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Higher-Order Strongly-Generalized Convex Functions

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Abstract: In this paper, we define and introduce some new concepts of the higher order strongly-generalized convex functions involving an arbitrary function. Some properties of the higher order strongly-generalized convex functions are investigated under suitable conditions. We have proved that the optimality conditions of higher order strongly generalized can be characterized by a class of variational inequalities, which is called higher-order strongly variational inequality. It is shown that the parallelogram laws for Banach spaces can be obtained as applications of higher-order strongly-generalized affine convex functions. Results obtained in this paper can be viewed as refinement and improvement of previously-known results.

Keywords: Convex functions, monotone operators, higher-order convex functions, strongly convex functions

1 Introduction

Strongly convex functions were introduced and studied by Polyak [1], which play an important part in the optimization theory and related areas. Applications of the strongly convex functions in complementarity problems, characterization of the inner product, exponential stability of primal-dual gradient dynamics are shown in [2,3,4,5]. Awan et al[8] have derived Hermite-Hadamard type inequalities for various classes of strongly convex functions, which provide upper and lower estimate for the integrand. For more applications and properties of the strongly convex functions, see [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16] and the references therein.

Lin and Fukushima [17] introduced the concept of higher-order strongly convex functions and studied its applications in mathematical program with equilibrium constraints. Mishra and Sharma [18] derived the Hermite-Hadamard type inequalities for higher-order strongly convex functions.

To be more precise, a function F on the closed convex set K is said to be a higher-order strongly convex [17], if there exists a constant $\mu \geq 0$, such that

$$F((u+t(v-u)) \leq (1-t)F(u)+tF(v) -\mu\varphi(t)\|v-u\|^{p}, \forall u,v \in K, t \in [0.1], p \geq 1, \quad (1)$$

where

$$\varphi(t) = t(1-t). \quad (2)$$

If $p = 2$, then higher-order strongly convex functions become strongly convex functions with the same $\varphi(t)$ as defined in (2).

We have noticed that the function $\varphi(.)$ in (2) is not correct and must be modified. Characterizations of the higher-order strongly convex functions discussed in Lin and Fukushima [17] are not correct. These facts and observations motivated Mohsen et al [11] to consider higher-order strongly convex function by modifying the function $\varphi(.)$. Continuing the same ideas, we introduce a new class of higher-order strongly convex functions involving an arbitrary bifunction. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. The minimum of the differentiable higher-order strongly-generalized convex functions is characterized by a class of variational inequalities, which is called higher-order strongly variational inequality. Several special cases are discussed as applications of new results. Higher-order strongly convex functions can be used to characterize the uniformly reflex Banach spaces. We have also deduced the weakly parallelogram laws for the L^p-spaces, which were discussed in [19,20,21,22,23] from the concept of higher-order strongly-generalized affine convex functions. This result can be viewed itself an elegant and interesting applications of the higher-order strongly-generalized convex functions.

In this paper, we have introduced and investigated some

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properties of the higher-order strongly-generalized convex functions along with their applications. A new class of higher-order variational inequalities is introduced, which is itself an open problem. The interested researchers may develop some numerical methods for solving these variational inequalities.

2 Formulations and basic facts

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively.

Definition 1.[11,19] *A set K in H is said to be a convex set, if*

$$u+t(v-u)\in K, \qquad \forall u,v\in K, t\in[0,1].$$

We now introduce some new classes of higher-order strongly convex functions and higher-order strongly-generalized affine convex functions.

