Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/060107

On a Space-Fractional Heat Equation with Non-homogeneous Fractional Time Poisson Process

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Received: 23 Dec. 2018, Revised: 12 Feb. 2019, Accepted: 13 Mar. 2019
Published online: 1 Jan. 2020

Abstract: The present paper addresses the following stochastic heat fractional integral equation (SHFIE):

$$\frac{\partial}{\partial t}u(x,t) = -(-\Delta)^{\alpha/2}u(x,t) + \sigma(u(x,t))\mathscr{N}_\lambda^{\beta,\nu}(t),\ x\in\mathbb{R}^d,\ t\geq 0,$$

with $\beta > 0$, $\nu \in (0,1]$, $\alpha \in (0,2]$. The operator $-(-\Delta)^{\alpha/2}$ is the generator of an isotropic stable process and $\mathscr{N}_\lambda^{\beta,\nu}(t)$ is the Riemann–Liouville non-homogeneous fractional integral process. The mean and variance for the process $\mathscr{N}_\lambda^{\beta,\nu}(t)$ for some specific rate functions were computed. Also, the growth moment bounds for the class of heat equation perturbed with the non-homogeneous fractional time Poisson process were given. In addition, the paper shows that the solution grows exponentially for some small time interval $t \in [t_0,T]$, $T < \infty$ and $t_0 > 1$. To explain, the result establishes that the energy of the solution grows at least as $c_3(t+t_0)^{-(\beta+a\nu)}\exp(c_4t)$ and at most as $c_1t^{-(\beta+a\nu)}\exp(c_2t)$ for different conditions on the initial data, where c_1, c_2, c_3 and c_4 are some positive constants depending on T.

Keywords: Energy moment growth bounds, fractional heat kernel, non-homogeneous fractional time Poisson process, random field solution, Riemann–Liouville fractional integral process.

1 Introduction

Several studies have investigated classes of heat equations driven by different stochastic processes (Lévy noise processes), such as the Brownian motion, fractional Brownian motion, Gaussian white noise process (Wiener process) and coloured noise process, see [1,2,3,4,5] and their references. However, the study of heat equations perturbed by classes of Poisson processes reveal few or no results, see [6]. Motivated by the probabilistic properties [7], modeling and other physical applications of the fractional Poisson process, and the paper by Orsingher and Polito [8] who investigated the integral of the fractional Poisson process, we apply the integral process to a class of fractional heat equation.

For over two decades, there has been an increase in the use of fractional Poisson process, which is a generalization of the standard Poisson process. The process was first introduced and studied by Repin and Saichev [9], followed by Laskin [10], Mainardi and his co-authors [11,12,13,14], Beghin and Orsingher [15,16,8], its representation in terms of stable subordinator [11,12,17,18,19,20,21,22] and its connection to Lévy process [23]. See the above–mentioned papers and their references for a complete study on fractional Poisson process and its fractional distributional properties. See also the article [24] addressing non-homogeneous fractional Poisson processes which involve replacing the time parameter in the fractional Poisson process with some suitable function of time. There are several physical motivations to consider fractional Poisson process. It has geophysical applications; for example, it can be used in the area of hydrology and Seismology to model earthquake inter-arrival times. In other words, the fractional Poisson process, which lacks the memoryless property of the standard Poisson process, can represent and model sequence of catastrophic events, [25,26,27,9,28].

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Now, consider the following class of heat equation with non-homogeneous fractional time Poisson process

$$\frac{\partial}{\partial t}u(x,t) = -(-\Delta)^{\alpha/2}u(x,t) + \sigma(u(x,t))\mathcal{N}_{\lambda}^{\beta,\nu}(t),\ x \in \mathbb{R}^d,\ t \geq 0, \tag{1}$$

with non-random initial function $u(x,0) = u_0(x)$, $\sigma : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous, and $\mathcal{N}_{\lambda}^{\beta,\nu}(t)$ is the Riemann–Liouville non-homogeneous fractional integral process. We also explore the equivalent class of the above–mentioned equation for the Riemann–Liouville fractional integral process $\mathcal{N}^{\beta,\nu}(t)$

$$\frac{\partial}{\partial t}u(x,t) = -(-\Delta)^{\alpha/2}u(x,t) + \sigma(u(x,t))\mathcal{N}^{\beta,\nu}(t),\ x \in \mathbb{R}^d,\ t \geq 0, \tag{2}$$

Let $\gamma > 0$ and define the Riemann–Liouville fractional integral by

$$I_t^{\gamma} f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s)ds.$$

Now, we define Riemann–Liouville non-homogeneous fractional integral process $\mathcal{N}_{\lambda}^{\beta,\nu}(t)$ for $\beta > 0$ in terms of the above integral by

$$\mathcal{N}_{\lambda}^{\beta,\nu}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} N_{\lambda}^{\nu}(s)ds \tag{3}$$

where $N_{\lambda}^{\nu}(t), t \geq 0$ is the non-homogeneous fractional Poisson process of order $0 < \nu \leq 1$.

Now assume the following global Lipschitz continuity condition on σ as follows:

Condition 1. *There exists a finite positive constant, Lip_{σ} such that for all $x, y \in \mathbb{R}$, we have*

$$|\sigma(x) - \sigma(y)| \leq Lip_{\sigma}|x-y|.$$

For convenience, we set $\sigma(0) = 0$.

For the lower bound result, we assume the following extra condition on σ

Condition 2. *There is a finite positive constant, L_{σ} such that for all $x \in \mathbb{R}$, we have*

$$\sigma(x) \geq L_{\sigma}|x|.$$

Definition 1. *The mild solution to equation (1), in the sense of Walsh integral, [29], is given by:*

$$u(t,x) = \int_{\mathbb{R}^d} p(t,x,y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} p(t-s,x,y)\sigma(u(s,y))\mathcal{N}_{\lambda}^{\beta,\nu}(s)dyds, \tag{4}$$

where $p^{\alpha}(t,.,.)$ is the fractional heat kernel. We also impose the following integrability condition on the solution u:

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \mathbb{E}|u(t,x)| < \infty,$$

Definition 2. *The process $\{u(t,x)\}_{x\in\mathbb{R}^d,t>0}$ is a mild solution of equation (2) if almost surely, the following is satisfied:*

$$u(t,x) = \int_{\mathbb{R}^d} p(t,x,y)u_0(y)dy + \lambda \int_0^t \int_{\mathbb{R}^d} p(t-s,x,y)\sigma(u(s,y),h)\mathcal{N}^{\beta,\nu}(s)dyds, \tag{5}$$

with $p(t,.,.)$ the fractional heat kernel. If, in addition to the above condition, $\{u(t,x)\}_{x\in\mathbb{R}^d,t>0}$ satisfies:

$$\sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}^d} \mathbb{E}|u(t,x)| < \infty,$$

for all $T > 0$, $\{u(t,x)\}_{x\in\mathbb{R}^d,t>0}$ is a random field solution to (2).

