Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/140219

Some New Hypergeometric Transformations via Fractional Calculus Technique

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Received: 2 Aug. 2019, Revised: 1 Oct. 2019, Accepted: 22 Oct. 2019
Published online: 1 Mar. 2020

Abstract: In this paper, we establish a new hypergeometric transformation involving Gauss function using fractional calculus technique. We also obtain some other known and new identities involving Gauss function as special cases of our main result. As application of our result, we give two new hypergeometric transformations involving the products of the hypergeometric functions.

Keywords: Fractional derivative operator; Hypergeometric transformation; Gauss's Function; Generalized hypergeometric function.

1 Introduction and preliminaries

The Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ ([1, p.22 eq(1), p.32 Q.N.(8) and Q.N.(9)], see also [2, p.23, eq(22) and eq(23)]), is defined by

$$(\alpha)_p := \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\dots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \tag{1}$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

The generalized hypergeometric function ${}_pF_q$ ([1, Art.44, pp.73-74], see also [3]), is defined by

$${}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \dots, \alpha_p; \\ \beta_1, \beta_2, \dots, \beta_q;\end{matrix}\ z\right] = {}_pF_q\left[\begin{matrix}(\alpha_p); \\ (\beta_q);\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n} \frac{z^n}{n!}, \tag{2}$$

$\big(p, q \in \mathbb{N}_0;\ \ p \leqq q+1\ ;\ \ p \leqq q$ and $|z| < \infty$; $p = q+1$ and $|z| < 1$; $p = q+1, |z| = 1$ and $\Re(\omega) > 0$; $p = q+1, |z| = 1, z \neq 1$ and $-1 < \Re(\omega) \leq 0\big)$ where

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j,$$

$$\big(\alpha_j \in \mathbb{C}\ (j = 1, 2, \dots, p); \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-(j = 1, 2, \dots, q)\big).$$

The classical summation theorems for the series ${}_2F_1$ such as Gauss, Kummer, and Bailey play an important role in the theory of hypergeometric series. For the extensions of these classical summation theorems, we refer to [4,5].

By employing the classical summation theorems, Bailey [6] obtained a large number of very interesting results (including some results due to Ramanujan, Gauss, Kummer, and Whipple) involving products of generalized hypergeometric series.

On the other hand, from the theory of differential equations, Kummer [7] established following very interesting and useful result known in the literature as Kummer's second theorem

$$e^{-\frac{x}{2}}{}_1F_1\left[\begin{matrix}a; \\ 2a;\end{matrix}\ x\right] = {}_0F_1\left[\begin{matrix}\overline{\quad\quad}; \\ a+\frac{1}{2};\end{matrix}\ \frac{x^2}{16}\right]. \tag{3}$$

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In 1836, Kummer gave the following hypergeometric transformation [7, p. 82, Entry 72]

$$
{}_2F_1\left[\begin{matrix} a,b; & \frac{1+z}{2} \\ \frac{1+a+b}{2}; & \end{matrix}\right] = \frac{\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\,{}_2F_1\left[\begin{matrix} \frac{a}{2},\frac{b}{2}; & z^2 \\ \frac{1}{2}; & \end{matrix}\right] + \frac{2\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}\,{}_2F_1\left[\begin{matrix} \frac{a+1}{2},\frac{b+1}{2}; & z^2 \\ \frac{1}{2}; & \end{matrix}\right] \tag{4}
$$

$\left(\frac{a+b+1}{2} \in \mathbb{C}\setminus\mathbb{Z}^-; |z|<1, \text{ or } z\in\{-1,1\}, \Re(a+b)<1, \text{ or } z\in\{-i,i\}, \Re(a+b)<3; i=\sqrt{(-1)}\right).$

It is also worthy to mention here that the above formula (4) was discovered by S. Ramanujan independently [8, p. 64, Entry 21], and also recorded in [9, p. 65, Equation 2.1.5(28), p. 111, Equation 2.11(3)].

Further in equation (4), if we set $a=b=\frac{1}{2}$, then we get the following result of Ramanujan [8, p. 96]

$$
{}_2F_1\left[\begin{matrix} \frac{1}{2},\frac{1}{2}; & \frac{1+z}{2} \\ 1; & \end{matrix}\right] = \Delta_1\,{}_2F_1\left[\begin{matrix} \frac{1}{4},\frac{1}{4}; & z^2 \\ \frac{1}{2}; & \end{matrix}\right] + \Delta_2\,{}_2F_1\left[\begin{matrix} \frac{3}{4},\frac{3}{4}; & z^2 \\ \frac{3}{2}; & \end{matrix}\right], \tag{5}
$$

where $\Delta_1 = \dfrac{\Gamma\left(\frac{1}{2}\right)}{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}$, and $\Delta_2 = \dfrac{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^3}$.