Definition 2. *A function F on the convex set K is said to be higher-order strongly-generalized convex with respect to the bifunction $\xi(.,.)$, if there exists a constant $\mu>0$, such that*

$$F(u+t(v-u))\leq(1-t)F(u)+tF(v) \\ -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p, \forall u,v\in K, t\in[0,1], p\geq 1. \qquad (3)$$

A function F is said to higher-order strongly-generalized concave with respect to the bifucntion $\xi(.-.)$, if and only if, $-F$ is higher-order strongly-generalized convex function with respect to the bifucntion $\xi(.-.)$.
If $t=\frac{1}{2}$, then

$$F\left(\frac{u+v}{2}\right)\leq\frac{F(u)+F(v)}{2}-\mu\frac{1}{2^p}\|\xi(v-u)\|^p, \forall u,v\in K, p\geq 1. \qquad (4)$$

The function F is said to be higher-order strongly J-convex function.

Definition 3. *A function F on the convex set K is said to be a higher-order strongly-generalized affine convex function with respect to the bifucntion $\xi(.-.)$, if there exists a constant $\mu>0$, such that*

$$F(u+t(v-u))=(1-t)F(u)+tF(v) \\ -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^2, \forall u,v\in K, t\in[0,1], p\geq 1. \qquad (5)$$

Note that if a function is both higher-order strongly-generalized convex and higher-order strongly=generalized concave, then it is higher-order strongly-generalized affine convex function.

A function F is called higher-order strongly-generalized quadratic equation with respect to the bifucntion $\xi(.-.)$, if there exists a constant $\mu>0$, such that

$$F\left(\frac{u+v}{2}\right)=\frac{F(u)+F(v)}{2} \\ -\mu\frac{1}{2^p}\|\xi(v-u)\|^p, \forall u,v\in K, t\in[0,1]. \qquad (6)$$

This function F is also called higher-order strongly-generalized affine J-convex function.
We now discuss some special cases.
I. If $p=2$, then the higher-order strongly-generalized convex function becomes strongly convex functions, that is,

$$F(u+t(v-u))\leq(1-t)F(u)+tF(v) \\ -\mu t(1-t)\|\xi(v-u)\|^2, \forall u,v\in K, t\in[0,1].$$

Such type of strongly convex functions were studied by Adamek [6]. For the properties of the strongly convex functions in variational inequalities and equilibrium problems, see Noor [13] and the references therein.

Definition 4. *A function F on the convex set K is said to be higher-order strongly-generalized quasi convex with respect to the bifucntion $\xi(.-.)$, if there exists a constant $\mu>0$ such that*

$$F(u+t(v-u)\leq\max\{F(u),F(v)\} \\ -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p, \\ \forall u,v\in K, t\in[0,1].$$

Definition 5. *A function F on the convex set K is said to be higher-order strongly-generalized log-convex with respect to the bifunction $\xi(.-.)$, if there exists a constant $\mu>0$ such that*

$$F(u+t(v-u))\leq(F(u))^{1-t}(F(v))^t \\ -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p, \\ \forall u,v\in K, t\in[0,1],$$

where $F(\cdot)>0$.

From the above definitions, we have

$$\begin{aligned} F(u+t(v-u)) &\leq (F(u))^{1-t},(F(v))^t \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p \\ &\leq (1-t)F(u)+tF(v) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p \\ &\leq \max\{F(u),F(v)\} \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p. p\geq 1. \end{aligned}$$

This shows that every higher-order strongly-generalized log-convex function is a higher-order strongly-generalized convex function and every higher-order strongly-generalized convex function is a higher-order strongly-generalized quasi-convex function. However, the converse is not true.
For appropriate and suitable choice of the arbitrary bifuction $\xi(v-u)$ one can obtain several new and known classes of strongly convex functions and their variant forms as special cases of higher-order generalized-strongly convex functions. This shows that the class of higher-order generalized-strongly convex functions is quite broad and a unifying one.