Define the following norm for the mild solution u as follows:

$$\|u\|_{1,\beta} = \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} e^{-\beta t}\mathbb{E}|u(x,t)|,\ for\ \beta > 0.$$

The present paper is outlined as follows. Section Two presents the summary statement of theorems of the main results. Section Three surveys some basic preliminary concepts on the fractional process, including estimates on the mean and variance of the Riemann–Liouville fractional integral process and its non-homogeneous counterpart. Some auxiliary results for existence and uniqueness result were obtained in Section Four. The energy moment growth estimates and proofs of main results are illustrated in Section Five. Section Six is devoted to conclusion.

2 Main results

This section begins with the statements of our main results. The first result follows by assuming that u_0 is bounded above:

Theorem 1. *Suppose that Condition 1 holds and u_0 is bounded above, then there exists $t_0 > 1$ such that for all $t_0 < t < T < \infty$, we have*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}|u(x,t)| \leq c_3 t^{-(\beta+\nu)} \exp(c_4 t)$$

with $c_3 = T^{(1-d)/\alpha+\beta+\nu}$, and $c_4 = \frac{\lambda C Lip_\sigma}{\Gamma(\beta+\nu+1)} T^{(1-d)/\alpha+\beta+\nu}$.

Next, we assume u_0 as a measurable function $u_0 : \mathbb{R}^d \to \mathbb{R}_+$ which is positive on a set of positive measure:

Definition 3. *The initial condition u_0 is assumed to be a bounded non-negative function such that*

$$\int_A u_0(x)dx > 0, \text{ for some } A \subset \mathbb{R}^d.$$

Thus, with the new assumption on the initial condition u_0, we have the following lower bound estimate:

Theorem 2. *Given that Condition 2 together with $\|u_0\|_{L^1(B(0,1))} > 0$ hold. Then, there exists $t_0 > 1$ such that for all $t_0 < t < T < \infty$, we have*

$$\inf_{x\in B(0,1)} \mathbb{E}|u(t,x)| \geq c_5 (t+t_0)^{-(\beta+\nu)} \exp(c_6 t), \text{ for all } t \in [t_0, T],$$

where $c_5 = c(t_0)(2t_0)^{\beta+\nu}(T+\eta)^{-1/\alpha}$, and $c_6 = \frac{\lambda c_2 L_\sigma}{\Gamma(\beta+\nu+1)}(2t_0)^{\beta+\nu} c_1 T^{-1/\alpha}$.

The graphs, on the next page, demonstrate the behaviours of the solution for some given values of $\beta > 0$, $\nu \in (0,1]$ in the given time intervals. To explain, the graphs of $t^{-(\beta+\nu)}e^{ct}$, $\beta = 2$, $c = 2$, $\nu = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{10}$ for the following different time intervals $1 \leq t \leq 2$, $1 \leq t \leq 5$, $1 \leq t \leq 10$, $1 \leq t \leq 100$ and $1 \leq t \leq 1000$.
We also give equivalent results for the non-homogeneous fractional time process for the Weibull's rate function as follow:

Theorem 3. *Given that Condition 1 holds and u_0 bounded above, then there exists $t_0 > 1$ such that for all $t_0 < t < T < \infty$, we have*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}|u(x,t)| \leq c_1 t^{-(\beta+a\nu)} \exp(c_2 t)$$

with $c_1 = T^{\beta+a\nu}\varepsilon$, and $c_2 = \frac{b^{-a\nu}C}{\Gamma(\nu+1)} \frac{\Gamma(1+a\nu)}{\Gamma(\beta+a\nu+1)} Lip_\sigma T^{\beta+a\nu}$.

Theorem 4. *Suppose that Condition 2 together with $\|u_0\|_{L^1(B(0,1))} > 0$ hold. Then there exists $t_0 > 1$ such that for all $t_0 < t < T < \infty$, we have*

$$\inf_{x\in B(0,1)} \mathbb{E}|u(t,x)| \geq c_3 (t+t_0)^{-(\beta+a\nu)} \exp(c_4 t), \text{ for all } t \in [t_0, T],$$

where $c_2 = c(t_0)(2t_0)^{\beta+a\nu}(T+\eta)^{-1/\alpha}$, and $c_3 = \frac{b^{-a\nu}C}{\Gamma(\nu+1)} \frac{\Gamma(1+a\nu)}{\Gamma(\beta+a\nu+1)} c_1 T^{-1/\alpha}$.

3 Preliminaries

We provide some basic concept of the fractional Poisson process. Detailed explanations and properties of the process are involved in the above–mentioned cited references.

