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A Novel Search Algorithm for a Multi Searchers Random Walk

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Abstract: A lost target is a random walker on one of n disjoint lines, and the purpose is to detect the target as fast as possible. On each line there are two searchers starting the search from the origin of their line, where they follow the so called coordinate search.
One common measure of effectiveness for the search process is the expected time of search. This type of search has been addressed and solved in the literature for the case where the target is located. In this paper, we introduce a new model of search and prove that the expected value of the first meeting time between one of the searchers and the lost target is finite. Also, we show the existence of the optimal search plan which minimizes the expected value. An application is also given to illustrate our results.

Keywords: Random walker, Linear search, Expected value, Optimal search plan, Stochastic process

1 Introduction

Searching for a lost target either located or moved is often a time-critical issue when the target is very important. The prime focus of search problems is finding the lost target in the shortest amount of time. The searchers can detect the target in the intersection point of their paths. The searching for lost targets has important applications such as: the search for a damaged unit in a large linear system, like the telephone lines and mining system, see [1]. Linear search problem for located or moved lost target has recently has recently been analyzed in the literature (see, for instance [2], [3] and [8]).
Our study illustrates the coordinated search technique, which asks how two or more searchers start together at a point in a search definite area .Thomas's searchers (parents) regroup back at the car (center) to see if the other one found the lost target (child), see [4]. The extension of this case, where the time lost in regrouping depends on the amount already searched was studied, see [5]. The coordinated search was studied on the line when the target was symmetric and asymmetric see [6] and [7]. The case of a located target on a known region like underground gas or petroleum supplies is studied see [8], like missing boats, submarines, a Bayesian approach would formulate for a target whose prior distribution and probabilistic motion model are known and generalized form of the approach for multi-vehicle search, see [9] and [10]. The tracking problem commenced with a simple feedback motion tracking algorithm, and has evolved with the developments of a number of recursive filtering techniques, see [11]. The distributed search frameworks which apply Bayesian techniques can be found, where the agents cooperate and learn policies by selecting actions and rewards from a discrete group see [12]. The same framework implemented with dynamic programming to find quasi-optimal trajectories based again on a discrete set is discussed, see [13]. The coordinated control solutions are different from the cooperative control solutions. In the coordinated control problems, decision maker's plans are - individually based on their current knowledge of the world, e.g. target state, and only exchange observed information via the Bayesian sensor network, ensuring that each platform shares a common global picture of the world, see [14]. Recently, the coordinated search strategy for a random walk moving target on one of two disjoint lines was explained, see [15].

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More recently, the coordinated search technique in the 3-D space that finds a three dimensional randomly located target in a known zone by two and four searchers was introduced, see [16] and [17].
This paper is organized as follows: In Section 2, we describe the formulation of our problem, and some conditions ensuring the expected value of the first meeting time between one of the searchers and the target are introduced. In Section 3, the existence of the optimal search plan which minimizes the expected value of the first meeting time is proved. In Section 4, the efficiency of this model using a numerical example is exhibited. Finally, the paper concludes with a discussion about the results and some directions for future research.

2 Problem Formulation

Undoubtedly, one of the most complicated problems faced by engineers in industrial gas companies is the discovery of hydrogen gas quickly and separating it from other gases mixed in the air. Hydrogen gas is converted, in a mixture with the air, into an explosive gas. Therefore, scientists develop different hydrogen sensors that can work in difficult working conditions (e.g. high temperature or concentrated gas) where the sensor records the high temperature by detecting the presence of hydrogen and its quantity in the air. Searching for random walk moving hydrogen gas molecules (the lost target) inside one of n disjoint systems of pipes (lines) using $2n$ sensors (searchers) has several applications in industrial fields, see Figure (1).

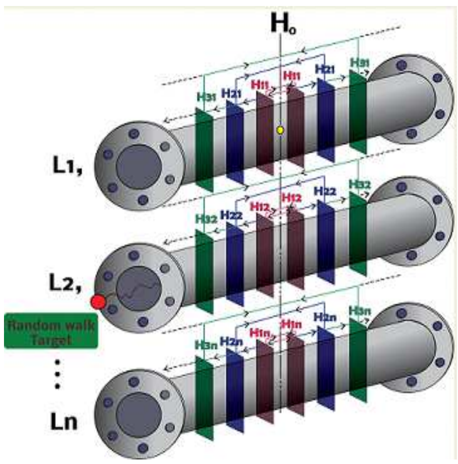

Fig. 1: Coordinated search for detecting a random walk target inside disjoint system of pipes.

In semi-coordinated search technique, every two searchers cooperate with each other to find out the location of the lost target quickly, where the motion of each searcher on each line is independent. Our goal is to find the existence of a search plan which minimizes the expected value of the first meeting time of the searchers to return to the origin after one of them has met the target.

2.1 The Searching Framework

The space of search: n disjoint lines.
The target: The target moves with a random walk motion on one of n disjoint straight lines.
The means of search: Looking for the lost target is performed by two searchers on each line. The searchers start searching for the target from the origins of the lines $H_0 = O_\xi = 0), \xi = 1,2,...,n$ with continuous paths and with equal speeds. In addition, the search spaces (n straight lines) are separated into many distances.