Definition 6. *An operator $T:K\rightarrow H$ is said to be:*

1. *higher-order strongly monotone, if and only if, there exist an arbitrary bifunction $\xi(.,.)$ and a constant $\alpha > 0$ such that*

$$\langle Tu - Tv, u - v\rangle \geq \alpha\{\|\xi(v-u)\|^p + \|\xi(u-v)\|^p\}, \forall u, v \in K.$$

2. *higher-order strongly pseudomonotone, if and only if, there exist an arbitrary bifunction $\xi(.,.)$ and a constant $\nu > 0$, such that*

$$\langle Tu, v-u\rangle + \nu\|\xi(v-u)\|^p \geq 0 \Rightarrow \langle Tv, v-u\rangle \geq 0, \forall u, v \in K.$$

3. *higher-order strongly relaxed pseudomonotone, if and only if, there exist an arbitrary bifunction $\xi(.,.)$ and constant $\mu > 0$, such that*

$$\langle Tu, v-u\rangle \geq 0 \Rightarrow -\langle Tv, u-v\rangle + \mu\|\xi(v-u)\|^p \geq 0, \forall u, v \in K.$$

Definition 7. *A differentiable function F on the convex set K is said to be higher-order strongly pseudo convex function with respect to the bifucntion $\xi(.-.)$, if and only if, if there exists a constant $\mu > 0$, such that*

$$\langle F'(u), v-u\rangle + \mu\|\xi(v-u)\|^p \geq 0 \Rightarrow F(v) \geq F(u), \forall u, v \in K.$$

3 Main results

In this section, we consider some basic properties of higher-order strongly-generalized convex functions.

Theorem 1. *Let F be a differentiable higher-order strongly-generalized convex function on the convex set K and let the bifunction $\xi(.,.)$ be homogeneous. Then the function F is higher-order strongly-generalized convex function, if and only if,*

$$F(v) - F(u) \geq \langle F'(u), v-u\rangle + \mu\|\xi(v-u)\|^p, \forall v, u \in K. \quad (7)$$

Proof. Let F be a higher-order strongly-generalized convex function on the convex set K. Then

$$F(u+t(v-u)) \leq (1-t)F(u) + tF(v) - \mu\{t^p(1-t) + t(1-t)^p\}\|\xi(v-u)\|^p, \forall u, v \in K,$$

which can be written as

$$F(v) - F(u) \geq \{\frac{F(u+t(v-u)-F(u)}{t}\} + \{t^{p-1}(1-t) + (1-t)^p\}\|\xi(v-u)\|^p.$$

Taking the limit in the above inequality as $t \to 0$,, we have

$$F(v) - F(u) \geq \langle F'(u), v-u)\rangle + \mu\|v-u\|^p, \forall u, v \in K.$$

which is (7), the required result.

Conversely, let (7) hold. Then, $\forall u, v \in K, t \in [0,1]$,

$$v_t = u + t(v-u) \in K.$$

Since the function $\xi(.,.)$ is homogeneous, we have

$$F(v) - F(v_t) \geq \langle F'(v_t), v - v_t)\rangle + \mu\|\xi(v-v_t)\|^p = (1-t)F'(v_t), v-u\rangle + \mu(1-t)^p\|\xi(v-u)\|^p. \quad (8)$$

In a similar way, we have

$$F(u) - F(v_t) \geq \langle F'(v_t), u - v_t)\rangle + \mu\|\xi(u-v_t)\|^p = -tF'(v_t), v-u\rangle + \mu t^p\|\xi(v-u)\|^p. \quad (9)$$

Multiplying (8) by t and (9) by $(1-t)$ and adding the resultant, we have

$$F(u+t\eta(v,u)) \leq (1-t)F(u) + tF(v) - \mu\{t^p(1-t) + t(1-t)^p\}\|\xi(v-u)\|^p, \forall u, v \in K, t \in [0,1],$$

showing that F is a higher-order strongly-generalized convex function.