1 t 2
m=1

1 t 5

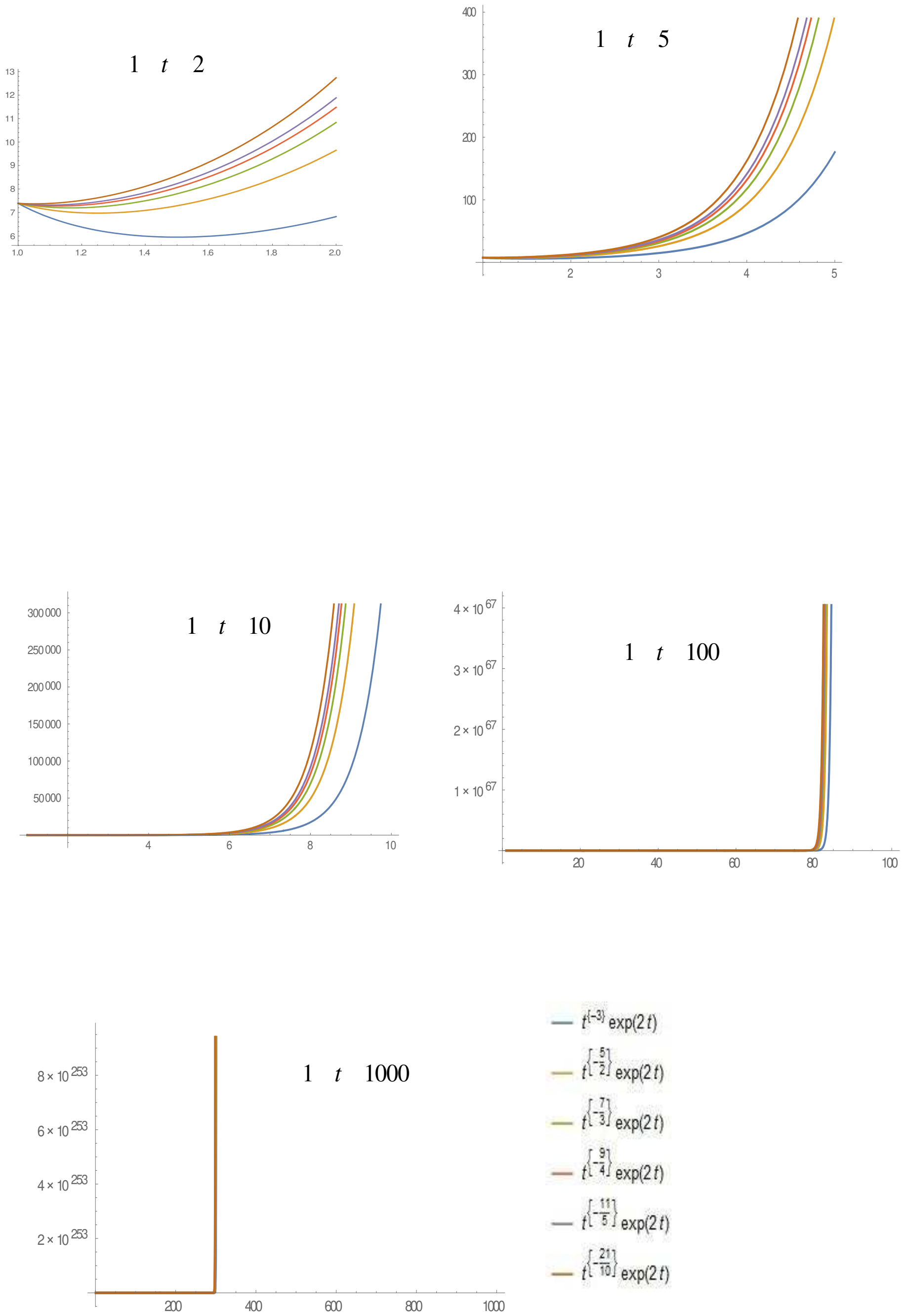

Fig. 1: Graphs showing behaviours of the solution over time intervals

3.1 Homogeneous Fractional Poisson process

For the standard Poisson process $\{N(t)\}_{t\geq 0}$ with intensity $\lambda > 0$, the probability distribution satisfies the following difference-differential equation, see [15,16,22],

$$\frac{d}{dt}p(n,t) = -\lambda\left(p(n,t) - p(n-1,t)\right),\ n \geq 1$$

with $p_n(0) = 0$ if $n = 0$ and is zero for $n \geq 1$. The solution is given by

$$p(n,t) = \mathbb{P}[N(t,\lambda) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

. Waiting time distribution function for the process is given by $\phi(t) = \lambda e^{-\lambda t}$, $\lambda > 0$, $t \geq 0$ and its moment generating function is given by

$$\mathbb{E}[e^{sN(t)}] = \exp(\lambda t(e^s - 1))\ s \in \mathbb{R}.$$

Definition 4(Fractional Poisson process). *Fractional Poisson process is a renewal process with inter-times between events represented by Mittag-Leffler distributions, see [16,25,22]. The fractional Poisson process $N^\nu(t)$, $0 < \nu \leq 1$ satisfies*

$$\begin{aligned} D_t^\nu p_\nu(n,t) &= -\lambda\left(p_\nu(n,t) - p_\nu(n-1,t)\right) \\ D_t^\nu p_\nu(0,t) &= -\lambda p_\nu(0,t) \end{aligned} \tag{6}$$

with $p_\nu(n,0) = 1$ if $n = 0$ and zero for $n \geq 1$. The symbol D_t^ν denotes the fractional derivative in the sense of Caputo-Dzhrbashyan, defined by

$$D_t^\nu f(t) = \begin{cases} \frac{1}{\Gamma(1-\nu)}\int_0^t \frac{f'(s)}{(t-s)^\nu}ds,\ 0 < \nu < 1 \\ f'(t),\ \nu = 1. \end{cases}$$

The solution is given by

$$p_\nu(n,t) = \mathbb{P}[N^\nu(t) = n] = \frac{(\lambda t^\nu)^n}{n!}E_{\nu,1}^{(n)}(-\lambda t^\nu) = \frac{(\lambda t^\nu)^n}{n!}\sum_{k=0}^{\infty}\frac{(n+k)!}{k!}\frac{(-\lambda t^\nu)^k}{\Gamma\left(\nu(k+n)+1\right)}.$$

Its waiting time distribution function is given by $\phi_\nu(t) = \lambda t^{\nu-1}E_{\nu,1}(-\lambda t^\nu)$ where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},\ \alpha,\beta \in \mathscr{C},\ \mathscr{R}(\alpha),\mathscr{R}(\beta) > 0,\ z \in \mathbb{R}$$

is the Mittag-Leffler function.

Theorem 5([24]). *Consider the fractional Poisson process $\{N^\nu(t)\}_{t\geq 0}$, $\nu \in (0,1]$. The moment generating function of the process $N^\nu(t)$ is expressed as follows:*

$$\mathbb{E}[e^{sN^\nu(t)}] = E_{\nu,1}(\lambda(e^s - 1)t^\nu)\ s \in \mathbb{R}.$$

The mean and the variance of $N^\nu(t)$ are given by

$$\mathbb{E}[N^\nu(t)] = \frac{\lambda t^\nu}{\Gamma(\nu+1)},\ \ Var[N^\nu(t)] = \frac{2(\lambda t^\nu)^2}{\Gamma(2\nu+1)} - \frac{(\lambda t^\nu)^2}{\Gamma^2(\nu+1)} + \frac{\lambda t^\nu}{\Gamma(\nu+1)}.$$

The pth order moment of the fractional process is given by

$$\mathbb{E}[N^\nu(t)]^p = \sum_{k=0}^{\infty} S_\nu(p,k)(\lambda t^\nu)^k,$$

where $S_\nu(p,k)$ is a fractional Stirling number.