2.2 The Searching Technique

In this paper, the target is assumed without goal and the motion of the searchers is deterministic. We have $2n$ searchers that start looking for the lost target from the origins of n lines, where the two searchers S_1 and S_2 start together looking for the lost target from O_1 on L_1, and the two searchers S_3 and S_4 start together looking for the lost target from O_2 on the second line L_2, and so on until the searchers S_{2n-1} and S_{2n} start together looking for the lost target from O_n on the line L_n. The two searchers S_1 and S_2 start together looking for the lost target from O_1 on L_1, where the searcher S_1 moves to the right and goes from the O_1 to H_{11}, and the searcher S_2 searchers to the left and goes from O_1 to $-H_{11}$, the two searchers S_1 and S_2 search to H_{11} and $-H_{11}$ in the same time G_{11}, then they come back to O_1 again in the same time of G_{21}. If one of the two searchers does not find the lost target, then the two searchers S_1 and S_2 begin the new cycle search for the lost target, where they go from O_1 to H_{21} and $-H_{21}$ respectively and they will reach H_{21} and $-H_{21}$ in the same time of G_{31}. Then, they come back to O_1 again in the same time of G_{41} and so on. In the same way, other searchers follow the same strategy of search on the other lines, taking into consideration the difference in distance and time needed by each two searchers on each line. The $2n$ searchers return to the $O_1, O_2, ..., O_n$ after searching successively common distances until the target is found, (see Figure (2)).

A target is assumed to move randomly on one of the n disjoint lines according to a stochastic process $\{S(t), t \in I^+\}$, $I^+ = \{0,1,2,...\}$. Assume that $\{Z_i\}_{i\geq 0}$ is a sequence of independent identically distributed random variables such as for any $i \geq 1 : p(Z_i) = p$ and $p(Z_i = -1) = 1 - p = q$, where $p, q > 0$. For $t_0, t \in I^+$, $S(t) \sum_{i=1}^{t} Z_i, S(0) = 0$.

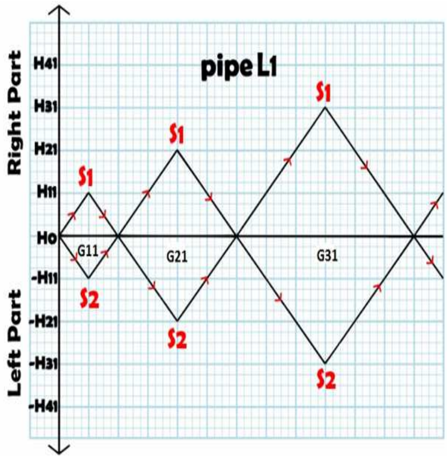

Fig. 2: The relationship between the distances of the searchers on n lines and the time of search.

The searchers $S_1, S_2, ..., S_n$ were assumed to begin their search path from $O_1, O_2, ..., O_n$ on $L_1, L_2, ..., L_n$ respectively, with speed $V_1, V_2, ..., V_n$, the searchers $S_{2\xi-1}$ and S_ξ follow the following search path which is given by the functions:

$$\|\phi_{(2\xi-1)\xi} : R^+ \to R \text{ and } \bar{\phi}_{(2\xi)\xi} : R^+ \to R \text{ or } L_\xi, \xi = 1, 2, ..., n$$

Such that:

$$|\phi_{(2\xi-1)\xi}(t_1) - \phi_{(2\xi-1)\xi}(t_2)| = |\bar{\phi}_{(2\xi)\xi} - \phi_{(2\xi)\xi}(t_2)| \leq V_\xi |t_1 - t_2|, \quad (1)$$
$$\forall \ \sigma = 1, 2, ..., n \text{ and } t_1, t_2 \in I^+,$$

where $V_\xi, \xi = 1, 2, ..., n$ are constants in R^+ and $\phi_{(2\xi-1)\xi}(0) = \bar{\phi}_{(2\xi)\xi)}(0) = 0$. Let all search paths of the two searchers $S_{2\xi-1}$ and S_ξ on L_ξ which satisfy condition (1), be represented by $\Phi_{V_{(2\xi-1)\xi}}$ and $\Phi_{V_{(2\xi)\xi}}$ respectively. We represent the path of $S_{2\xi-1}$ and S_ξ by $\phi_\xi = (\phi_{2\xi-1\xi}, \phi_{2\xi\xi}) \in \Phi_\xi$, where $\Phi_\xi = \{(\phi_{2\xi-1\xi}, \phi_{2\xi\xi}) : \phi_{2\xi-1\xi} \in \Phi_{V_{2\xi-1\xi}}, \phi_{2\xi\xi} \in \Phi_{V_{2\xi\xi}}\}, \xi = 1, 2, ..., n.$

The search plans of the n searchers are represented by $\hat{\phi} = (\phi_1, \phi_2, \phi_3, ..., \phi_n) \in \hat{\Phi}$, where $\hat{\Phi}_i$ the set of all search plans.

We assume that $Z_0 = X_\xi$ of the target which moves on L_ξ, such that $\sum_{\xi=1}^{n} p(Z_0 = X_\xi) = 1$. There is a known probability measure $v_1 + v_2 + ... + v_n = 1$ on $L_1 \cup L_2 \cup ... \cup L_n$, which described the location of the target, where v_ξ is the probability measure induced by the position of the target on L_ξ. The first meeting time in I^+ is defined by:

$$\tau_{\hat{\phi}} = \inf\{t : \phi_{2\xi-1\xi}(t) = X_\xi + S(t) \text{ or } \phi_{2\xi\xi} = X_\xi + (S)\},$$

where Z_0 is a random variable representing the initial position of the target and valued in $2I$ (or $2I+1$) and independent of $S(t), t > 0$.