In 1995, Rathie and Nagar [10] obtained two results closely related to Kummer's second theorem (3), one of those results is given below

$$
e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix} a; & x \\ 2a+1; & \end{matrix}\right] = {}_0F_1\left[\begin{matrix} \text{---}; & \frac{x^2}{16} \\ a+\frac{3}{2}; & \end{matrix}\right] - \frac{x}{2(2a+1)}\,{}_0F_1\left[\begin{matrix} \text{---}; & \frac{x^2}{16} \\ a+\frac{3}{2}; & \end{matrix}\right]. \tag{6}
$$

In 2008, Rathie and Pogany [11] established a new summation formula for ${}_3F_2(1/2)$ and as an application, the following result which is known as an extension of the Kummer's second theorem (3)

$$
e^{-\frac{x}{2}}\,{}_2F_2\left[\begin{matrix} a,1+d; & x \\ 2a+1,d; & \end{matrix}\right] = {}_0F_1\left[\begin{matrix} \text{---}; & \frac{x^2}{16} \\ a+\frac{3}{2}; & \end{matrix}\right] - \frac{xd(1-2a/d)}{2(2a+1)}\,{}_0F_1\left[\begin{matrix} \text{---}; & \frac{x^2}{16} \\ a+\frac{3}{2}; & \end{matrix}\right], \tag{7}
$$

provided $d \neq 0,-1,-2,-3,\ldots$.
It is to be noted that if we set $d=2a$ in equation (7) then it reduces to Kummer's second theorem (3).

In 2018, Qureshi and Baboo [12] gave a novel unified and generalised result on Kummer type transformation as follows

$$
{}_2F_1\left[\begin{matrix} a,b; & \frac{1+z}{2} \\ \frac{1+a+b-2m}{2}; & \end{matrix}\right] = \frac{2^{a+b-2}\Gamma\left(\frac{1+a+b}{2}\right)}{\sqrt{\pi}\,\Gamma(a)\Gamma(b)} \left[\sum_{p=0}^{\infty} \frac{\left(\frac{1+a+b}{4}\right)_p \left(\frac{3+a+b}{4}\right)_p z^{2p}}{\left(\frac{1}{2}\right)_p \left(\frac{1+a+b-2m}{4}\right)_p \left(\frac{3+a+b-2m}{4}\right)_p p!} \times \sum_{r=0}^{m} \binom{m}{r} \frac{2^r\Gamma\left(\frac{a+2p+r}{2}\right)\Gamma\left(\frac{b+2p+r}{2}\right)}{\left(\frac{1+a+b-2m+4p}{2}\right)_r} + \frac{2(1+a+b)z}{(1+a+b-2m)} \times \sum_{p=0}^{\infty} \frac{\left(\frac{3+a+b}{4}\right)_p \left(\frac{5+a+b}{4}\right)_p z^{2p}}{\left(\frac{3}{2}\right)_p \left(\frac{3+a+b-2m}{4}\right)_p \left(\frac{5+a+b-2m}{4}\right)_p p!} \sum_{r=0}^{m} \binom{m}{r} \frac{2^r\Gamma\left(\frac{a+2p+r+1}{2}\right)\Gamma\left(\frac{b+2p+r+1}{2}\right)}{\left(\frac{3+a+b-2m+4p}{2}\right)_r}\right] \tag{8}
$$

$\left(a,b,\frac{1+a+b-2m}{2},\frac{1+a+b-2m}{2} \in \mathbb{C}\setminus\mathbb{Z}_0^-; |z|<1; m\in\mathbb{N}_0\right).$

2 Pochhammer contour integral representation for fractional derivative

The most familiar Pochhammer contour integral representation for fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [13], (see also [14, 15, 16]), that is,

$$
\mathscr{D}_z^{\alpha}\{z^p f(z)\} = \frac{1}{\Gamma(-\alpha)} \int_0^z f(\xi)\xi^p(\xi-z)^{\alpha-1}d\xi \tag{9}
$$

$(\Re(\alpha)<0, \Re(p)>1)$,
where the integration is done along a straight line from 0 to z in the ξ-plane. By integrating by parts m-times, we obtain

$$
\mathscr{D}_z^{\alpha}\{z^p f(z)\} = \frac{d^m}{dz^m}\mathscr{D}_z^{\alpha-m} z^p f(z). \tag{10}
$$

This allows us to modify the restriction $\Re(\alpha)<0$ to $\Re(\alpha)<m$ (see [17]).