Theorem 2. *Let F be differentiable higher-order strongly-generalized convex function on the convex set K. Then $F'(.)$ is a higher-order strongly monotone operator.*

Proof. Let F be a higher-order strongly-generalized convex function on the convex set K. Then, from Theorem 1. we have

$$F(v) - F(u) \geq \langle F'(u), v-u\rangle + \mu\|\xi(v-u)\|^p, \quad \forall u, v \in K. \quad (10)$$

Changing the role of u and v in (10), we have

$$F(u) - F(v) \geq \langle F'(v), u-v)\rangle + \mu\|\xi(u-v)\|^p, \forall u, v \in K. \quad (11)$$

Adding (10) and (11), we have

$$\langle F'(u) - F'(v), u-v\rangle \geq \mu\{\|v-u\|^p + \|\xi(u-v)\|^p\}, \forall u, v \in K, \quad (12)$$

which shows that $F'(.)$ is a higher-order strongly monotone operator.

We remark that the converse of Theorem 2 is not true. In this direction, we have the following result.

Theorem 3. *Let the differential operator $F'(.)$ of a differentiable higher-order strongly-generalized convex function F be a higher-order strongly monotone operator. If the bifunction $\xi(.-.)$ be symmetric and homogeneous, then*

$$F(v) - F(u) \geq \langle F'(u), v-u\rangle + 2\mu\frac{1}{p}\|v-u\|^p, \forall u, v \in K. \quad (13)$$

Proof. Let $F'(.)$ be a higher order strongly monotone operator. Then, from (12), we have

$$\langle F'(v), u-v\rangle \geq \langle F'(u), u-v)\rangle + \mu\{\|\xi(v-u)\|^p + \|\xi(u-v)\|^p\}, \forall u, v \in K. \quad (14)$$

Since K is an convex set, $\forall u, v \in K$, $t \in [0,1]$, $v_t = u + t(v-u) \in K$. Taking $v = v_t$ in (14). Since the bifunction $\xi(.,.)$ is symmetric and homogeneous, we have

$$\langle F'(v_t), u-v_t\rangle \leq \langle F'(u), u-v_t\rangle - 2\mu\|\xi(v-u)\|^p$$
$$= -t\langle F'(u), v-u\rangle - 2\mu t^p\|\xi(v-u)\|^p,$$

which implies that

$$\langle F'(v_t), v-u\rangle \geq \langle F'(u), v-u\rangle + 2\mu t^{p-1}\|\xi(v-u)\|^p. \quad (15)$$

Consider the auxiliary function

$$g(t) = F(u+t(v-u)), \forall u, v \in K,$$

from which, we have

$$g(1) = F(v), \quad g(0) = F(u).$$

Then, from (15), we have

$$g'(t) = \langle F'(v_t, v-u\rangle \geq \langle F'(u), v-u\rangle + 2\mu t^{p-1}\|\xi(v-u)\|^p. (16)$$

Integrating (16) between 0 and 1, we have

$$g(1) - g(0\ ) = \int_0^1 g'(t)dt$$
$$\geq \langle F'(u), v-u\rangle + 2\mu\frac{1}{p}\|\xi(v-u)\|^p.$$

Thus it follows that

$$F(v) - F(u) \geq \langle F'(u), v-u\rangle + 2\mu\frac{1}{p}\|\xi(v-u)\|^p,$$

which is the required (13).

We note that, if $p=2$, then Theorem 3 can be viewed as the converse of Theorem 2.

We now give a necessary condition for higher-order strongly-generalized pseudo-convex function.

Theorem 4. *Let $F'(.)$ be a higher-order strongly relaxed pseudomonotone operator. If the bifunction $\xi(.,.)$ is homogeneous, then F is a higher-order strongly-generalized pseudo-connvex function.*

Proof. Let $F'(.)$ be a higher-order strongly relaxed pseudomonotone operator. Then, $\forall u, v \in K$,

$$\langle F'(u), v-u\rangle \geq 0.$$

implies that

$$\langle F'(v), v-u\rangle \geq \mu\|\xi(v-u)\|^p, \forall u, v \in K. \quad (17)$$