3.2 Non-homogeneous fractional Poisson process

The non-homogeneous fractional Poisson process is obtained through replacing the time variable in the fractional Poisson process of renewable type with an appropriate function of time - $\Lambda(t)$.

Definition 5(Non-homogeneous Poisson process). *A counting process $\{N_\lambda(t)\}_{t\geq 0}$ is said to be a non-homogeneous Poisson process with intensity function $\lambda(t):[0,\infty)\to[0,\infty)$ if*

$$N_\lambda(t)=N(\Lambda(t)),\ t\geq 0.$$

The non-homogeneous Poisson process is specified either by its intensity function $\lambda(t)$ or its expectation function $\Lambda(t)=\mathbb{E}[N_\lambda(t)]$. When the intensity function $\lambda(t)$ exists, one denotes

$$\Lambda(t,s)=\int_s^t\lambda(y)dy$$

where the function $\Lambda(t)=\Lambda(0,t)$ is known as the rate function or cumulative rate function. The stochastic process $N_\lambda(t)$ has independent but not necessarily stationary increments: let $0\leq s<t$, then the Poisson marginal distributions of N_λ is given by

$$\mathbb{P}[N_\lambda(t+s)-N_\lambda(t)=n]=\frac{e^{-(\Lambda(t+s)-\Lambda(t))}(\Lambda(t+s)-\Lambda(t))^n}{n!},\ n\in\mathbb{Z}_+.$$

Remark. The following examples represent rate functions:

–Weibull's rate function:

$$\Lambda(t)=\left(\frac{t}{b}\right)^a,\ \lambda(t)=\frac{a}{b}\left(\frac{t}{b}\right)^{a-1},\ a\geq 0, b>0,$$

–Gompertz's rate function:

$$\Lambda(t)=\frac{a}{b}e^{bt}-\frac{a}{b},\ \lambda(t)=ae^{bt},\ a,b>0,$$

–Makeham's rate function:

$$\Lambda(t)=\frac{a}{b}e^{bt}-\frac{a}{b}+\mu t,\ \lambda(t)=ae^{bt}+\mu,\ a>0, b>0, \mu\geq 0.$$

Definition 6(Non-homogeneous fractional Poisson process). *The non-homogeneous fractional Poisson process is defined as*

$$N_\lambda^\nu(t)=N^\nu(\Lambda(t)),\ t\geq 0,\ 0<\nu\leq 1$$

where $N^\nu(t)$ is the fractional Poisson process and $\Lambda(t)$ is the rate (or cumulative rate) function.

One observes that when $\lambda(t)=\lambda^{1/\nu},\ t\geq 0,\ \ \Lambda(t)=\lambda^{1/\nu}t$ and the non-homogeneous fractional Poisson process easily gives the fractional Poisson process. The probability mass function of the non-homogeneous fractional Poisson process is given by

$$p_\nu(n,\Lambda(t))=\mathbb{P}[N_\lambda^\nu(\Lambda(t))=n]=\frac{(\Lambda(t))^{n\nu}}{n!}\sum_{k=0}^\infty\frac{(n+k)!}{k!}\frac{(-\Lambda(t)^\nu)^k}{\Gamma\big(\nu(k+n)+1\big)}.$$

Theorem 6([24]). *Let $0<s\leq t<\infty$, $q=1/\Gamma(1+\nu)$ and $d=\nu q^2B(\nu,1+\nu)$ then the mean and variance of the process $N_\lambda^\nu(t)$ are given by*

$$\mathbb{E}[N_\lambda^\nu(t)]=q\Lambda^\nu(t),\ Var[N_\lambda^\nu(t)]=q\Lambda^\nu(t)\big(1-q\Lambda^\nu(t)\big)+2d\Lambda^{2\nu}(t),$$

where $B(a,b)$ is a Bessel function.

Now, we return to equation (3) and compute the expectation of $\mathscr{N}^{\beta,\nu}(t)$ for the fractional Poisson process $N^\nu(t)$ and $\mathscr{N}_\nu^{\beta,\nu}(t)$ for the non-homogeneous fractional Poisson process using some specific rate functions.

Lemma 1. *Consider the Riemann-Liouville fractional integral process $\mathscr{N}^{\beta,\nu}(t)$, $\beta>0, 0<\nu\leq 1$, we have*

$$\mathbb{E}[\mathscr{N}^{\beta,\nu}(t)]=\frac{\lambda t^{\beta+\nu}}{\Gamma(\beta+\nu+1)}.$$

Proof. From Theorem 5, we have

$$\begin{aligned}\mathbb{E}[\mathcal{N}^{\beta,\nu}(t)] &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mathbb{E}[N^{\nu}(s)]ds \\ &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\lambda s^{\nu}}{\Gamma(\nu+1)}ds \\ &= \frac{\lambda}{\Gamma(\beta)\Gamma(\nu+1)}\int_0^t (t-s)^{\beta-1}s^{\nu}ds \\ &= \frac{\lambda}{\Gamma(\beta)\Gamma(\nu+1)}\frac{t^{\beta+\nu}\Gamma(\beta)\Gamma(\nu+1)}{\Gamma(\beta+\nu+1)}.\end{aligned}$$

Lemma 2. *Consider the Riemann–Liouville fractional integral process $\mathcal{N}^{\beta,\nu}(t)$, $\beta>0, 0<\nu\le 1$, we have*

$$Var[\mathcal{N}^{\beta,\nu}(t)] = \frac{\lambda t^{\beta+\nu}}{\Gamma(\beta+\nu+1)} + \frac{\lambda^2}{\Gamma(1+2\nu+\beta)}\left\{2-\frac{\Gamma(1+2\nu)}{\Gamma^2(\nu+1)}\right\}t^{(2\nu+\beta)}$$