Another representation for the fractional derivative is based on the Cauchy integral formula. This representation has been widely used in many interesting papers (see, for example [17, 18, 19, 20]).

Definition 1. Let $f(z)$ be analytic in a simply-connected region $\mathscr{R}$ of the complex z-plane. Let $g(z)$ be regular and univalent on $\mathscr{R}$ and let $g^{-1}(0)$ be an interior point of $\mathscr{R}$. Then, if α is not a negative integer, p is not an integer, and z is in $\mathscr{R}$ $\{g^{-1}(0)\}$, we define the fractional derivative of order α of $[g(z)]^p f(z)$ with respect to $g(z)$ by

$$\mathscr{D}^{\alpha}_{g(z)}\{[g(z)]^p f(z)\} = \frac{e^{-i\pi p}\Gamma(1+\alpha)}{4\pi\sin(\pi p)} \times \int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \frac{f(\xi)[g(\xi)]^p g'(\xi)}{[g(\xi)-g(z)]^{\alpha+1}} d\xi. \tag{11}$$

For non-integers α and p, the functions $[g(\xi)]^p$ and $[g(\xi)-g(z)]^{-\alpha-1}$ in the integrand have two branch lines which begin, respectively, at $\xi = z$ and $\xi = g^{-1}(0)$, and both branches pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = \{C_1 \bigcup C_2 \bigcup C_3 \bigcup C_4\}$ at any other point. Here $F(a)$ denotes the principal value of the integrand in (11) at the beginning and the ending point of the Pochhammer contour $P(a)$ which is closed on the Riemann surface of the multiple-valued function $F(\xi)$.

Remark 1. In Definition 1, the function $f(z)$ must be analytic at $\xi = g^{-1}(0)$. However, it is interesting to note here that, if we could also allow $f(z)$ to have an essential singularity at $\xi = g^{-1}(0)$, then equation (11) would still be valid.

Remark 2. Since Pochhammer contour never crosses the singularities at $\xi = g^{-1}(0)$ and $\xi = z$ in (11), then we know that the integral is analytic for all p and for all α and for z in $\mathscr{R}$ $\{g^{-1}(0)\}$. Indeed, in this case, the only possible singularities of $\mathscr{D}^{\alpha}_{g(z)}\{[g(z)]^p f(z)\}$ are $\alpha = -1,-2,-3,\ldots$ and $p = 0,\pm 1,\pm 2,\ldots$, which can be directly identified from the coefficient of the integral (11). However, by integrating by parts N times the integral in (11) by two different ways, we can show that $\alpha = -1,-2,\ldots$ and $p = 0,1,2,\ldots$ are removable singularities (see, for details, [21]).

It is well known that (see, [22], [23, p. 83, Eq. (2.4)])

$$\mathscr{D}^{\alpha}_{z} z^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} \quad (\Re(p) > -1). \tag{12}$$

Adopting the Pochhammer representation for the fractional derivative modifies the restriction to the case when p is non-negative integer.

A large number of transformation formulas involving hypergeometric functions were obtained, recently, by using the so-called Beta integral method [24,25,26,27]. The beta function $B(\alpha,\beta)$ is defined by the following integral representation

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt \tag{13}$$

$(\Re(\alpha) > 0, \Re(\beta) > 0))$.
The so-called Beta integral method consists essentially of an integral from 0 to 1 of expressions which contain terms in the form $z^a(1-z)^b$ to obtain new transformations formulas.

Fractional calculus has various applications in the fields of natural science and technology, including chemistry, biology, physics, engineering, as well as pure and applied mathematics [28,29,30,31,32]. Many physical phenomena can be described more effectively via fractional calculus. Most of them are portrayed via nonlinear fractional differential equations. Thus, scientists in various branches of science have tried to solve them.