Since K is a convex set, $\forall u, v \in K, \quad t \in [0,1]$,
$v_t = u + t(v-u) \in K$.
Taking $v = v_t$ in (17), we have

$$\langle F'(v_t), v-u\rangle \geq \mu t^{p-1}\|\xi(v-u)\|^p. \quad (18)$$

Consider the auxiliary function

$$g(t) = F(u+t(v-u)) = F(v_t), \quad \forall u, v \in K, t \in [0,1].$$

Then, using (18), we have

$$g'(t) = \langle F'(v_t), v-u)\rangle \geq \mu t^{p-1}\|\xi(v-u)\|^p.$$

Integrating the above relation between 0 to 1, we have

$$g(1) - g(0) = \int_0^1 g'(t)dt \geq \frac{\mu}{p}\|\xi(v-u)\|^p,$$

that is,

$$F(v) - F(u) \geq \frac{\mu}{p}\||\xi(v-u)\|^p), \forall u, v \in K,$$

showing that F is a higher-order strongly-generalized pseudo-convex function.

Definition 8. *A function F is said to be sharply higher-order strongly-generalized pseudo convex with respect to the bifucntion $\xi(.-.)$, if there exists a constant $\mu > 0$, such that*

$$\langle F'(u), v-u\rangle \geq 0$$
$$\Rightarrow$$
$$F(v) \geq F(v+t(u-v))$$
$$+\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p, \forall u, v \in K, p \geq 1.$$

Theorem 5. *Let F be a sharply higher-order strongly-generalized pseudo convex function on K with a constant $\mu > 0$. Then*

$$\langle F'(v), v-u\rangle \geq \mu\|\xi(v-u)\|^p, \forall u, v \in K.$$

Proof. Let F be a sharply higher-order strongly-generalized pseudo convex function on K. Then

$$F(v) \geq F(v+t(u-v))$$
$$+\ \mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p, \forall u, v \in K, t \in [0,1],$$

from which, we have

$$\{\frac{F(v+t(u-v)) - F(v)}{t}\} + \mu\{t^{p-1}(1-t) + (1-t)^p\}\|\xi(v-u)\|^p \geq 0.$$

Taking limit in the above inequality, as $t \to 0$, we have

$$\langle F'(v), v-u\rangle \geq \mu\|\xi(v-u)\|^p, \forall u, v \in K,$$

which is the required result.

Definition 9. *A function F is said to be a pseudo convex function, if there exists a strictly positive bifunction $B(.,.)$, such that*

$$F(v) < F(u)$$
$$\Rightarrow$$
$$F(u+t(v-u)) < F(u) + t(t-1)B(v,u), \forall u, v \in K, t \in [0,1].$$

Theorem 6. *If the function F is higher-order strongly-generalized convex function such that $F(v) < F(u)$, then the function F is higher-order strongly-generalized pseudo convex.*

Proof. Since $F(v) < F(u)$ and F is higher-order strongly-generalized convex function, then $\forall u, v \in K, \quad t \in [0,1]$, we have

$$\begin{aligned} F(u+t(v-u)) &\leq F(u)+t(F(v)-F(u)) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p \\ &< F(u)+t(a-t)(F(v)-F(u)) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p \\ &= F(u)+t(t-1)(F(u)-F(v)) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p \\ &< F(u)+t(t-1)B(u,v) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|v-u\|^p, \forall u,v \in K, \end{aligned}$$

where $B(u,v) = F(u) - F(v) > 0$, which is the required result.

We now discuss the optimality for the differentiable higher-order strongly-generalized convex functions, which is the main motivation of our next result.