At the beginning of the search, we suppose that the lost target is existing on any integer point on L_1 or L_2 or ... or L_n, but more than $H_{1\xi}$ or less than $-H_{1\xi}$. Let $\tau_{\phi_{2\xi-1\xi}}$ be the first meeting time between $S_{2\xi-1}$ and the target and $\tau_{\phi_{2\xi\xi}}$ be the first meeting time between the target and $S_{2\xi}$ on L_ξ. The main objective is to find the search plan $\hat{\phi} = (\phi_1, \phi_2, \phi_3, ..., \phi_n) \in \hat{\Phi}$ such that $E(\tau_{\hat{\phi}}) < \infty$. In this case, $\hat{\phi}$ is said to be a finite search plan, and if $E(\tau^*_{\hat{\phi}}) < E(\tau_{\hat{\phi}}) \forall \hat{\phi} \in \hat{\Phi}$, where E denotes expectation, then $\hat{\phi}^*$ is called the optimal search plan. Given $n > 0$, if z is: $0 \leq k_1 \leq \frac{n+z}{2} \leq n$, where k_1 is integer, then:

$$p(S(n) = k_1) = \begin{cases} \binom{n}{k_1} p^{k_1} q^{n-k_1}, \\ 0 \quad \text{if} \quad k_1 \quad \text{does not exist.} \end{cases}$$

2.3 Finite Search Plan

Let $\lambda_1, \lambda_2, ..., \lambda_n$ and $\zeta_1, \zeta_2, ..., \zeta_n > 1$, be positive integers such that $\zeta_1, \zeta_2, ..., \zeta_n > 1$, $\lambda_1 = k\theta_1, \lambda_2 = k\theta_2, ..., \lambda_n = k\theta_n$, where $k = 1, 2, ...$ and $\theta_1, \theta_2, ..., \theta_n$ are the least positive integers and $V_\xi = 1, \xi = 1, 2, ..., n$.

The sequences $\{G_{i\xi}\}_{i\geq 0}$, $\{H_{i\xi}\}_{i\geq 0}$ for the searcher $S_{2\xi-1}$ on the line L_ξ have been defined (see[2] and [3]) to obtain the distances that the searcher cover according to the values of λ_ξ and ζ_ξ, and explains the relationship between distance and time of the searchers according to the following:

$$G_{i\xi} = 2^{\frac{1}{2}\{1-(1-)^{i+1}\}} \lambda_\xi (\zeta_\xi^{\frac{1}{2}+\frac{1}{4}-(-1)^i \frac{1}{4}} - 1), \quad (2)$$

$$H_{i\xi} = G_{2i\xi-1} = \frac{1}{2} G_{2i\xi} \qquad i \geq 1 \text{ on } L_\xi \quad (3)$$

We shall define the search path of the searchers for any $t \in I^+$ if $G_{i\xi} \leq t < G_{i\xi+1}$, then:

$$\phi_{2\xi-1\xi} = \left(\frac{1}{2} H_{\frac{i\xi+1}{2}}\right) + (-1)^{i+1} \left(\frac{1}{2} H_{\frac{i\xi+1}{2}}\right) + (-1)^i (t - G_{i\xi}), \quad (4)$$

and $\phi_{2-\xi-1\xi} = -\phi_{2\xi\xi}$, (see Figure2) The searchers $S_{2\xi-1}$ and $S_{2\xi}$ return to the origin of L_ξ after searching successively common distances $H_{1\xi}, H_{2\xi}, ...$ and $-H_{1\xi}, -H_{2\xi}, ...$, respectively, until the target is found.

Theorem 2.1. If $\hat{\phi} = (\phi_1, \phi_2, \phi_3, ..., \phi_n) \in \hat{\Phi}$ is the search plan defined above, then the expectation $E(\tau_{\hat{\phi}})$ is finite if

$$w_{1\xi}(x_\xi) = \sum_{i=1}^{\infty} (\zeta_\xi^i - 1) p(\bar{\varphi}_\xi(G_{2i\xi-1}) < -x_\xi),$$

$$w_{2\xi}(x_\xi) = \sum_{i=1}^{\infty} (\zeta_\xi^i - 1) p(\varphi_\xi(G_{2i\xi-1}) > -x_\xi),$$

$$w_{3\xi}(x_\xi) = \sum_{i=1}^{\infty} (\zeta_\xi^i(\zeta_\xi - 2) + 1) p(\bar{\varphi}_\xi(G_{2i\xi}) < -x_\xi),$$

$$w_{3\xi}(x_\xi) = \sum_{i=1}^{\infty} (\zeta_\xi^i(\zeta_\xi - 2) + 1) p(\varphi_\xi(G_{2i\xi}) < -x_\xi), \quad (5)$$

are finite $\forall \xi = 1, 2, ..., n$.