3 The well poised fractional calculus operator ${}_zO^{\alpha}_{\beta}$

In this section, we recall some important properties of the fractional calculus operator ${}_zO^{\alpha}_{\beta}$ that was introduced by Tremblay [33] as follows

$${}_zO^{\alpha}_{\beta} = \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta} D^{\alpha-\beta}_{z} z^{\alpha-1} \quad (\beta \in \mathbb{N}_0). \tag{14}$$

Here, we have some properties of the fractional calculus operator ${}_zO^{\alpha}_{\beta}$ as follows
1. Linearity:

$${}_zO^{\alpha}_{\beta}[\lambda_1 f(z) + \lambda_2 g(z)] = \lambda_1\, {}_zO^{\alpha}_{\beta} f(z) + \lambda_2\, {}_zO^{\alpha}_{\beta} g(z). \tag{15}$$

2. Identity:

$${}_zO^{\alpha}_{\beta} = I. \tag{16}$$

3. Reduction:

$${}_zO^{\alpha}_{\beta}\, {}_zO^{\alpha}_{\gamma} = {}_zO^{\alpha}_{\gamma}, \tag{17}$$

$${}_zO^{\alpha}_{\beta}\, {}_zO^{\gamma}_{\alpha} = {}_zO^{\gamma}_{\beta}. \tag{18}$$

4. Elementary cases:

$${}_zO^{\alpha}_{\beta} 1 = 1 \tag{19}$$

$${}_zO^{\alpha}_{\beta} z^n = \frac{(\alpha)_n}{(\beta)_n} z^n \tag{20}$$

$${}_zO^{\alpha}_{\beta}(1-z)^{-\gamma} = {}_2F_1\left[\begin{matrix}\gamma,\alpha; \\ \beta;\end{matrix} z\right]. \tag{21}$$

5. Useful cases:

$${}_zO^{\alpha}_{\beta}[z^{\lambda} f(z)] = \frac{\Gamma(\beta)\Gamma(\beta-\alpha+\theta)}{\Gamma(\alpha)\Gamma(\beta+\lambda)} z^{\lambda}\, {}_zO^{\alpha+\lambda}_{\beta+\lambda}[f(z)] \tag{22}$$

$${}_zO^{\alpha}_{\beta}[(w-z)^{\theta} f(z)]_{w=z} = \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)}{\Gamma(\beta-\alpha)\Gamma(\beta+\theta)} z^{\theta}\, {}_zO^{\alpha}_{\beta+\theta}[f(z)]. \tag{23}$$

It is admirable to mention that the fractional calculus operator ${}_zO^{\alpha}_{\beta}$ has a lot more interesting properties and

applications. Tremblay [33] introduced this operator in order to deal with special functions more accurately and obtained the new relations such as hypergeometric transformations (see also [34]).
For this work, one of the most important property of the operator ${}_zO_\beta^\alpha$ is given by the following relation

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)}\,{}_zO_\beta^{\alpha+\beta}z^\gamma\Big|_{z=1}. \tag{24}$$

This relation shows, in fact, that the so-called beta integral method consists in a fractional derivative evaluated at the point $z=1$.

4 Main results

Theorem 1. *The following transformation holds true*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{\left(\frac{1+a+b-2m}{2}\right)_n 2^n\,n!}\,{}_2F_1\left[\begin{matrix}\gamma-n,\alpha;\\ \beta;\end{matrix}-z\right]\\
&=\frac{2^{a+b-2}\Gamma\left(\frac{1+a+b}{2}\right)}{\sqrt{\pi}\Gamma(a)\Gamma(b)}\left[\sum_{p=0}^{\infty}\frac{\left(\frac{1+a+b}{4}\right)_p\left(\frac{3+a+b}{4}\right)_p}{\left(\frac{1+a+b-2m}{4}\right)_p\left(\frac{3+a+b-2m}{4}\right)_p}\right.\\
&\times\frac{z^{2p}}{\left(\frac{1}{2}\right)_p p!}\sum_{r=0}^{m}\binom{m}{r}\frac{2^r\Gamma\left(\frac{a+2p+r}{2}\right)\Gamma\left(\frac{b+2p+r}{2}\right)}{\left(\frac{1+a+b-2m+4p}{2}\right)_r}\\
&\times\frac{(\alpha)_{2p}}{(\beta)_{2p}}\,{}_2F_1\left[\begin{matrix}\gamma,\alpha;\\ \beta;\end{matrix}-z\right]\\
&+\frac{2\alpha(1+a+b)z}{\beta(1+a+b-2m)}\sum_{p=0}^{\infty}\frac{\left(\frac{3+a+b}{4}\right)_p\left(\frac{5+a+b}{4}\right)_p}{\left(\frac{3+a+b-2m}{4}\right)_p\left(\frac{5+a+b-2m}{4}\right)_p}\\
&\times\frac{z^{2p}}{\left(\frac{3}{2}\right)_p p!}\sum_{r=0}^{m}\binom{m}{r}\frac{2^r\Gamma\left(\frac{a+2p+r+1}{2}\right)\Gamma\left(\frac{b+2p+r+1}{2}\right)}{\left(\frac{3+a+b-2m+4p}{2}\right)_r}\\
&\left.\times\frac{(\alpha+1)_{2p}}{(\beta+1)_{2p}}\,{}_2F_1\left[\begin{matrix}\gamma,\alpha;\\ \beta;\end{matrix}-z\right]\right]\end{aligned} \tag{25}$$

