1}{2} I , \ \tfrac{3}{2} I), G(\tfrac{3}{2} I , \ \tfrac{1}{2} I)\right) \leq M(X_L) = G(X_L , X_L) \leq G \ \left(G(\tfrac{3}{2} I , \ \tfrac{1}{2} I) , G(\tfrac{1}{2} I , \ \tfrac{3}{2} I)\right) ,$$

since $G(\frac{1}{2} I , \ \frac{3}{2} I) \geq \frac{1}{2} I$ and $G(\frac{3}{2} I , \ \frac{1}{2} I) \leq \frac{3}{2} I$.

Applying the property of the mixed monotone of G again implies that

$$G \ \left(G(\tfrac{1}{2} I , \ \tfrac{3}{2} I), G(\tfrac{3}{2} I , \ \tfrac{1}{2} I)\right) \geq G(\tfrac{1}{2} I , \ \tfrac{3}{2} I), \tag{10}$$

$$G \ \left(G(\tfrac{3}{2} I , \ \tfrac{1}{2} I), G(\tfrac{1}{2} I , \ \tfrac{3}{2} I)\right) \leq G(\tfrac{3}{2} I , \ \tfrac{1}{2} I) \tag{11}$$

From (10) and (11), it follows that

$$G(\frac{1}{2} I , \ \frac{3}{2} I) \leq M(X_L) \leq G(\frac{3}{2} I , \ \frac{1}{2} I) .$$

Thus, our claim that

$$M\left(\left[G(\tfrac{1}{2}I\,,\;\tfrac{3}{2}I)\,,\,G(\tfrac{3}{2}I\,,\;\tfrac{1}{2}I)\right]\right) \subseteq \left[G(\tfrac{1}{2}I\,,\;\tfrac{3}{2}I)\,,\,G(\tfrac{3}{2}I\,,\;\tfrac{1}{2}I)\right].$$

holds.

Now, we have a continuous mapping M that maps the compact convex set $\left[G(\frac{1}{2}I\,,\;\frac{3}{2}I)\,,\,G(\frac{3}{2}I\,,\;\frac{1}{2}I)\right]$ into itself, from Schuader fixed point theorem we get that M has at least one fixed point in this set, but a fixed point of M is a solution of Eq. (1), and we proved already that Eq. (1) has a unique solution in Ψ. Thus, this solution must be in the set $\left[G(\frac{1}{2}I\,,\;\frac{3}{2}I)\,,\,G(\frac{3}{2}I\,,\;\frac{1}{2}I)\right]$.

That is,

$$X_L \in \left[\begin{array}{c} I-2\sum_{j=1}^{n}B_j^*B_j+\frac{2}{3}\sum_{i=1}^{m}A_i^*A_i\,, \\ I-\frac{2}{3}\sum_{j=1}^{n}B_j^*B_j+2\sum_{i=1}^{m}A_i^*A_i \end{array}\right].$$

Which completes the proof of 2) of the theorem.

Now, we present the convergence of the sequences $\{X_k\}$ and $\{Y_k\}$ to the maximal positive definite solution X_L of Eq.(1).

Theorem 5. *The sequences $\{X_k\}$ and $\{Y_k\}$ of positive definite matrices generated from the following iterative algorithm*

$X_0 = \frac{1}{2}I and Y_0 = \frac{3}{2}I.$

$$X_{k+1} = I-\sum_{j=1}^{n}B_j^*X_k^{-1}B_j+\sum_{i=1}^{m}A_i^*Y_k^{-1}A_i\,,$$

$$\begin{array}{l} Y_{k+1} = I-\sum_{j=1}^{n}B_j^*Y_k^{-1}B_j+\sum_{i=1}^{m}A_i^*X_k^{-1}A_i \\ \quad k=0\,,\,1\,,\,2\,,\,\ldots \end{array} \tag{12}$$

converge to X_L, that is,

$$\lim_{k\to\infty}\|X_k-X_L\|_{tr} = \lim_{k\to\infty}\|Y_k-X_L\|_{tr} = 0$$

with the error

$$\begin{array}{l} \max\{\,\|X_k-X_L\|_{tr}\;,\;\|\,Y_k-X_L\|_{tr}\} \\ \quad\le \frac{\eta^k}{1-\eta}\max\{\,\|X_1-X_0\|_{tr}\;,\;\|\,Y_1-Y_0\|_{tr}\}\,, \end{array}$$

where $0<\eta<1$.

