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An Inequality Related to s-φ-Convex Functions.

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Abstract: Using the notion of s-φ-convex functions as generalization of convex functions, we estimate the difference between the middle and right terms in Hermite- Hadamard-Fejer inequality for differentiable mappings.

Keywords: $\varphi-$convex function, $s-\varphi-$convex function, $s-$convex function, Hermite-Hadamard type inequalities.

1 Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. We note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (1) we refer the reader to the recent papers [1, 2,3,4,5,6,7].

The convex functions play a significant role in many fields, for example in biological system, economy, optimization and so on [8,9]. And many important inequalities are established for these class of functions. Also the evolution of the concept of convexity has had a great impact in the community of investigators. In recent years, for example, generalized concepts such as s-convexity (see[10]), h-convexity (see [11,12]), m-convexity (see [1,13]), MT- convexity (see[14]) and others, as well as combinations of these new concepts have been introduced.

The role of convex sets, convex functions and their generalizations are important in applied mathematics specially in nonlinear programming and optimization theory. For example in economics, convexity plays a fundamental role in equilibrium and duality theory. The convexity of sets and functions have been the object of many studies in recent years. But in many new problems encountered in applied mathematics the notion of convexity is not enough to reach favorite results and hence it is necessary to extend the notion of convexity to the new generalized notions. Recently, several extensions have been considered for the classical convex functions such that some of these new concepts are based on extension of the domain of a convex function (a convex set) to a generalized form and some of them are new definitions that there is no generalization on the domain but on the form of the definition. Some new generalized concepts in this point of view are pseudo-convex functions [15], quasi-convex functions [16], invex functions [17], preinvex functions and Hermite-Hadamard-Fėjer type inequalities for strongly (s,m)-convex functions with modulus c, in the second sense [2].

Hermite-Hadamard-Fėjer inequality, an interesting result related to convex functions has been proved in [18] as the following:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{2}$$

where $g : [a,b] \to \mathbb{R}^+ = [0,+\infty)$ is integrable and symmetric about $x = \frac{a+b}{2}$.

If in (2) we consider $g \equiv 1$ then we obtain Hermite-Hadamard inequality (1).

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On the other hand Gordji, Dragomir and Delavar in [19] proved an interesting result related to φ-convex function as the following:

Theorem 2. *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable function, $g:[a,b]\to\mathbb{R}^+$ is a continuous function and symmetric about $\frac{a+b}{2}$ and $|f'|$ is an φ-convex function where φ is bounded from above on $[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right|$$
$$\leq \frac{b-a}{4}\left[2|f'(b)|+|\varphi(f'(a),f'(b))|\right]\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)dudt,$$

whose demonstration is based on the following Lemma that is also very useful for us.

Lemma 1. *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable function, $g:[a,b]\to\mathbb{R}^+$ is a continuous function and symmetric about $\frac{a+b}{2}$ and f' is an integrable function on $[a,b]$. Then*

$$\frac{f(a)+f(b)}{2}\int_a^b g(x)dx-\int_a^b f(x)g(x)dx=$$
$$\frac{b-a}{4}\left\{\int_0^1\left(\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du\right)f'(\frac{1+t}{2}a+\frac{1-t}{2}b)dt\right.$$
$$\left.+\int_0^1\left(\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du\right)f'(\frac{1-t}{2}a+\frac{1+t}{2}b)dt\right\}.$$

Motivated by these studies and following the definition s-φ-convex functions introduced in [20] we would like to give a generalization to Theorem 2. For this, we remember the definition of s-φ-convex functions.

Definition 1. *Let $0<s\leq 1$. A function $f:I\subset\mathbb{R}\to\mathbb{R}$ is called $s-\varphi-$convex with respect to bifunction $\varphi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ (briefly $\varphi-$convex), if*

$$f(tx+(1-t)y)\leq f(y)+t^s\varphi(f(x),f(y)) \tag{3}$$

for all $x,y\in I$ and $t\in[0,1]$.

Example 1. Let $f(x)=x^2$, then f is convex and $\frac{1}{2}$-φ-convex with $\varphi(u,v)=2u+v$.