$$\left(a,b,\alpha,\beta,\gamma-n,\frac{1+a+b-2m}{2},\frac{1+a+b}{2}\in\mathbb{C}\setminus\mathbb{Z}_0^-;|z|<1;m,n\in\mathbb{N}_0\right).$$

Proof. On multiplying the equation (8) by $[1-(-z)]^{-\gamma}$ where γ is a complex number, expressing ${}_2F_1$ involved as a series, applying the operator ${}_zO_\beta^\alpha$ and changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the involved series, gives

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{\left(\frac{1+a+b-2m}{2}\right)_n 2^n\,n!}\,{}_zO_\beta^\alpha\{[1-(-z)]^{-\gamma-n}\}\\
&=\frac{2^{a+b-2}\Gamma\left(\frac{1+a+b}{2}\right)}{\sqrt{\pi}\Gamma(a)\Gamma(b)}\left[\sum_{p=0}^{\infty}\frac{\left(\frac{1+a+b}{4}\right)_p\left(\frac{3+a+b}{4}\right)_p}{\left(\frac{1+a+b-2m}{4}\right)_p\left(\frac{3+a+b-2m}{4}\right)_p}\right.\\
&\times\frac{1}{\left(\frac{1}{2}\right)_p p!}\sum_{r=0}^{m}\binom{m}{r}\frac{2^r\Gamma\left(\frac{a+2p+r}{2}\right)\Gamma\left(\frac{b+2p+r}{2}\right)}{\left(\frac{1+a+b-2m+4p}{2}\right)_r}\\
&\times{}_zO_\beta^\alpha\{z^{2p}[1-(-z)]^{-\gamma}\}\\
&+\frac{2(1+a+b)}{(1+a+b-2m)}\sum_{p=0}^{\infty}\frac{\left(\frac{3+a+b}{4}\right)_p\left(\frac{5+a+b}{4}\right)_p}{\left(\frac{3+a+b-2m}{4}\right)_p\left(\frac{5+a+b-2m}{4}\right)_p}\\
&\times\frac{1}{\left(\frac{3}{2}\right)_p p!}\sum_{r=0}^{m}\binom{m}{r}\frac{2^r\Gamma\left(\frac{a+2p+r+1}{2}\right)\Gamma\left(\frac{b+2p+r+1}{2}\right)}{\left(\frac{3+a+b-2m+4p}{2}\right)_r}\\
&\left.\times\,{}_zO_\beta^\alpha\{z^{2p+1}[1-(-z)]^{-\gamma}\}\right],\end{aligned} \tag{26}$$

which on using the following properties of ${}_zO_\beta^\alpha$

$${}_zO_\beta^\alpha\{[1-(-z)]^{-\gamma-n}\}={}_2F_1\left[\begin{matrix}\gamma-n,\alpha;\\ \beta;\end{matrix}-z\right],$$

$${}_zO_\beta^\alpha\{z^{2p}[1-(-z)]^{-\gamma}\}=\frac{(\alpha)_{2p}}{(\beta)_{2p}}z^{2p}\,{}_2F_1\left[\begin{matrix}\gamma,\alpha;\\ \beta;\end{matrix}-z\right]$$

and

$${}_zO_\beta^\alpha\{z^{2p+1}[1-(-z)]^{-\gamma}\}=\frac{(\alpha)_{2p+1}}{(\beta)_{2p+1}}z^{2p+1}\,{}_2F_1\left[\begin{matrix}\gamma,\alpha;\\ \beta;\end{matrix}-z\right],$$

we get our desired result (25).