Proof. We consider the iterative process (12). Applying this process generates the sequences of matrices $\{X_k\}$ and $\{Y_k\}$ as follows:

For $k=0$,

$$\begin{array}{l} X_1 = I-\sum_{j=1}^{n}B_j^*X_0^{-1}B_j+\sum_{i=1}^{m}A_i^*Y_0^{-1}A_i \\ \quad = I-2\sum_{j=1}^{n}B_j^*B_j+\frac{2}{3}\sum_{i=1}^{m}A_i^*A_i \ge \frac{1}{2}I = X_0 \end{array},$$

that is, $X_1 \ge X_0$.

Moreover, for some positive matrix $\tilde{X}$ for the Eq. (1), and from 2) of Theorem 4 we have

$$\tilde{X} \in \left[\begin{array}{c} I-2\sum_{j=1}^{n}B_j^*B_j+\frac{2}{3}\sum_{i=1}^{m}A_i^*A_i\,, \\ I-\frac{2}{3}\sum_{j=1}^{n}B_j^*B_j+2\sum_{i=1}^{m}A_i^*A_i \end{array}\right].$$

Which means that, $\frac{1}{2}I \le \tilde{X} \le \frac{3}{2}I$. i.e., $\tilde{X} \ge X_0 = \frac{1}{2}I$ and $\tilde{X} \le Y_0 = \frac{3}{2}I$.

Now, $$\begin{array}{l} \tilde{X} = I-\sum_{j=1}^{n}B_j^*\tilde{X}^{-1}B_j+\sum_{i=1}^{m}A_i^*\tilde{X}^{-1}A_i \\ \quad \ge I-\sum_{j=1}^{n}B_j^*X_0^{-1}B_j+\sum_{i=1}^{m}A_i^*Y_0^{-1}A_i = X_1 \end{array},$$

that is, $\tilde{X} \ge X_1$. Thus, $\tilde{X} \ge X_1 \ge X_0$.

Also, for some positive matrix $\tilde{Y}$ for the Eq. (1)

$$\begin{array}{l} Y_1 = I-\sum_{j=1}^{n}B_j^*Y_0^{-1}B_j+\sum_{i=1}^{m}A_i^*X_0^{-1}A_i \\ \quad = I-\frac{2}{3}\sum_{j=1}^{n}B_j^*B_j+2\sum_{i=1}^{m}A_i^*A_i \le \frac{3}{2}I = Y_0 \end{array},$$

that is, $Y_1 \le Y_0$.

Moreover,

$$\begin{array}{l} \tilde{Y} = I-\sum_{j=1}^{n}B_j^*\tilde{Y}^{-1}B_j+\sum_{i=1}^{m}A_i^*\tilde{Y}^{-1}A_i \\ \quad \le I-\sum_{j=1}^{n}B_j^*Y_0^{-1}B_j+\sum_{i=1}^{m}A_i^*X_0^{-1}A_i = Y_1 \end{array}.$$

So, we have, $\tilde{Y} \le Y_1$. Thus, $Y_0 \ge Y_1 \ge \tilde{Y}$.

Suppose that

$$\tilde{X} \ge X_k \ge X_{k-1} \text{ and } Y_{k-1} \ge Y_k \ge \tilde{Y}. \tag{13}$$

Now we will prove that $\tilde{X} \ge X_{k+1} \ge X_k$ and $Y_k \ge Y_{k+1} \ge \tilde{Y}$.

Using inequalities (13) yields that

$$\begin{array}{l} I-\sum_{j=1}^{n}B_j^*\tilde{X}^{-1}B_j+\sum_{i=1}^{m}A_i^*\tilde{X}^{-1}A_i \\ \quad \ge I-\sum_{j=1}^{n}B_j^*X_k^{-1}B_j+\sum_{i=1}^{m}A_i^*Y_k^{-1}A_i \\ \quad \ge I-\sum_{j=1}^{n}B_j^*X_{k-1}^{-1}B_j+\sum_{i=1}^{m}A_i^*Y_{k-1}^{-1}A_i\,. \end{array}$$

That is, $\tilde{X} \ge X_{k+1} \ge X_k$.