Example 2. Let $f(x)=x^n$ and $0<s\leq 1$, then f is convex and s-φ-convex with

$$\varphi(u,v)=\sum_{k=1}^{n}\binom{n}{k}v^{1-\frac{k}{n}}\left(u^{\frac{1}{n}}-v^{\frac{1}{n}}\right)^n.$$

Remark. In [20] Proposition 4 we have that if f is s-φ-convex and φ bounded from above on $f([a,b])\times f([a,b])$, then f is bounded and therefore integrable on $f([a,b])\times f([a,b])$.

With this definition we will present in section 2 the main results of this article, in section 3 we will give some applications related to the estimation of the error for the midpoint formula, and in section 4 we will give some applications to special means.

2 Main results

Based on Lemma 1 we obtain the main theorem.

Theorem 3. *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable function, $g:[a,b]\to\mathbb{R}^+$ is a continuous function and symmetric about $\frac{a+b}{2}$ and $|f'|$ is an s-φ-convex function where φ is bounded from above on $[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right|$$
$$\leq\frac{b-a}{4}\left[2|f'(b)|+K\varphi(|f'(a)|,|f'(b)|)\right]\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)dudt.$$

where $K=\max_{t\in[0,1]}|h(t)|$ and $h(t)=\left(\frac{1+t}{2}\right)^s+\left(\frac{1-t}{2}\right)^s$.

Proof. From Lemma 1 and the fact that $|f'|$ is s-φ-convex where φ is bounded above we have

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right|$$
$$\leq\frac{b-a}{4}\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)\left[|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)|\right.$$
$$\left.+|f'(\frac{1-t}{2}a+\frac{1+t}{2}b)|\right]dudt$$
$$\leq\frac{b-a}{4}\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)\left[2|f'(b)|+\left(\frac{1+t}{2}\right)^s\varphi(|f'(a)|,|f'(b)|)\right.$$
$$\left.+\left(\frac{1-t}{2}\right)^s\varphi(|f'(a)|,|f'(b)|)\right]dudt$$
$$=\frac{b-a}{4}\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)\left(2|f'(b)|+\right.$$
$$\left.\left[\left(\frac{1+t}{2}\right)^s+\left(\frac{1-t}{2}\right)^s\right]\varphi(|f'(a)|,|f'(b)|)\right)dudt$$
$$=\frac{b-a}{4}\left(2|f'(b)|+\left[\left(\frac{1+t}{2}\right)^s+\left(\frac{1-t}{2}\right)^s\right]\varphi(|f'(a)|,|f'(b)|)\right)$$
$$\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)dudt$$
$$\leq\frac{b-a}{4}\left[2|f'(b)|+K\varphi(|f'(a)|,|f'(b)|)\right]\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)dudt,$$

where $K=\max_{t\in[0,1]}|h(t)|$ and $h(t)=\left(\frac{1+t}{2}\right)^s+\left(\frac{1-t}{2}\right)^s$. This completes the proof.

Corollary 1. *In Theorem 3 if we choose $s=1$, then $K=1$ and we have the inequality of the Theorem 2 (Gordji, Dragomir and Delavar).*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right|\leq\frac{b-a}{4}$$
$$\left[2|f'(b)|+|\varphi(f'(a),f'(b))|\right]\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)dudt$$

Corollary 2. *In Theorem 3 if we choose $g=1$ and $\varphi(x,y)=x-y$, we have the following inequality for s-convex functions in the first sense*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{8}\left[(2-K)\left|f'(b)\right|+K|f'(a)|\right],$$

where $K=\max_{t\in[0,1]}|h(t)|$ and $h(t)=\left(\frac{1+t}{2}\right)^s+\left(\frac{1-t}{2}\right)^s$.