5 Corollaries and consequences

Corollary 1. *The following transformation holds true*

$$\begin{aligned}{}_2F_1\left[\begin{matrix}a,b;\\ \frac{1+a+b}{2};\end{matrix}\frac{1+z}{2}\right]&=\frac{\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\,{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{b}{2};\\ \frac{1}{2};\end{matrix}z^2\right]\\
&+\frac{2\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}\,{}_2F_1\left[\begin{matrix}\frac{a+1}{2},\frac{b+1}{2};\\ \frac{1}{2};\end{matrix}z^2\right],\end{aligned} \tag{27}$$

$$\left(\frac{a+b+1}{2}\in\mathbb{C}\setminus\mathbb{Z}^-;|z|<1,\text{ or }z\in\{-1,1\},\Re(a+b)<1,\right.$$
$$\left.\text{or }z\in\{-i,i\},\Re(a+b)<3;i=\sqrt{(-1)}\right).$$

Proof. On setting $m=0$, $\alpha=\beta$ and $\gamma=1$ in equation (25), we get Kummer transformation formula (27). We omit the details.

Corollary 2. *The following transformation holds true*

$${}_2F_1\left[\begin{matrix}a,b;\\ \frac{a+b-1}{2};\end{matrix}\frac{1}{2}\right]=\frac{\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}+\frac{2\Gamma\left(\frac{a+b-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}, \tag{28}$$

$$\left(\frac{a+b-1}{2} \in \mathbb{C} \setminus \mathbb{Z}^-\right).$$

Proof. On setting $m = 1$, $z = 0$, $\alpha = \beta$ and $\gamma = 1$ in equation (25), we get Kummer transformation formula (28). We omit the details.

Corollary 3. *The following transformation holds true*

$$ {}_2F_1\left[\begin{matrix} a, b; & \frac{1}{2} \\ \frac{a+b-3}{2}; & 2 \end{matrix}\right] = \frac{\Gamma\left(\frac{a+b-3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2} \times \left\{ \frac{(a+b)^2+4(ab-a-b)+3}{2\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)} + \frac{4\Gamma\left(\frac{a+b-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)} \right\} \quad (29)$$

$$\left(\frac{a+b-3}{2} \in \mathbb{C} \setminus \mathbb{Z}^-\right).$$

Proof. On setting $m = 2$, $z = 0$, $\alpha = \beta$ and $\gamma = 1$ in equation (25), we get Kummer transformation formula (29). We omit the details.

6 Applications

In this section, we get some new identities involving the product of the hypergeometric function as application of our result.

(1) On setting $m = 0$ in our main result (25), we get following interesting result

$$\sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{\left(\frac{1+a+b}{2}\right)_n 2^n n!} {}_2F_1\left[\begin{matrix} \gamma-n, \alpha; & \frac{1+z}{2} \\ \beta; & \end{matrix}\right] = \frac{2^{a+b-2}\Gamma\left(\frac{1+a+b}{2}\right)}{\sqrt{\pi}\Gamma(a)\Gamma(b)} \left\{ \Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right) \times {}_4F_3\left[\begin{matrix} \frac{a}{2}, \frac{b}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}; & z^2 \\ \frac{1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}; & \end{matrix}\right] {}_2F_1\left[\begin{matrix} \gamma, \alpha; & -z \\ \beta; & \end{matrix}\right] + \frac{2\alpha z}{\beta} \Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right) \times {}_4F_3\left[\begin{matrix} \frac{a}{2}, \frac{b}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}; & z^2 \\ \frac{1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}; & \end{matrix}\right] {}_2F_1\left[\begin{matrix} \gamma, \alpha; & -z \\ \beta; & \end{matrix}\right] \right\}. \quad (30)$$

(2) On setting $m = 1$ in our main result (25), we get following interesting result