Proof. Also, from Theorem 5, we have

$$\begin{aligned}Var[\mathcal{N}^{\beta,\nu}(t)] &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}Var[N^{\nu}(s)]ds \\ &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\left\{\frac{2(\lambda s^{\nu})^2}{\Gamma(2\nu+1)} - \frac{(\lambda s^{\nu})^2}{\Gamma^2(\nu+1)} + \frac{\lambda s^{\nu}}{\Gamma(\nu+1)}\right\}ds \\ &= \frac{\lambda t^{\beta+\nu}}{\Gamma(\beta+\nu+1)} + \frac{\lambda^2}{\Gamma(\beta)}\left\{\frac{2}{\Gamma(2\nu+1)} - \frac{1}{\Gamma^2(\nu+1)}\right\}\int_0^t (t-s)^{\beta-1}s^{2\nu}ds \\ &= \frac{\lambda t^{\beta+\nu}}{\Gamma(\beta+\nu+1)} + \frac{\lambda^2}{\Gamma(\beta)}\left\{\frac{2}{\Gamma(2\nu+1)} - \frac{1}{\Gamma^2(\nu+1)}\right\}t^{(2\nu+\beta)}\frac{\Gamma(1+2\nu)\Gamma(\beta)}{\Gamma(1+2\nu+\beta)}.\end{aligned}$$

Lemma 3. *For the Weibull's rate function $\Lambda(t) = \left(\frac{t}{b}\right)^a$:*

$$\begin{aligned}\mathbb{E}[\mathcal{N}_{\lambda}^{\beta,\nu}(t)] &= \frac{b^{-a\nu}}{\Gamma(\nu+1)}\frac{t^{\beta+a\nu}\Gamma(1+a\nu)}{\Gamma(\beta+a\nu+1)}, \\ Var[\mathcal{N}_{\lambda}^{\beta,\nu}(t)] &= \mathbb{E}[\mathcal{N}_{\lambda}^{\beta,\nu}(t)] + (2d-q^2)b^{-2a\nu}\frac{t^{\beta+2a\nu}\Gamma(1+2a\nu)}{\Gamma(\beta+2a\nu+1)},\end{aligned}$$

with q and d as given in Theorem 6.

Proof. Now, from Theorem 6, we have

$$\begin{aligned}\mathbb{E}[\mathcal{N}_{\lambda}^{\beta,\nu}(t)] &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mathbb{E}[N_{\lambda}^{\nu}(s)]ds \\ &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\frac{\Lambda(s)^{\nu}}{\Gamma(\nu+1)}ds \\ &= \frac{1}{\Gamma(\beta)\Gamma(\nu+1)}\int_0^t (t-s)^{\beta-1}\left(\frac{s}{b}\right)^{a\nu}ds \\ &= \frac{b^{-a\nu}}{\Gamma(\beta)\Gamma(\nu+1)}\int_0^t (t-s)^{\beta-1}s^{a\nu}ds \\ &= \frac{b^{-a\nu}}{\Gamma(\beta)\Gamma(\nu+1)}\frac{t^{\beta+a\nu}\Gamma(\beta)\Gamma(1+a\nu)}{\Gamma(\beta+a\nu+1)}.\end{aligned}$$

Remark. The mean of the non-homogeneous fractional process, $\mathbb{E}[\mathcal{N}_{\lambda}^{\beta,\nu}(t)] = \frac{1}{\lambda}\mathbb{E}[\mathcal{N}^{\beta,\nu}(t)]$ for the Weibull's rate function for $a=b=1$.

For the Gompertz and Makeham's rate functions, we were able to compute the expectations of $\mathcal{N}_{\lambda}^{\beta,\nu}(t)$ for $\nu=1$.

Lemma 4. *Given the Gompertz's rate function* $\Lambda(t) = \frac{a}{b}\left(e^{bt}-1\right)$, *we have*

$$\mathbb{E}[\mathcal{N}_\lambda^{\beta,1}(t)] = \frac{ab^{-(\beta+1)}}{\beta\Gamma(\beta)}\Big[-(bt)^\beta + \beta e^{bt}\big(\Gamma(\beta)-\Gamma(\beta,bt)\big)\Big].$$

Proof. Adopting similar steps, we obtain

$$\begin{aligned}\mathbb{E}[\mathcal{N}_\lambda^{\beta,1}(t)] &= \frac{a/b}{\Gamma(\beta)\Gamma(2)}\int_0^t (t-s)^{\beta-1}\left(e^{bs}-1\right)ds\\ &= \frac{a/b}{\Gamma(\beta)}\frac{b^{-\beta}\Big[-(bt)^\beta+\beta e^{bt}(\Gamma(\beta)-\Gamma(\beta,bt))\Big]}{\beta}.\end{aligned}$$

Lemma 5. *For the Makeham's rate function* $\Lambda(t) = \frac{a}{b}\left(e^{bt}-1+\frac{b\mu}{a}t\right)$, *we have*

$$\begin{aligned}\mathbb{E}[\mathcal{N}_\lambda^{\beta,1}(t)] = \frac{a}{b\beta(1+\beta)\Gamma(\beta)}t^\beta\Big\{&-(1+\beta)+\frac{b}{a}\mu t\\ &+(1+\beta)\text{HypergeometricPFQ}\Big[\{1\},\Big\{\frac{1}{2}+\frac{\beta}{2},1+\frac{\beta}{2}\Big\},\frac{b^2t^2}{4}\Big]\\ &+bt\ \text{HypergeometricPFQ}\Big[\{1\},\Big\{1+\frac{\beta}{2},\frac{3}{2}+\frac{\beta}{2}\Big\},\frac{b^2t^2}{4}\Big]\Big\}.\end{aligned}$$

Proof. Adopting the same steps, we obtain

$$\begin{aligned}\mathbb{E}[\mathcal{N}_\lambda^{\beta,1}(t)] &= \frac{a/b}{\Gamma(\beta)\Gamma(2)}\int_0^t (t-s)^{\beta-1}\left(e^{bs}-1+\frac{b\mu}{a}s\right)ds\\ &= \frac{a/b}{\beta(1+\beta)\Gamma(\beta)}t^\beta\Big\{-1-\beta+\frac{b}{a}\mu t\\ &\quad+(1+\beta)\text{HypergeometricPFQ}\Big[\{1\},\Big\{\frac{1}{2}+\frac{\beta}{2},1+\frac{\beta}{2}\Big\},\frac{b^2t^2}{4}\Big]\\ &\quad+bt\ \text{HypergeometricPFQ}\Big[\{1\},\Big\{1+\frac{\beta}{2},\frac{3}{2}+\frac{\beta}{2}\Big\},\frac{b^2t^2}{4}\Big]\Big\}.\end{aligned}$$