Proof. Assume that $X_1, X_2, ..., X_n$ are independent of $S(t), t > 0$, if $X_\xi > 0$, then $X_\xi + S(t) > \phi_{2\xi-1\xi}(t)$ until the first meeting between $S_{2\xi-1\xi}$ and the target on L_ξ, also, if $X_\xi < 0$, then $X_\xi + S(t) < \phi_{2\xi\xi}(t)$ until the first meeting between $S_{2\xi\xi}$ and the target on L_ξ. Hence, for any $i \geq 0$ $p(\tau_{\hat{\phi}} > t) = p(\tau_{\phi_1} > t \text{ or } \tau_{\phi_2} > t \text{ or } ... \text{ or } \tau_{\phi_n} > t)$,

$$\begin{aligned}
E(\tau_{\hat{\phi}}) &= \int_0^\infty p(\tau_{\hat{\phi}} > t) dt \\
&\geq \sum_{i=0}^{\infty} \Bigg[\int_{G_{i1}}^{G_{i1+1}} p(\tau_{\phi_1} > G_{i1}) dt + \int_{G_{i2}}^{G_{i2+1}} p(\tau_{\phi_2} > G_{i2}) dt \\
&\quad + ... + \int_{G_{in}}^{G_{i1+1}} p(\tau_{\phi_n} > G_{in}) dt \Bigg] \qquad (6) \\
&= \sum_{i=1}^{\infty} (2^{\frac{1}{2}[(1-(-1)^{i+2}]} \lambda_1 (\zeta_1^{\frac{i+1}{2} + \frac{1}{4} - (-1)^{i+1}\frac{1}{4}} - 1) \\
&\quad - 2^{\frac{1}{2}[1-(-1)^{i+1}]} \lambda_1 (\zeta_1^{\frac{1}{2} + \frac{1}{4} - (-1)^{i}\frac{1}{4}} - 1) p(\tau_{\phi_1} > G_{i1}) \\
&\quad + 2^{\frac{1}{2}[1-(-1)^{i+2}]} \lambda_2 (\zeta_2^{\frac{i+1}{2} + \frac{1}{4} - (-1)^{i+1}\frac{1}{4} - 1)} \\
&\quad - 2^{\frac{1}{2}[1-(-1)^{i+1}]} \lambda_2 (\zeta_2^{\frac{i}{2} + \frac{1}{4}(-1)^{i}\frac{1}{4}} - 1) p(\tau_{\phi_2} > G_{i2}) \\
&\quad + ... + (2^{\frac{1}{2}[1-(-1)^{i+2}]} \lambda_n (\zeta_n^{\frac{i}{2} + \frac{1}{4} - (-1)^{i+1}\frac{1}{4}} - 1) \\
&\quad - 2^{\frac{1}{2}[1-(-1)^{i+1}]} \lambda_n (\zeta_n^{\frac{i}{2} + \frac{1}{4} - (-1)^{i}\frac{1}{4}} - 1) p(\tau_{\phi_n} > G_{in}) \\
&= \lambda_1 [((\zeta_1 - 2) + 1) p(\tau_{\phi_1} > 0) + (\zeta_1 - 1) p(\tau_{\phi_1} > \\
&\quad G_{11}) + (\zeta_1(\zeta_1 - 2) + 1) p(\tau_{\phi_1} > G_{21}) \\
&\quad + (\zeta_1^2 - 1) p(\tau_{\phi_1} > G_{31}) + (\zeta_1^2(\zeta_1 - 2) + 1) \\
&\quad p(\tau_{\phi_1} > G_{41}) + (\zeta_1^3 - 1) + 1) p(\tau_{\phi_1} > G_{51}) \\
&\quad + (\zeta_1^3(\zeta_1 - 2) + 1) p(\tau_{\phi_1} > G_{61}) + ...] \\
&+ \lambda_2 [((\zeta_2 - 2) + 1) p(\tau_{\phi_2} > 0) + (\zeta_2 - 1) p(\tau_{\phi_2} > \\
&\quad G_{12}) + (\zeta_2(\zeta_2 - 2) + 1) p(\tau_{\phi_2} > G_{22}) \\
&\quad + (\zeta_2^2 - 1) p(\tau_{\phi_2} > G_{32}) + (\zeta_2^2(\zeta_2 - 2) + 1) \\
&\quad p(\tau_{\phi_2} > G_{42}) + (\zeta_2^3 - 1) p(\tau_{\phi_2}) > G_{52}) \\
&\quad + (\zeta_2^3(\zeta_2 - 2) + 1) p(\tau_{\phi_2}) > G_{62}) + ...] + ... \\
&\lambda_n [((\zeta_n - 2) + 1) p(\tau_{\phi_n} > 0) + (\zeta_n - 1) p(\tau_{\phi_n} > \\
&\quad G_{1n}) + (\zeta_n(\zeta_n - 2) + 1) p(\tau_{\phi_n} > G_{2n}) \\
&\quad + (\zeta_n^2 - 1) p(\tau_{\phi_n} > G_{3n}) + (\zeta_n^2(\zeta_n - 2) + 1) \\
&\quad p(\tau_{\phi_n} > G_{4n}) + (\zeta_n^3 - 1) p(\tau_{\phi_n}) > G_{5n}) \\
&\quad + (\zeta_n^3(\zeta_n - 2) + 1) p(\tau_{\phi_n}) > G_{6n}) + ...], \qquad (7)
\end{aligned}$$

to solve the equation (6), the value of $p(\tau_{\phi_1}) > G_{2i1-1})$, $p(\tau_{\phi_2}) > G_{2i2-1})$, ..., $p(\tau_{\phi_n} > G_{2in-1})$ and the value of $p(\tau_{\phi_1} > G_{2i1}), p(\tau_{\phi_2}), ..., p(\tau_{\phi_n} > G_{2in})$

$$\begin{aligned}
p(\tau_{\phi_\xi} > G_{2i\xi}) &\leq \int_{-\infty}^{0} p(x_\xi + S(G_{2i\xi-1}) < -H_{i\xi}/X_\xi \\
&= x_\xi) v_\xi(dx) + \int_0^\infty p(x_\xi + S(G_{2i\xi-1}) \\
&> H_{i\xi}/X_\xi = x_\xi) v_\xi(dx).
\end{aligned}$$