$$\sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{\left(\frac{1+a+b}{2}\right)_n 2^n n!} {}_2F_1\left[\begin{matrix} \gamma-n, \alpha; & \frac{1+z}{2} \\ \beta; & \end{matrix}\right] = \frac{2^{a+b-2}\Gamma\left(\frac{a+b-1}{2}\right)}{\sqrt{\pi}\Gamma(a)\Gamma(b)} \left\{ \Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right) \times {}_5F_4\left[\begin{matrix} \frac{3+a+b}{4}, \frac{a}{2}, \frac{b}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}; & z^2 \\ \frac{a+b-1}{4}, \frac{1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}; & \end{matrix}\right] {}_2F_1\left[\begin{matrix} \gamma, \alpha; & -z \\ \beta; & \end{matrix}\right] \right.$$
$$+ \frac{2\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\left(\frac{a+b-1}{4}\right)} \times {}_5F_4\left[\begin{matrix} \frac{3+a+b}{4}, \frac{a+1}{2}, \frac{b+1}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2}; & z^2 \\ \frac{a+b+1}{4}, \frac{1}{2}, \frac{\beta}{2}, \frac{\beta+1}{2}; & \end{matrix}\right] {}_2F_1\left[\begin{matrix} \gamma, \alpha; & -z \\ \beta; & \end{matrix}\right]$$
$$+ \frac{2\alpha z(a+b+1)\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\beta(a+b-1)} \times {}_5F_4\left[\begin{matrix} \frac{5+a+b}{4}, \frac{a+1}{2}, \frac{b+1}{2}, \frac{\alpha+1}{2}, \frac{\alpha+2}{2}; & z^2 \\ \frac{a+b+1}{4}, \frac{3}{2}, \frac{\beta+1}{2}, \frac{\beta+2}{2}; & \end{matrix}\right] {}_2F_1\left[\begin{matrix} \gamma, \alpha; & -z \\ \beta; & \end{matrix}\right]$$
$$+ \frac{4\alpha z(a+b+1)\Gamma\left(\frac{a+2}{2}\right)\Gamma\left(\frac{b+2}{2}\right)}{\left(\frac{a+b-1}{4}\right)\beta(a+b-1)} \times {}_6F_5\left[\begin{matrix} \frac{5+a+b}{4}, \frac{a+b-1}{4}, \frac{a+2}{2}, \frac{b+2}{2}, \frac{\alpha+1}{2}, \frac{\alpha+2}{2}; & z^2 \\ \frac{a+b+1}{4}, \frac{a+b+1}{4}, \frac{3}{2}, \frac{\beta+1}{2}, \frac{\beta+2}{2}; & \end{matrix}\right] \left. \times {}_2F_1\left[\begin{matrix} \gamma, \alpha; & -z \\ \beta; & \end{matrix}\right] \right\}. \quad (31)$$

7 Conclusion and observation:

We have established a new result on identity involving Gauss function using fractional calculus technique. The results as application of our main result are simple but interesting. Both two identities in applications are derived just setting $m = 0$, and $m = 1$ in (25). For different values of m, one can get another different new identities involving Gauss function.

Acknowledgement

The author thanks the constructive comments and suggestions by editor and anonymous referees, which have contributed to the improvement of the presentation of this paper.
The author is also thankful to Saudi Electronic University for providing financial support.

References

[1] E.D. Rainville, *Special functions*, The Macmillan Co. Inc., New York 1960. Reprinted by Chelsea Publ. Co. Bronx, New York, 1971.