Remark. For Gomertz and Makeham's rate functions,

$$Var[\mathcal{N}_\lambda^{\beta,1}(t)] = \mathbb{E}[\mathcal{N}_\lambda^{\beta,1}(t)].$$

Now, we present some properties of the fractional heat kernel $p(t,x)$ that will be used in the proof of our results, see [30]:

$$\begin{aligned}p(t,x) &= t^{-\frac{d}{\alpha}}p(1,t^{-\frac{1}{\alpha}}x)\\ p(st,x) &= t^{-\frac{d}{\alpha}}p(s,t^{-\frac{1}{\alpha}}x).\end{aligned}$$

For all $t \geq s$,

$$p(t,x) = p(t,|x|) \geq \left(\frac{s}{t}\right)^{\frac{d}{\alpha}}p(s,x).$$

Proposition 1. *Suppose* $p(t,x)$ *is the transition density of a strictly* α*-stable process. Given that* $p(t,0)\leq 1$ *and* $a\geq 2$, *then*

$$p\big(t,\frac{1}{a}(x-y)\big) \geq p(t,x)p(t,y)\ \forall x,y\in\mathbb{R}^d.$$

The transition density is also known to satisfy the following Chapman-Kolmogorov equation,

$$\int_{\mathbb{R}^d} p(t,x)p(s,x)dx = p(t+s,0).$$

Lemma 6. *Let* $p(t,x)$ *denotes the heat kernel for a strictly stable process of order* α, *then the following estimate holds:*

$$p(t,x,y) \asymp t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}} \quad \textit{for all} \quad t>0 \quad \textit{and} \quad x,y\in\mathbb{R}^d.$$

4 Some auxiliary results

For the condition on the existence and uniqueness result for the stable process, we obtain the following result:

Theorem 7. *Suppose that $\mathscr{C}_{d,\alpha,\beta,\lambda,\nu} < \frac{1}{Lip_\sigma}$ for some positive constant Lip_σ together with condition 1, then there is a random field solution u that is unique up to modification.*

The proof of the above–mentioned theorem is based on the following Lemma 7 and Lemma 8, see Theorem 4.1.1 of [31]. Now, let

$$\mathscr{A}u(t,x) := \int_0^t \int_{\mathbb{R}^d} p(t-s,x-y)\sigma(u(s,y))\mathscr{N}^{\beta,\nu}(s)dyds,$$

and

$$\mathscr{A}_\lambda u(t,x) := \int_0^t \int_{\mathbb{R}^d} p(t-s,x-y)\sigma(u(s,y))\mathscr{N}_\lambda^{\beta,\nu}(s)dyds.$$

Then, the following Lemma(s) follow:

Lemma 7. *Given that u is a predictable solution and $\|u\|_{1,\beta} < \infty$ for all $\beta > 0$ and suppose that $\sigma(u)$ satisfies condition 1, then*

$$\|\mathscr{A}^\alpha u\|_{1,\beta} \leq \mathscr{C}_{d,\alpha,\beta,\lambda,\nu}[\sigma(0) + Lip_\sigma \|u\|_{1,\beta}]$$

where $\mathscr{C}_{d,\alpha,\beta,\lambda,\nu} := \frac{2\lambda C}{\Gamma(\beta+\nu+1)} \frac{d+\alpha}{d+\alpha-1} \frac{\Gamma(\gamma+1)}{\beta^{\gamma+1}}$.

Proof. By Lemma 1, we have

$$\begin{aligned}\mathbb{E}|\mathscr{A}u(t,x)| &= \int_0^t \int_{\mathbb{R}^d} |p(t-s,x-y)|\mathbb{E}|\sigma(u(s,y))| \frac{\lambda s^{\beta+\nu}}{\Gamma(\beta+\nu+1)}dyds \\ &\leq \frac{\lambda}{\Gamma(\beta+\nu+1)} \int_0^t \int_{\mathbb{R}^d} s^{\beta+\nu}|p(t-s,x,y)|[\sigma(0) + \mathrm{Lip}_\sigma \mathbb{E}|u(s,y)|]dyds.\end{aligned}$$

Next, multiply by $\exp(-\beta t)$ to get

$$\begin{aligned}e^{-\beta t}\mathbb{E}|\mathscr{A}u(t,x)| &\leq \frac{\lambda}{\Gamma(\beta+\nu+1)} \int_0^t \int_{\mathbb{R}^d} s^{\beta+\nu} e^{-\beta(t-s)}|p(t-s,x-y)| \\ &\quad \times e^{-\beta s}\left[\sigma(0) + \mathrm{Lip}_\sigma \mathbb{E}|u(s,y)|\right] dyds \\ &\leq \frac{\lambda}{\Gamma(\beta+\nu+1)} \sup_{s\geq 0}\sup_{y\in\mathbb{R}^d} \left\{e^{-\beta s}[\sigma(0) + \mathrm{Lip}_\sigma \mathbb{E}|u(s,y)|]\right\} \\ &\quad \times \int_0^t \int_{\mathbb{R}^d} s^{\beta+\nu} e^{-\beta(t-s)}|p(t-s,\, x-y)|dyds.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\|\mathscr{A}u\|_{1,\beta} &\leq \frac{\lambda}{\Gamma(\beta+\nu+1)}[\sigma(0) + \mathrm{Lip}_\sigma \|u\|_{1,\beta}] \\ &\quad \times \sup_{t\geq 0}\sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} s^{\beta+\nu} e^{-\beta(t-s)}|p(t-s,\, x-y)|dyds \\ &\leq \frac{\lambda}{\Gamma(\beta+\nu+1)}[\sigma(0) + \mathrm{Lip}_\sigma \|u\|_{1,\beta}] \int_0^\infty \int_{\mathbb{R}^d} s^{\beta+\nu} e^{-\beta s}|p(s,y)|dyds \\ &\leq \frac{\lambda}{\Gamma(\beta+\nu+1)}[\sigma(0) + \mathrm{Lip}_\sigma \|u\|_{1,\beta}] \int_0^\infty \int_{\mathbb{R}^d} s^{\beta+\nu} e^{-\beta s}\left\{C\left(\frac{s}{|y|^{d+\alpha}} \wedge s^{-\frac{d}{\alpha}}\right)\right\}dyds,\end{aligned}$$