Therefore,

$$\begin{aligned}
p(\tau_{\phi_\xi} > G_{2i\xi-1}) &\leq \int_{-\infty}^{0} p(x\bar{\varphi}_\xi(G_{2i\xi-1}) < -x_\xi) v_\xi(dx) \\
&+ \int_0^\infty p(\varphi_\xi(G_{2i\xi-1}) > x_\xi) v_\xi(dx), \qquad (8) \\
&\xi = 1, 2, ..., n.
\end{aligned}$$

And

$$\begin{aligned}
p(\tau_{\phi_\xi} > G_{2i\xi 1}) &\leq \int_{-\infty}^{0} p(x_\xi + S(G_{2i\xi}) < -H_{2i\xi}) v_\xi(dx) \\
&+ \int_0^\infty p(x_\xi + S(G_{2i\xi}) > H_{i\xi}) v_\xi(dx).
\end{aligned}$$

We obtain

$$\begin{aligned}
p(\tau_{\phi_\xi} > G_{2i\xi 1-1}) &\leq \int_{-\infty}^{0} p(x\bar{\varphi}_\xi(G_{2i\xi}) < -x_\xi) v_\xi(dx) \\
&+ \int_0^\infty p(\varphi_\xi(G_{2i\xi}) > x_\xi) v_\xi(dx), \qquad (9) \\
&\xi = 1, 2, ..., n.
\end{aligned}$$

Substituting (7) and (8) in (6),

$$\begin{aligned}
E(\tau_{\hat{\phi}}) \leq\ & \lambda_1 [((\zeta_1 - 2) + 1) p(\tau_{\phi_1} > 0) + (\zeta_1 - 1) \\
& p(\tau_{\phi_1} > G_{11}) + (\zeta_1(\zeta_1 - 2) + 1) p(\tau_{\phi_1} > G_{21}) \\
& + (\zeta_1^2 - 1) p(\tau_{\phi_1} > G_{31}) + (\zeta_1^2(\zeta_1 - 2) + 1) \\
& p(\tau_{\phi_1} > G_{41}) + (\zeta_1^3 - 1) p(\tau_{\phi_1} > G_{51}) + (\zeta_1^3(\zeta_1 \\
& - 2) + 1) p(\tau_{\phi_1} > G_{61}) + ...] \\
& \lambda_2 [((\zeta_2 - 2) + 1) p(\tau_{\phi_2} > 0) + (\zeta_2 - 1) \\
& p(\tau_{\phi_2} > G_{12}) + (\zeta_2(\zeta_2 - 2) + 1) p(\tau_{\phi_2} > G_{22}) \\
& + (\zeta_2^2 - 1) p(\tau_{\phi_1} > G_{32}) + (\zeta_2^2(\zeta_2 - 2) + 1) \\
& p(\tau_{\phi_2} > G_{42}) + (\zeta_2^3 - 1) p(\tau_{\phi_2} > G_{52}) + (\zeta_2^3(\zeta_2 \\
& - 2) + 1) p(\tau_{\phi_2} > G_{62}) + ...] + ... \\
& \lambda_n [((\zeta_n - 2) + 1) p(\tau_{\phi_n} > 0) + (\zeta_n - 1) \\
& p(\tau_{\phi_n} > G_{1n}) + (\zeta_n(\zeta_n - 2) + 1) p(\tau_{\phi_n} > G_{2n}) \\
& + (\zeta_n^2 - 1) p(\tau_{\phi_n} > G_{3n}) + (\zeta_n^2(\zeta_n - 2) + 1) \\
& p(\tau_{\phi_n} > G_{4n}) + (\zeta_n^3 - 1) p(\tau_{\phi_n} > G_{5n}) + (\zeta_n^3(\zeta_n
\end{aligned}$$

$$-2)+1)p(\tau_{\phi_n} > G_{6n})+...].$$

Hence

$$E(\tau_{\hat{\phi}}) \leq \sum_{\xi=1}^{n} \lambda_\xi [((\zeta_\xi - 2)+1)p(\tau_\xi > 0) + \sum_{\xi=1}^{n} \int_{-\infty}^{0} w_{1\xi}(x_\xi) v_\xi(dx_\xi) + \int_{-\infty}^{0} w_{2\xi}(x_\xi) v_\xi(dx_\xi) + \int_{-\infty}^{0} w_{3\xi}(x_\xi) v_\xi(dx_\xi) + \int_{-\infty}^{0} w_{4\xi}(x_\xi) v_{,\xi}(dx_\xi)$$

where,

$$w_{1\xi}(x_\xi) = \sum_{i=1}^{\infty} (\zeta_\xi^i - 1) p(\bar{\varphi}_\xi(G_{2i\xi-1}) < -x_\xi),$$

$$w_{2\xi}(x_\xi) = \sum_{i=1}^{\infty} (\zeta_\xi^i - 1) p(\varphi_\xi(G_{2i\xi-1}) > -x_\xi),$$

$$w_{3\xi}(x_\xi) = \sum_{i=1}^{\infty} (\zeta_\xi^i(\zeta_\xi^i - 2)+1) p(\bar{\varphi}_\xi(G_{2i\xi}) < -x_\xi),$$

and

$$w_{3\xi}(x_\xi) = \sum_{i=1}^{\infty} (\zeta_\xi^i(\zeta_\xi^i - 2)+1) p(\varphi_\xi(G_{2i\xi}) > -x_\xi).$$