Similarly, we can prove that

$$\begin{array}{l} I-\sum_{j=1}^{n}B_j^*\tilde{Y}^{-1}B_j+\sum_{i=1}^{m}A_i^*\tilde{Y}^{-1}A_i \\ \quad \le I-\sum_{j=1}^{n}B_j^*Y_k^{-1}B_j+\sum_{i=1}^{m}A_i^*X_k^{-1}A_i \\ \quad \le I-\sum_{j=1}^{n}B_j^*Y_{k-1}^{-1}B_j+\sum_{i=1}^{m}A_i^*X_{k-1}^{-1}A_i\,. \end{array}$$

That is, $Y_k \ge Y_{k+1} \ge \tilde{Y}$.

Therefore, the inequalities (13) are true for all $k=0\,,\,1\,,\,2\,,\,\ldots$.

Hence, the sequence $\{X_k\}$ is monotonically increasing and bounded from above by some positive definite solution $\tilde{X}\,(X_k \ge \tilde{X})$ of Eq. (1) and the sequence $\{Y_k\}$ is monotonically decreasing and bounded from below by some positive definite solution $\tilde{Y}\,(Y_k \le \tilde{Y})$ of Eq. (1) .It follows that $\lim_{k\to\infty}X_k = \tilde{X}$ and $\lim_{k\to\infty}Y_k = \tilde{Y}$ exist. Taking limits of (12) gives that $\tilde{X} = \tilde{Y} = X_L$.

Thus, the proof of the theorem is completed.

4 Numerical Examples

In this portion, numerical examples are presented to confirm the correctness of Theorem 4 and the existence for the unique maximal positive definite solution $X_L$of Eq. (1) using the iterative process (12) in Theorem 5 We have performed the algorithms in MATLAB (writing our own programs) and we have also run the programs on a PC Pentium IV.

Example 1. Consider the nonlinear matrix Eq. (1) with

$$A_1=\begin{pmatrix}-0.04 & -0.02 & 0.02\\ -0.02 & -0.04 & 0.06\\ -0.08 & -0.02 & -0.04\end{pmatrix},\ A_2=\begin{pmatrix}0.06 & 0.02 & 0.002\\ 0.02 & 0.06 & 0.04\\ -0.04 & 0.02 & 0.06\end{pmatrix},$$

$$B_1=\begin{pmatrix}-0.12 & -0.03 & 0.03\\ -0.03 & -0.12 & -0.09\\ -0.09 & -0.03 & -0.12\end{pmatrix} \text{ and } B_2=\begin{pmatrix}0.05 & 0.01 & -0.02\\ 0.01 & 0.05 & 0.02\\ 0.02 & 0.01 & 0.05\end{pmatrix}.$$

The presumptions of Theorem 4 hold, that is,

i)$\sum_{j=1}^{n}\|B_j\|^2 = 0.018277 < \frac{1}{4^2}$ and $\sum_{i=1}^{m}\|A_i\|^2 = 0.04827 < \frac{1}{4^2}$,

ii) $eig\left(\frac{3}{2}I-6\sum_{j=1}^{n}B_j^*B_j+2\sum_{i=1}^{m}A_i^*A_i\right) = (1.2396\ ,\ 1.4224\ ,\ 1.4692)$,

which satisfies inequality (7) of Theorem 4.

iii) $eig\left(\frac{3}{2}I-6\sum_{i=1}^{m}A_i^*A_i+2\sum_{j=1}^{n}B_j^*B_j\right) = (1.4488\ ,\ 1.4905\ ,\ 1.5151)$,

which satisfies inequality (8) of Theorem 4 Now, We consider the sequences $\{X_k\}$ and $\{Y_k\}$ defined in (12) in Theorem 5 with $X_0=\frac{1}{2}I$and $Y_0=\frac{3}{2}I$.

We have the errorsfor every iteration k,

$$R(X_k)=\left\|X_k-I+\sum_{j=1}^{n}B_j^*X_k^{-1}B_j-\sum_{i=1}^{m}A_i^*X_k^{-1}A_i\right\|,$$

$$R(Y_k)=\left\|Y_k-I+\sum_{j=1}^{n}B_j^*Y_k^{-1}B_j-\sum_{i=1}^{m}A_i^*Y_k^{-1}A_i\right\|.$$

after 12 iterations, we get

$$X_L\approx X_{12}=Y_{12} =\begin{pmatrix}0.98709 & -0.0066787 & -0.01099\\ -0.0066787 & 0.98765 & -0.013558\\ -0.01099 & -0.013558 & 0.983675\end{pmatrix},$$

with $R_{12}=2.318617\,e\ -016$.

The eigenvalues of X_L are $(0.96473\ ,\ 0.99369\ ,\ 1)$.