where the last inequality follows by Lemma 6. Let's assume that $\frac{s}{|y|^{d+\alpha}} \leq s^{-\frac{d}{\alpha}}$ which holds only when $|y|^{\alpha} \geq s$. Therefore

$$\begin{aligned}
\|\mathscr{A}u\|_{1,\beta} &\leq \frac{\lambda C}{\Gamma(\beta+\nu+1)}[\sigma(0)+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\int_0^\infty s^{\beta+\nu}e^{-\beta s}\left\{s\int_{|y|\geq s^{1/\alpha}}\frac{dy}{|y|^{d+\alpha}}+s^{-d/\alpha}\int_{|y|<s^{1/\alpha}}dy\right\}ds\\
&= \frac{\lambda C}{\Gamma(\beta+\nu+1)}[\sigma(0)+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\\
&\quad\times\int_0^\infty s^{\beta+\nu}e^{-\beta s}\left\{s\left(-\int_{-\infty}^{s^{1/\alpha}}y^{-(d+\alpha)}dy+\int_{s^{1/\alpha}}^\infty y^{-(d+\alpha)}dy\right)+2s^{(1-d)/\alpha}\right\}ds\\
&= \frac{\lambda C}{\Gamma(\beta+\nu+1)}[\sigma(0)+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\\
&\quad\times\int_0^\infty s^{\beta+\nu}e^{-\beta s}\left\{s\left(-\frac{y^{-(d+\alpha-1)}}{1-d-\alpha}\Big|_{-\infty}^{s^{1/\alpha}}+\frac{y^{-(d+\alpha-1)}}{1-d-\alpha}\Big|_{s^{1/\alpha}}^{\infty}\right)+2s^{(1-d)/\alpha}\right\}ds\\
&= \frac{\lambda C}{\Gamma(\beta+\nu+1)}[\sigma(0)+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\int_0^\infty s^{\beta+\nu}e^{-\beta s}\left\{s\left(-\frac{2}{1-d-\alpha}s^{(1-d-\alpha)/\alpha}\right)+2s^{(1-d)/\alpha}\right\}ds\\
&= \frac{\lambda C}{\Gamma(\beta+\nu+1)}[\sigma(0)+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\int_0^\infty s^{\beta+\nu}e^{-\beta s}\left\{\frac{2}{d+\alpha-1}s^{1+(1-d-\alpha)/\alpha}+2s^{(1-d)/\alpha}\right\}ds.
\end{aligned}$$

Thus,

$$\|\mathscr{A}u\|_{1,\beta} \leq \frac{2\lambda C}{\Gamma(\beta+\nu+1)}[\sigma(0)+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\frac{d+\alpha}{d+\alpha-1}\int_0^\infty s^{\gamma}e^{-\beta s}ds,$$

where $\gamma := (1-d)/\alpha+\beta+\nu$. Hence,

$$\|\mathscr{A}u\|_{1,\beta} \leq \frac{2\lambda C}{\Gamma(\beta+\nu+1)}[\sigma(0)+\mathrm{Lip}_\sigma\|u\|_{1,\beta}]\frac{d+\alpha}{d+\alpha-1}\frac{\Gamma(\gamma+1)}{\beta^{\gamma+1}}.$$

Lemma 8. *Let u and v be two predictable random field solutions satisfying $\|u\|_{1,\beta}+\|v\|_{1,\beta}<\infty$ for all $\beta>0$, and given that $\sigma(u)$ satisfies Condition 1, then*

$$\|\mathscr{A}u-\mathscr{A}v\|_{1,\beta} \leq \mathscr{C}_{d,\alpha,\beta,\lambda,\nu}Lip_\sigma\|u-v\|_{1,\beta}.$$

Proof. Similar steps as Lemma 7

Now, we obtain the following estimates for the Weibull's rate function:

Lemma 9. *Given that u is a predictable random field solution and $\|u\|_{1,\beta}<\infty$ for all $\beta>0$ and $\sigma(u)$ satisfies Condition 1, then*

$$\|\mathscr{A}_\lambda u\|_{1,\beta} \leq \mathscr{C}_{d,\alpha,\beta,\nu}[\sigma(0)+Lip_\sigma\|u\|_{1,\beta}]$$

where $\mathscr{C}_{d,\alpha,\beta,\nu,a,b} := \frac{2Cb^{-a\nu}}{\Gamma(\nu+1)}\frac{\Gamma(1+a\nu)}{\Gamma(\beta+a\nu+1)}\frac{d+\alpha}{d+\alpha-1}\frac{\Gamma(\gamma+1)}{\beta^{\gamma+1}}$ *with* $\gamma := (1-d)/\alpha+\beta+a\nu$.

Lemma 10. *Let u and v be two predictable random field solutions satisfying $\|u\|_{1,\beta}+\|v\|_{1,\beta}<\infty$ for all $\beta>0$ and let $\sigma(u)$ satisfy Condition 1, then*

$$\|\mathscr{A}_\lambda u-\mathscr{A}_\lambda v\|_{1,\beta} \leq \mathscr{C}_{d,\alpha,\beta,\nu,a,b}Lip_\sigma\|u-v\|_{1,\beta}.$$

5 Moment growths

The proofs of the energy moment growth of our random field solutions are presented in this section. We note that the mild solution is given by

$$u(x,t) = (P_tu_0)(x)+\mathscr{A}u(x,t),$$

where

$$(P_tu_0)(x) = \int_{\mathbb{R}^d}p(t,x-y)u(0,y)dy.$$

We first give some growth bounds on the semigroup $(P_tu_0)(x)$ and show that it grows or decays but only polynomially fast with time. Assume that the initial function u_0 is bounded, we obtain the following:

Lemma 11([31]). *There exists a positive constant c_0 such that*

$$|(P_t u_0)(x)| \leq 2c_0 \frac{d+\alpha}{d+\alpha-1} t^{\frac{(1-d)}{\alpha}}.$$

Next, if one assumes that u_0 is positive on a set of positive measures, we obtain:

Proposition 2([31]). *There is a $T > 0$ and a constant $c_1 > 0$ such that for all $t > T$ and all $x \in B(0, t^{\frac{1}{\alpha}})$,*

$$(P_t u_0)(x) \geq \frac{c_1}{t^{\frac{d}{\alpha}}}.$$

One can also obtain a more general result using Proposition 2 as follows:

Proposition 3([31]). *Given the assumption on the initial function u_0. Then for $t_0 \geq 1$, $\eta > 0$, there exists $c(t_0) > 0$ such that*

$$\int_{\mathbb{R}^d} p(t+t_0, x, y) u_0(y) dy \geq c(t_0) p(t+\eta, x).$$

5.1 Proofs of Main Results

This section is devoted to the proofs of main results.