Lemma 2.1. For any $k \geq 0$, let $a_n \geq 0$ for $n \geq 0$, and $a_{n+1} \leq 0$. Let $\{d_n\}_{n\geq 0}$ be a strictly increasing sequence of integers with $d_0 = 0$,

$$\sum_{n=k}^{\infty} (d_{n+1} - d_n) a_{d_{n+1}} \leq \sum_{n=d_k}^{\infty} a_k \leq \sum_{n=k}^{\infty} (d_{n+1} - d_n),$$

(see([2]). **Theorem 2.2.** The chosen search plan satisfies:

$$w_{1\xi}(x_\xi) \leq w_{5\xi}(|x_\xi|),$$
$$w_{2\xi}(x_\xi) \leq w_{6\xi}(|x_\xi|),$$
$$w_{3\xi}(x_\xi) \leq w_{7\xi}(|x_\xi|),$$

and

$$w_{4\xi}(x_\xi) \leq w_{8\xi}(|x_\xi|),$$

where, $w_{5\xi}(|x_\xi|)$, $w_{6\xi}(|x_\xi|)$, $w_{7\xi}(|x_\xi|)$ and $w_{8\xi}(|x_\xi|)$ are linear functions.

Proof. This theorem will be proved for $w_{2\xi}(x_\xi)$, and in a similar way, the other cases can be proved

$$w_{2\xi}(x_\xi) = \sum_{i=1}^{\infty} (\zeta_\xi^i - 1) p(\varphi_\xi(G_{2i\xi-1}) > -x_\xi,$$

(i) if $x_\xi \geq 0$, then $w_{2\xi}(x_\xi) \geq w_{2\xi}(0)$,

(ii) if $x_\xi > 0$, then $w_{2\xi}(x_\xi) = w_{2\xi}(0) + \sum_{i=0}^{\infty} (\zeta_\xi^i - 1)$ $p(-x_\xi < \varphi_\xi(G_{2i\xi-1}) \geq 0)$, from Theorem 2.2 (see[1]), then

$$w_{2\xi}(0) \sum_{i=1}^{\infty} (\zeta_\xi^i - 1) p(\varphi_\xi(G_{2i\xi-1} > 0) \sum_{i=1}^{\infty} (\zeta_\xi^i - 1) \varepsilon^{G_{2i\xi-1}},$$

$0 < \varepsilon < 1$.

Let us define the following

(1) $V_\xi(n) = \varphi_\xi(n\theta_\xi) \backslash 2 = \sum_{i=1}^{n} W_{i\xi}$, where $\{W_{i\xi}\}$ is a sequence of (i. i. d. r. v.),

(2) $d_{\xi^n} = G_{i\xi^{2n-1} \backslash \theta_\xi} = k(\zeta_\xi^n - 1)$,

(3) $a_\xi(n) = \frac{n}{n+k} p(-x_\xi \backslash 2 < V_\xi(n) \leq 0) = \sum_{j=0}^{|x_\xi| \backslash 2} p[-(j+1) < V_\xi(n) \leq \{-j\}]$,

(4) m_ξ is an integer such that $d_{m\xi} = b_{1\xi}|x_\xi| + b_{2\xi}$,

(5) $\alpha_\xi = \frac{\zeta_\xi}{(\zeta_\xi - 1)k}$,

and

(6) $U_{1\xi}(j, j+1) = \sum_{n=0}^{\infty} p[-(j+1) < V_\xi(n) \leq (-j)]$.

Then $U_{1\xi}(j, j+1)$ satisfies the condition of the renewal equation (see[1]).

If $n > d_{m_\xi}$, then By Theorem 2.1 (see[1]) is non increasing

$$\begin{aligned} w_{2\xi}(x_\xi) - w_{(2\xi)}(0) &= \sum_{i=0}^{\infty} (\zeta_\xi^i - 1) p(-x_\xi < \varphi_\xi(G_{2i\xi-1}) \leq 0) \\ &= \sum_{n=1}^{n_\xi} \zeta_\xi^n a(d_{\xi n}) + \sum_{n=n_{\xi+1}}^{\infty} \zeta_\xi^n a(d_{\xi n}) \\ &\leq \sum_{n=1}^{n_\xi} \zeta_\xi^n + \alpha_\xi \sum_{n=n_{\xi+1}} (d_{\xi n} - d_{\xi n} - 1) a(d_{\xi n}) \\ &\leq \sum_{n=1}^{n_\xi} \zeta_\xi^n + + \alpha_\xi \sum_{n=d_{m\xi}} a_\xi(n) \\ &\leq \sum_{n=1}^{n\xi} \zeta_\xi^n + \alpha_\xi \sum_{n=d_{m_\xi}}^{\infty} \sum_{i=0}^{|x_\xi|)} p[-(j+1) < V_\xi(n) \leq (-j)] \\ &\leq \sum_{n=1}^{n_\xi} \zeta_\xi^n + \alpha_\xi \sum_{n=d_{m_\xi}}^{\infty} U_{1\xi}(j, j+1). \end{aligned}$$