Example 2. Consider the nonlinear matrix Eq. (1) with

$$A_1=\begin{pmatrix}0.0002 & -0.0004 & 0.0001 & 0.15\\ 0.0004 & -0.0016 & -0.0002 & -0.0008\\ -0.0012 & -0.0063 & 0.0014 & -0.0018\\ 0.0034 & -0.0012 & -0.0001 & -0.0015\end{pmatrix},$$

$$A_2=\begin{pmatrix}-0.0005 & 0.0002 & 0.0024 & -0.0003\\ 0.0002 & -0.002 & 0.0005 & 0.0001\\ 0.0004 & 0.0003 & -0.0004 & -0.0034\\ -0.0021 & 0.0004 & 0.0006 & 0.0007\end{pmatrix},$$

$$B_1=\begin{pmatrix}-0.09 & -0.18 & 0.06 & 0.003\\ -0.003 & 0.012 & -0.039 & -0.015\\ -0.039 & -0.063 & 0.012 & -0.0072\\ -0.033 & -0.006 & -0.09 & 0.0003\end{pmatrix} \text{ and}$$

$$B_2=\begin{pmatrix}0.003 & -0.002 & -0.001 & 0.002\\ 0.0001 & 0.004 & -0.013 & 0.0005\\ -0.013 & 0.012 & -0.0014 & -0.0104\\ 0.0101 & 0.0012 & 0.003 & -0.0012\end{pmatrix}.$$

The presumptions of Theorem 4 are satisfied, that is,

i)$\sum_{j=1}^{n}\|B_j\|^2 = 0.02252 < \frac{1}{4^2}$ and $\sum_{i=1}^{m}\|A_i\|^2 = 0.050544 < \frac{1}{4^2}$,

ii) $eig\left(\frac{3}{2}I-6\sum_{j=1}^{n}B_j^*B_j+2\sum_{i=1}^{m}A_i^*A_i\right) = (1.1992\ ,\ 1.4353\ ,\ 1.4979\ ,\ 1.5428)$,

which satisfies inequality (7) of Theorem 4.

iii) $eig\left(\frac{3}{2}I-6\sum_{i=1}^{m}A_i^*A_i+2\sum_{j=1}^{n}B_j^*B_j\right) = (1.6\ ,\ 1.5006\ ,\ 1.5215\ ,\ 1.3657)$,

which satisfies inequality (8) of Theorem 4. Now, We consider the sequences $\{X_k\}$ and $\{Y_k\}$ defined in (12) in Theorem 5 with $X_0=\frac{1}{2}I$and $Y_0=\frac{3}{2}I$.

We get the errorsfor every iteration k,

$$R(X_k)=\left\|X_k-I+\sum_{j=1}^{n}B_j^*X_k^{-1}B_j-\sum_{i=1}^{m}A_i^*X_k^{-1}A_i\right\|,$$

$$R(Y_k)=\left\|Y_k-I+\sum_{j=1}^{n}B_j^*Y_k^{-1}B_j-\sum_{i=1}^{m}A_i^*Y_k^{-1}A_i\right\|.$$

after 10 iterations, we get

$$X_L\approx X_{10}=Y_{10} =\begin{pmatrix}0.98894 & -0.018837 & 0.0028039 & -0.00017124\\ -0.018837 & 0.96304 & 0.011557 & 0.00031524\\ 0.0028039 & 0.011557 & 0.98662 & -0.00064303\\ -0.00017124 & 0.00031524 & -0.00064303 & 1.0224\end{pmatrix},$$

$$eig\,(X_L)=(0.94952\ ,\ 0.98939\ ,\ 0.99967,\ 1.0224)$$

with $R_{10}=1.0489617\,e\ -018$.

5 Conclusion

In this paper, the coupled fixed-point theory on ordered metric spaces is used to find the solution of the nonlinear matrix equation $X-\sum_{i=1}^{m}A_i^*X^{-1}A_i+\sum_{j=1}^{n}B_j^*X^{-1}B_j=I$. An iterative process is proposed to compute the unique maximal positive definite solution of this nonlinear matrix equation. Finally, numerical examples are given to show the applicability and the effectiveness of our technique.

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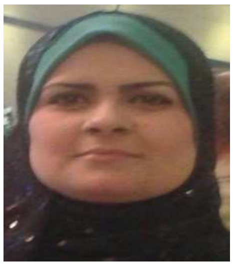

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