[2] H.M. Srivastava and H.L. Manocha; *A treatise on generating functions*, Halsted Press (Ellis Horwood limited), New York, 1984.
[3] W.N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935; Reprinted by Stechert Hafner, New York, 1964.
[4] Y.S. Kim, M.A. Rakha and A.K. Rathie, Extensions of certain classical summation theorems for the series $_2F_1$, $_3F_2$, and $_4F_3$ with applications in Ramanujan's summations, *International Journal of Mathematics and Mathematical Sciences*, **2010** (2010), 26 pages.
[5] M.A. Rakha and A.K. Rathie, Generalization of classical summation theorems for the series $_2F_1$, and $_3F_2$ with applications , *Integral Transform and Special Functions* , **22(11)** (2011), 823-840.
[6] W.N. Bailey, Products of generalized hypergeometric series, *Proc. London Math. Soc.*, **28(1)** (1928), 242-250.
[7] E.E. Kummer, "Uber die hypergeometriche Reihe", *Journal fur Die Reine Und Angewandte Mathematik*, **15** (1836), 39-83.
[8] B.C. Berndt, *Ramanujan's Notebook II*, Springer-Verleg, New York, 1989.
[9] A. Erdélyi and W. Magnus, Oberhettinger F. and Tricomi F.G., *Higher Transcendal Functions-I*, McGraw-Hill, New York, 1953.
[10] A.K. Rathie and V. Nagar, On Kummer's second theorem involving product of generalized hypergeometric series, *Le Matematiche*, **50(1)** (1995), 35-38.
[11] A.K. Rathie and T.K. Pogany, New summation formula for $_3F_2(1/2)$ and Kummer type second transformation of $_2F_2(x)$, *Mathematical Communications*, **13(1)** (2008), 63-66.
[12] M.I. Qureshi and M.S. Baboo, A hypergeometric transormation for Gauss function $_2F_1[a,b;\frac{1+a+b-2m}{2};\frac{1+z}{2}]$ and its application in Clausen's function, *Math. Sci. Lett.*, **7(1)** (2018), 7-11.
[13] M. Riesz, L'intégrale de Riemann-Liouville et le problème de Cauchy, *Acta Math.* **81** (1949), 1-223.
[14] A. Erdélyi, An integral equation involving Legendre polynomials, *SIAM J. Appl. Math.*, **12** (1964), 15-30.
[15] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies (Jan van Mill ed.), Vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.
[16] J. Liouville, Mémoire sur le calcul des différentielles á indices quelconques, *J. de l Ecole Polytechnique*, **13** (1832), 71-162.
[17] T.J. Osler, The fractional derivative of a composite function, *SIAM J. Math. Anal.*, **1** (1970), 288-293.
[18] T.J. Osler, *Leibniz rule, the chain rule and Taylor's theorem for fractional derivatives*, Ph.D. thesis, New York University, 1970.
[19] T.J. Osler, Fractional derivatives and Leibniz rule, *Amer. Math. Monthly*, **78** (1971), 645-649.
[20] T.J. Osler, Leibniz rule for the fractional derivatives and an application to infinite series, *SIAM J. Appl. Math.*, **18** (1970), 658-674.
[21] J.-L. Lavoie, T.J. Osler and R. Tremblay, *Fundamental properties of fractional derivatives via Pochhammer integrals*, Lecture Notes in Mathematics, Springer-Verlag, 1976.
[22] S. Gaboury and A.K. Rathie, Further hypergeometric identities deducible by fractional calculus, *Commun. Korean Math. Soc.*, **29(3)** (2014), 429-437.
[23] A.R. Miller and K.S. Ross, *An Introduction of the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[24] J. Choi, A.K. Rathie and H.M. Srivastava, Certain hypergeometric identities deducible by using the beta integral method, *Bull. Korean Math. Soc.*, **50** (2013), 1673-1681.
[25] A.K. Ibrahim, M.A.Rakha and A.K. Rathie, On certain hypergeometric identities deducible by using the beta integral method, *Adv. Diff. Equ.*, **361** (2013), 1-8.
[26] C. Krattenthaler and K.S. Rao, Automatic generation of hypergeometric identities by the beta integral method, *J. Comput. Appl. Math.*, **160** (2003), 159-173.
[27] C. Wei, X. Wang and Y. Li, Certain transformations for multiple hypergeometric functions, *Adv. Diff. Equ.*, **360** (2013), 1-13.
[28] S. Oweyd, M.A. Abdou, A. Abdel-Aty and H. Dutta, Optical solitons solutions for perturbed time fractional nonlinear Schrodinger equation via two strategic algorithms, *AIMS Math.*, **5(3)** (2020), 2057-2070.
[29] S. Oweyd, M.A. Abdou, A. Abdel-Aty, W. Alharbi and R. Nekhili, Numerical and approximate solutions for coupled time fractional nonlinear evolutions equations via reduced differential transform method, *Chaos, Solitons and Fractals*, **131** (2020), Article Id. 109474.
[30] S. Oweyd, M.A. Abdou, A. Abdel-Aty and S.S. Ray, New Optical Soliton Solutions of Nolinear Evolution Equation Describing Nonlinear Dispersion, *Commun. Theor. Phys.*, **71(9)** (2019), 1063-1068.
[31] S. Oweyd, M.A. Abdou, A. Abdel-Aty, A.A. Ibrahim, R. Nekhili and D. Baleanu, New optical soliton solutions of space-time fractional nonlinear dynamics of microtubules via three integration schemes, *J. Intel. Fuzzy Syst.*, (2019), 1-8.
[32] G.M. Ismail, H.R. Abdl-Rahim, A. Abdel-Aty, R. Kharabsheh, W. Akharbi and M. Abdel-Aty, An analytical solution for fractional oscillator in a resisting medium, *Chaos, Solitons and Fractals*, **130** (2020), Article Id. 109395.
[33] R. Tremblay, *Une contribution à la théorie de la dérivé frationnaire*, Ph.D. thesis., Laval University, Canada, 1974.
[34] Y.S. Kim, New Transformations for Hypergeometric Functions Deducible by Fractional Calculus, *Bull. Korean Math. Soc.*, **33(4)** (2018), 1239-1248.

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