Since $U_{1\xi}(j, j+1)$ satisfies the condition of the renewal equation, hence $U_{1\xi}(j, j+1)$ is bounded fo all j by constant, so

$$w_{2\xi}(x_\xi) \leq w_{2\xi}(0) + N_{1\xi} + N_{2\xi}[x_\xi] = w_{5\xi}(|x_\xi|).$$

Theorem 2.3. If $\hat{\phi} = (\phi_1, \phi_2, \phi_3, ..., \phi_n) \in \hat{\Phi}$ is a finite search plan, then $E|Z_0|$ is finite. **Proof.** If $E(\tau_{\hat{\phi}}) < \infty$, then $p(\tau_{\hat{\phi}})$ is finite)= 1 and so

$$p(\tau_{\phi_1} \text{ is finite}) + p(\tau_{\phi_2} \text{ is finite}) + ... + p(\tau_{\phi_n} \text{ is finite}) = 1,$$

then one can conclude that $p(\tau_{\phi_1}$ is finite$) = 1$, $p(\tau_{\phi_2}$ is finite$) = 0$, ... and $p(\tau_{\phi_n}$ is finite$) = 0$, or

$p(\tau_{\phi_2}$ is finite$) = 1$, $p(\tau_{\phi_1}$ is finite$) = 0$, ... and $p(\tau_{\phi_n}$ is finite$) = 0$, or $p(\tau_{\phi_n}$ is finite$) = 1$, $p(\tau_{\phi_1}$ is finite$) = 0$, ... and $p(\tau_{\phi_2}$ is finite$) = 0$, ... and $p(\tau_{\phi_{n-1}}$ is finite $= 0, \ldots$.
On the line L_ξ is $p(\tau_{\phi_\xi}$ is finite$)= 1$, then, $X_\xi = \phi(\tau_{\phi_\xi}) - S(\tau_{\phi_\xi})$, with probability one and hence $X|X_\xi| \leq E(\tau_{\phi_\xi}) + E|S(\tau_{\phi_\xi})|$.
If $E(\tau_{\phi_\xi}) < \infty$, but $|S(\tau_{\phi_\xi})| \leq \tau_{\phi_\xi}$, then $4E|(\tau_{\phi_\xi})| \leq E(\tau_{\phi_\xi})$ and $E|X_\xi| < \infty$.

3 Existence of an Optimal Search Plan

Let $\hat{\phi}_n \in \hat{\Phi}(t)$ be a sequence of search plan, $\hat{\phi}_n$ converges to $\hat{\phi}$ as n tends to ∞ iff for any $t \in I^+, \hat{\phi}_n(t)$ converges to $\hat{\phi}(t)$ uniformly on every compact subset.
Theorem 3.1. For any $t \in I^+$, let $S(t)$ be a process. The mapping $\hat{\phi} \to E(\tau_{\hat{\phi}}) \in R^+$ is lower semi-continuous on $\hat{\Phi}(t)$.
Proof. Let $I(\hat{\phi}, t)$ be the indicator function of the set $\{\tau_{\hat{\phi}} \geq t\}$, by the Fatou-Lebesgue theorem (see[16]), then:

$$E(\tau_{\hat{\phi}}) = E\left[\sum_{t=1}^{\infty} I(\hat{\phi}, t)\right] = E\left[\sum_{t=1}^{\infty} \lim_{i\to\infty} \inf(\hat{\phi}_n, t)\right] \leq \lim_{i\to\infty} \inf E(\tau_{\hat{\phi}}),$$

for any sequence $\hat{\phi}_n \to \hat{\phi}$ in $\hat{\Phi}(t)$ sequentially compact (see[3]). Thus the mapping $\hat{\phi} \to E(\tau_{\hat{\phi}})$ is lower semi-continuous on $\hat{\Phi}(t)$, then this mapping attains its minimum.

4 Application

In one of industrial gas companies, the discovery of hydrogen gas molecules from other gases mixed in air is needed, it moves randomly with a random walk motion in one of three similar straight pipes. The expected value of the first meeting time between one of the sensors which follows the above coordinated search plan and the hydrogen gas molecules depends on the values of the distances $H_{i\xi}, i = 1, 2, \ldots,;\ \xi = 1, 2, 3$ that the sensors $S_n, n = 1, 2, \ldots, 6$ cover. Since $\lambda_1, \lambda_2, \lambda_3$ and $\zeta_1, \zeta_2, \zeta_3$ are positive integer numbers greater than one and $k = 1$, then for different values of $\lambda_1, \lambda_2, \lambda_3$ and $\zeta_1, \zeta_2, \zeta_3$we can use (2) to obtain the value of $G_{2i\xi-1}$ of $G_{2i\xi}$. Hence, we use (3) to obtain the values of $H_{i\xi}$ (the distance) which the sensors should cover, see Table 1.

Note that the sign refers to the movement in the negative direction. From the property of random walks, the hydrogen (target), the intersection points between the path of one sensor and the motion of the hydrogen gas molecules are the integer points of the distance $H_{i\xi}, i = 1, 2, \ldots; \xi = 1, 2, 3$.

Let the hydrogen gas molecules move with a random walk $Y_i = Y_{i-1} + u(t)$ where $Y(0) = 0$ and $u(t)$ is an independent identically distributed standard normal random variable with mean 0 and variance 1. Also, the initial point of the motion of hydrogen molecules is any random point and the speeds of the sensors and the target (is equal to one).