Corollary 3. *In Corollary 2 if we choose $s = 1$, we have the following inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[|f'(b)| + |f'(a)|\right],$$

for convex functions that is equivalent to Theorem 1.2 in [19]

3 Applications

We give an error estimate for mid point formula that is generalization of Theorema 2.6 in [19].
Suppose that d is a partition $a = x_0 < x_1 < \ldots < x_n = b$ of interval $[a,b]$. Consider formula

$$\int_a^b f(x)g(x)dx = T(f,g,d) + E(f,g,d),$$

where

$$T(f,g,d) = \sum_{i=0}^{n-1} \frac{f(x_i)+f(x_{i+1})}{2}\int_{x_i}^{x_{i+1}} g(x)dx,$$

and $E(f,g,d)$ is the approximation error.

Theorem 4. *Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable function, $g : [a,b] \to \mathbb{R}^+$ is a continuous function and symmetric with respect to $\frac{a+b}{2}$ and $|f'|$ is an s-φ-convex function where φ is bounded from above on $[a,b]$. Then*

$$|E(f,g,d)| \leq \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)}{4}\Big[2|f'(x_{i+1})|$$
$$+ K\varphi(|f'(x_i)|, |f'(x_{i+1})|)\Big]\int_0^1 \int_{\frac{1+t}{2}x_i+\frac{1-t}{2}x_{i+1}}^{\frac{1-t}{2}x_i+\frac{1+t}{2}x_{i+1}} g(x)dxdt.$$

Proof. It is enough to apply Theorem 3 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the partition d for interval $[a,b]$, and to sum all achieved inequalities over i and then using triangle inequality.

4 Applications to special means

We now consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$. We take:

(1) Arithmetic mean:

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}^+;$$

(2) Generalized log-mean:

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} - \{-1,0\}, \alpha,\beta \in \mathbb{R}^+.$$

Proposition 1. *Let $0 < a < b$ and $s \in (0,1)$. Then we have*

$$|A^s(a,b) - L_s^s(a,b)| \leq \frac{s(s-1)(b-a)^2}{16(s+3)}\left[\frac{s}{3}a^{s-2}\right.$$
$$\left. + \frac{2}{(s+1)(s+2)}b^{s-2} + \frac{(s+1)(s+2)(s+6)-6}{3(s+1)(s+2)}\left(\frac{a+b}{2}\right)^{s-2}\right].$$

Proof. The assertion follows from Corollary 2 applied to the s-φ-convex function in the first sense
$f : [0,1] \to [0,1]$, $f(x) = x^s$ with $\varphi(x,y) = x - y$.

Proposition 2. *Let $0 < a < b$ and $s \in (0,1)$. Then we have*

$$|A^s(a,b) - L_s^s(a,b)| \leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$
$$\left[\left(s^2(s-1)a^{q(s-2)} + s(s-1)A^{q(s-2)}(a,b)\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(s(s-1)A^{q(s-2)}(a,b) + s(s-1)b^{q(s-2)}\right)^{\frac{1}{q}}\right].$$

Proof. The assertion follows from Corollary 2 applied to the s-φ-convex function in the first sense
$f : [0,1] \to [0,1]$, $f(x) = x^s$ with $\varphi(x,y) = x - y$.

Proposition 3. *Let $0 < a < b$ and $s \in (0,1)$. Then we have*

$$|A^s(a,b) - L_s^s(a,b)| \leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left[\left(\frac{s^2(s-1)}{3(s+3)}a^{q(s-2)}\right.\right.$$
$$\left. + \frac{s(s-1)}{s+3}A^{q(s-2)}(a,b)\right)^{\frac{1}{q}} + \left(\frac{[(s+1)(s+2)(s+3)-6]s(s-1)}{3(s+1)(s+2)(s+3)}A^{q(s-2)}(a,b)\right.$$
$$\left.\left. + \frac{2s(s-1)}{(s+1)(s+2)(s+3)}b^{q(s-2)}\right)^{\frac{1}{q}}\right].$$

Proof. The assertion follows from Corollary 2 applied to the s-φ-convex function in the first sense
$f : [0,1] \to [0,1]$, $f(x) = x^s$ with $\varphi(x,y) = x - y$.

5 Conclusion

In this paper we have established Hermite-Hadamard type inequalities given by Erdem-Ogunmez-Budak in [3] for the case s-φ-convex functions. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly-introduced functions in various fields of pure and applied sciences,for example: quantum calculus, integral equations among others.

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