Theorem 7. *Let F be a differentiable higher-order strongly-generalized convex function with modulus $\mu > 0$. If $u \in K$ is the minimum of the function F, then*

$$F(v) - F(u) \geq \mu\|\xi(v-u)\|^p, \quad \forall u,v \in K. \tag{19}$$

Proof. Let $u \in K$ be a minimum of the function F. Then

$$F(u) \leq F(v), \forall v \in K. \tag{20}$$

Since K is a convex set, so, $\forall u, v \in K, \quad t \in [0,1]$,

$$v_t = (1-t)u + tv \in K.$$

Taking $v = v_t$ in (20), we have

$$\begin{aligned} 0 &\leq \lim_{t\to 0}\{\frac{F(u+t(v-u))-F(u)}{t}\} \\ &= \langle F'(u), v-u\rangle. \end{aligned} \tag{21}$$

Since F is differentiable higher-order strongly-generalized convex function, so

$$\begin{aligned} F(u+t(v-u)) &\leq F(u)+t(F(v)-F(u)) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\xi(v-u)\|^p, \forall u,v \in K, \end{aligned}$$

from which, using (21), we have

$$\begin{aligned} F(v)-F(u) &\geq \lim_{t\to 0}\{\frac{F(u+t(v-u))-F(u)}{t}\} \\ &\quad +\mu\{t^{p-1}(1-t)+(1-t)^p\}\|\xi(v-u)\|^p \\ &= \langle F'(u), v-u\rangle + \mu\|\xi(v-u)\|^p, \end{aligned}$$

which is the required result (19).

Remark: We would like to mention that, if

$$\langle F'(u), v-u\rangle + \mu\|\xi(v-u)\|^p \geq 0, \quad \forall u,v \in K, p > 1 \tag{22}$$

then $u \in K$ is the minimum of the function F.
The inequality of the type (22) is called the higher-order strongly variational inequality, which appears to be new one. It is an open problem to develop the numerical methods for solving higher-order strongly variational inequalities.

It is well known that each strongly convex functions is of the form $f + \|.\|^2$, where f is a convex function. Similar result can be proved for the higher-order strongly-generalized strongly convex functions using the technique of Noor and Noor [14].

Theorem 8. *Let f be a higher-order strongly-generalized affine function. Then F is a higher-order strongly-generalized convex function, if and only if, $G = F - f$ is a convex function.*

4 Applications

In this section, we show that the characterizations of uniformly Banach spaces involving the notion of strong convexity are given.

Bynum [20] and Chen et al [21,22,23] have studied the properties and applications of the parallelogram laws for the Banach spaces. Xu [19] obtained some characteristics of p-uniform convexity and q-uniform smoothness of a Banach space via the convex functions $\|.\|^p$ and $\|.\|^q$, respectively. We show that these results follow directly from the definitions of the higher order strongly convex functions, which can be viewed as novel application.
Setting $F(u) = \|u\|^p, \quad \xi(v-u) = v-u$ in Definition2 with $t = \frac{1}{2}$, we have

$$\|u+v\|^p + \mu\|v-u\|^p = 2^{p-1}\{\|u\|^p + \|v\|^p\}, \forall u,v \in K \tag{23}$$

which is known as the parallelogram for the l^p-spaces. For the applications of the parallelogram laws for Banach spaces in prediction theory and applied sciences, see [11, 19,20,21,22,23] and the references therein.

5 Conclusion

In this paper, we have introduced and studied a new class of convex functions with respect to an arbitrary function, which is called higher-order strongly-generalized convex function. It is shown that several new and known classes of strongly convex functions can be obtained as special cases of these higher-order strongly-generalized convex functions. We have studied the basic properties of these functions. We have shown that one can derive the parallelogram laws for Banach spaces, which have applications in prediction theory and stochastic analysis. These parallelogram laws can be used to characterize the uniform convexity and uniform smoothness of Banach spaces. The interested readers may explore the applications and other properties of the higher-order strongly-generalized convex functions in various fields of pure and applied sciences.