Figures. (3), (4) and (5) show the first meeting time between the path of the sensors and the motion of the moving hydrogen molecules where, the target moves on the first pipe, second pipe and the third pipe.

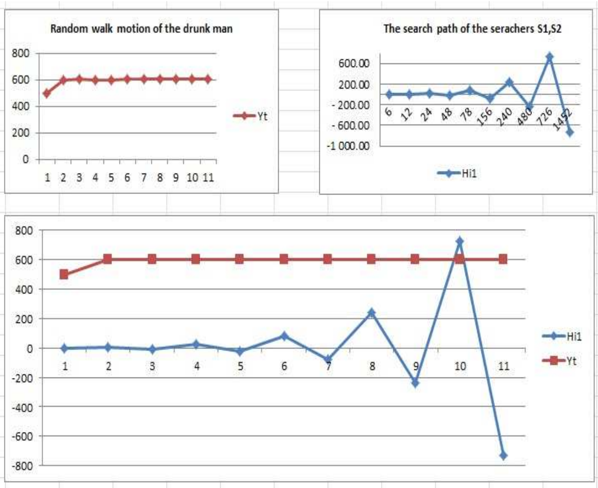

Fig. 3: The first meeting time between the sensor S_1 and the hydrogen molecules on L_1.

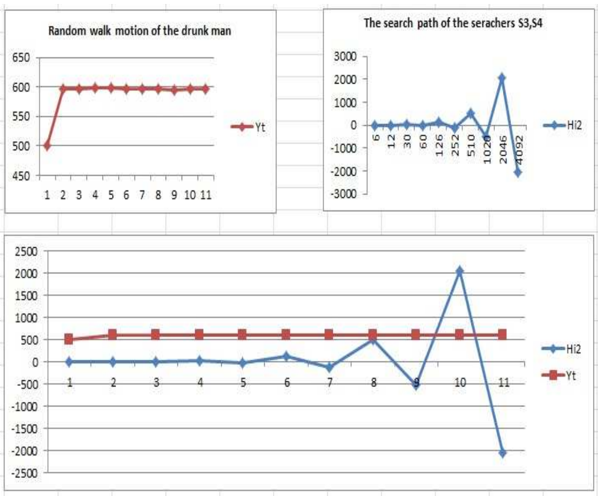

Fig. 4: The first meeting time between the sensor S_3 and the hydrogen molecules on L_2.

Table 1: The relation between the distance and the time of search

	i	1	2	3	4	5	6	7	8	9	10
						$\lambda_1 = 3,\ \zeta_1 = 3$					
	G_{2i1}	12	48	156	480	1452	4368	13116	39360	118092	354288
H_{i1}	Search Distance	6	24	78	240	726	2184	6558	19680	59046	177144
	Return Distance	-6	-24	-78	-240	-726	-2184	-6558	-19680	-59046	-177144
						$\lambda_1 = 2,\ \zeta_1 = 4$					
	G_{2i1}	12	60	252	1020	4092	16380	65532	262140	1048572	4194300
H_{i1}	Search Distance	6	30	126	510	2046	8190	32766	131070	524286	2097150
	Return Distance	-6	-30	-126	-510	-2046	-8190	-32766	-131070	-524286	-2097150
						$\lambda_1 = 5,\ \zeta_1 = 3$					
	G_{2i1}	20	80	260	800	2420	7280	21860	65600	196820	5940480
H_{i1}	Search Distance	10	40	130	400	1210	3640	10930	32800	98410	295240
	Return Distance	-10	-40	-130	-400	-1210	-3640	-10930	-32800	-98410	295240

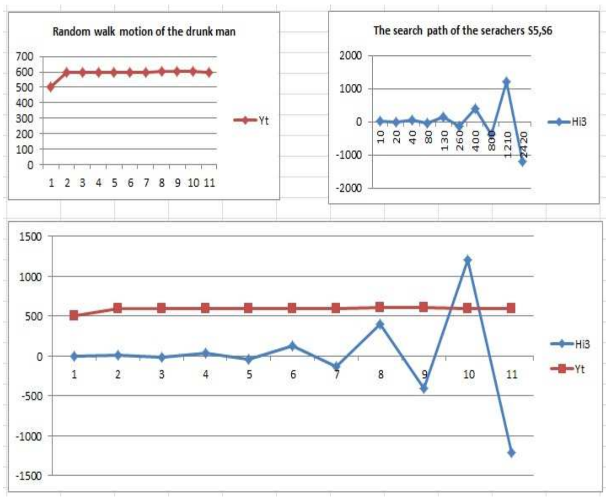

Fig. 5: The first meeting time between the sensor S_5 and the hydrogen molecules on L_3.

5 Conclusion

The multiplicative semi-coordinated linear search strategy for a random walk target on one of several real lines is illustrated, where the initial position of the lost target is represented by a random variable Z_0 as in Figures (1) and (2).

The main objective of Theorems 1, 2 and 3 is explained, and a finite search plan exists under some conditions; the existence of an optimal search plan which minimizes the expected value of the first meeting time between one of the searchers and the target is confirmed. In this model, the motion of the searchers on n lines are independent; this helps us to find the lost target without wasting time and cost. A real life application to clarify the efficiency of this model is introduced.

In future work, one can generalize the multiplicative semi-coordinated linear search plan for a randomly moving target.

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