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References

[1] B. T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, Soviet Math. Dokl. Vol. 7, pp. 2-75(1966).
[2] S. Karamardian, The nonlinear complementarity problems with applications, Part 2, J. Optim. Theory Appl. Vol. 4, No. 3, pp. 167-181(1969).
[3] K. Nikodem and Z. S. Pales, Characterizations of inner product spaces by strongly convex functions, Banach J. Math. Anal. Vol. 1, No. 2, pp. 83-87(2011).
[4] D. L. Zu and P. Marcotte, Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities. SIAM Journal on Optimization, Vol. 6, No. 3, pp. 714-726(1996).
[5] G. Qu and N. Li. On the exponentially stability of primal-dual gradeint dynamics, IEEE Control Syst. Letters, Vol. 3, No. 1, pp. 43-48(2019).
[6] M. Adamek, On a problem connected with strongly convex functions, Math. Inequ. Appl.,Vol. 19, No.4, pp. 1287-1293.(2016).
[7] H. Angulo, J. Gimenez, A. M. Moeos and K. Nikodem, K.(2011). On strongly h-convex functions, Ann. Funct. Anal. Vol. 2, No. 2, pp. 85-91(2011).
[8] M. U. Awan, M. A. Noor, T.-S. Du and K. I. Noor, New refinemnts of fractional Hermite-Hadamard inequality, RACSAM, Vol. 113, pp. 21-29(2019).
[9] A. Azcar, J. Gimnez, K. Nikodem and J. L. Snchez, On strongly midconvex functions, Opuscula Math. Vol. 31, No.1, pp. 15-16(2011).
[10] T. Lara, N. Merentes and K. Nikodem, Strongly h-convexity and separation theorems, Int. J. Anal. Vol. 2016, pp. 1-5(2016): 7160348.
[11] B. B. Mohsen, M. A. Noor, K. I. Noor and M. Postolache, Strongly convex functions of higher-order involving bifunction, Mthematics, Vol. 7. N0. 11, pp.1-12 (2019): 1028, https://doi.org/10.3390/math7111028
[12] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications, Springer-Verlag, New York, (2018).
[13] M. A. Noor, Fundamentals of equilibrium problems. Math. Inequal. Appl. Vol. 9, No. 3, pp. 529-566(2006).
[14] M. A. Noor and K. I. Noor, On generalized strongly convex functions involving bifunction, Appl. Math. Inform. Sci. Vol. 13, No. 3, pp. 411-416(2019).
[15] W. Oettli and M. Thera, On maximal monontoncity of perturbed mapping, Bollettino Union. Math. Ital. Vol. 79, pp. 47-55(1995).
[16] J. Pecric, F. Proschan and Y. I. Tong, Convex Functions, Partial Ordering and Statistical Applications, Academic Press, New York, USA.(1992).
[17] G. H. Lin and M. Fukushima, Some exact penalty results for nonlinear programs and mathematical programs with equilibrium constraints, J. Optim. Theory Appl. Vol. 118, No. 1, pp. 67-80(2003).
[18] .S. M. Mishra and N. Sharma, On strongly generalized convex functions of higher-order, Math. Inequalit. Appl. Vol. 22, No. 1, 111-121(2019).
[19] H-K, Xu, Inequalities in Banach spaces with applications, Nonl. Anal.Theory, Meth. Appl.Vol. 16, No. 12, pp. 1127-1138(1991).
[20] W.L. Bynum, Weak parallelogram laws for Banach spaces. Can. Math. Bull. Vol. 19, pp. 269-275(1976).
[21] R. Cheng, C.B. Harris, Duality of the weak parallelogram laws on Banach spaces. J. Math. Anal. Appl. Vol.404, pp. 64-70(2013).
[22] R. Cheng and W. T. Ross, Weak parallelogram laws on Banach spaces and applications to prediction, Period. Math. Hung. Vol. 71, pp. 45-58(2015).
[23] R. Cheng, J. Mashreghi and W. T. Ross, Optimal weak parallelogram constants for L_p space, Math. Inequal. Appl. Vol. 21, No.4, pp. 1047-1058(2018).

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