Proof. (Proof of Theorem 1). We let $\sigma(0) = 0$ and start with writing

$$\begin{aligned}
\mathbb{E}|u(x,t)| &= |(P_t u_0)(x)| + \int_0^t \int_{\mathbb{R}^d} |p(t-s, x-y)| \mathbb{E}|\sigma(u(s,y))| \frac{\lambda}{\Gamma(\beta+\nu+1)} s^{\beta+\nu} dyds \\
&\leq 2c_0 \frac{d+\alpha}{d+\alpha-1} t^{(1-d)/\alpha} + \frac{\lambda}{\Gamma(\beta+\nu+1)} \mathrm{Lip}_\sigma \int_0^t s^{\beta+\nu} \int_{\mathbb{R}^d} |p(t-s, x-y)| \mathbb{E}|u(s,y)| dyds \\
&\leq c_1 t^{(1-d)/\alpha} + \frac{\lambda C \mathrm{Lip}_\sigma}{\Gamma(\beta+\nu+1)} \int_0^t s^{\beta+\nu} \sup_{y\in\mathbb{R}^d} \mathbb{E}|u(s,y)| \int_{\mathbb{R}^d} |p(t-s, x-y)| dyds \\
&\leq c_1 t^{(1-d)/\alpha} + \frac{\lambda C \mathrm{Lip}_\sigma}{\Gamma(\beta+\nu+1)} \int_0^t (t-s)^{(1-d)/\alpha} f_{\beta,\nu}(s) ds,
\end{aligned}$$

where we defined $f_{\beta,\nu}(t) = t^{\beta+\nu} \sup_{x\in\mathbb{R}^d} \mathbb{E}|u(t,x)|$. Then, for $t_0 < t < T$ and $t - s \leq t < T$,

$$\begin{aligned}
f_{\beta,\nu}(t) &\leq c_1 t^{(1-d)/\alpha+\beta+\nu} + \frac{\lambda c_2 \mathrm{Lip}_\sigma}{\Gamma(\beta+\nu+1)} t^{\beta+\nu} \int_0^t (t-s)^{(1-d)/\alpha} f_{\beta,\nu}(s) ds \\
&\leq c_1 T^{(1-d)/\alpha+\beta+\nu} + \frac{\lambda c_2 \mathrm{Lip}_\sigma}{\Gamma(\beta+\nu+1)} T^{(1-d)/\alpha+\beta+\nu} \int_0^t f_{\beta,\nu}(s) ds.
\end{aligned}$$

Then, by Gronwall's inequality, we obtain

$$f_{\beta,\nu}(t) \leq c_3 \exp(c_4 t);\ c_3 = c_1 T^{(1-d)/\alpha+\beta+\nu}, \text{ and } c_4 = \frac{\lambda c_2 \mathrm{Lip}_\sigma}{\Gamma(\beta+\nu+1)} T^{(1-d)/\alpha+\beta+\nu}.$$

Proof. (Proof of Theorem 2). Take the first moment of the solution to obtain

$$\begin{aligned}
\mathbb{E}|u(x,t+t_0)| &= |(P_{t+t_0} u_0)(x)| + \int_0^{t+t_0} \int_{\mathbb{R}^d} |p(t+t_0-s, x-y)| \mathbb{E}|\sigma(u(s,y))| \frac{\lambda}{\Gamma(\beta+\nu+1)} s^{\beta+\nu} dyds \\
&\geq c(t_0) p(t+\eta, x) + \frac{\lambda \mathrm{L}_\sigma}{\Gamma(\beta+\nu+1)} \int_{t_0}^{t+t_0} s^{\beta+\nu} \int_{\mathbb{R}^d} |p(t+t_0-s, x-y)| \mathbb{E}|u(s,y)| dyds \\
&\geq c(t_0) p(t+\eta, x) + \frac{\lambda C \mathrm{L}_\sigma}{\Gamma(\beta+\nu+1)} \int_{t_0}^{t+t_0} s^{\beta+\nu} \inf_{y\in B(0,1)} \mathbb{E}|u(s,y)| \int_{B(0,1)} |p(t+t_0-s, x-y)| dsdy.
\end{aligned}$$

Make the following change of variable $s-t_0$. Then, set $v(t,x) := u(t+t_0,x)$ for a fixed $t_0>0$ together with Proposition 3 to write

$$\mathbb{E}|v(x,t)| \geq c(t_0)p(t+\eta,x)+\frac{\lambda \mathrm{CL}_\sigma}{\Gamma(\beta+\nu+1)}\int_0^t (s+t_0)^{\beta+\nu}\inf_{y\in B(0,1)}\mathbb{E}|v(s,y)|\int_{B(0,1)}(t-s)^{-1/\alpha}dsdy.$$

Now define $g_{\beta,\nu}(t)=(t+t_0)^{\beta+\nu}\inf_{x\in B(0,1)}\mathbb{E}|v(t,x)|$ for fixed $t_0>0$, then for $t_0<t<T$,

$$\begin{aligned} g_{\beta,\nu}(t) &\geq c(t_0)(t+t_0)^{\beta+\nu}(t+\eta)^{-1/\alpha}+\frac{\lambda \mathrm{CL}_\sigma}{\Gamma(\beta+\nu+1)}(t+t_0)^{\beta+\nu}\int_0^t g_{\beta,\nu}(s)\int_{B(0,1)}(t-s)^{-1/\alpha}dsdy\\ &\geq c(t_0)(2t_0)^{\beta+\nu}(T+\eta)^{-1/\alpha}+\frac{\lambda \mathrm{CL}_\sigma}{\Gamma(\beta+\nu+1)}(2t_0)^{\beta+\nu}c_1T^{-1/\alpha}\int_0^t g_{\beta,\nu}(s)ds \end{aligned}$$

since $t_0\leq t\leq T$, $0\leq s<t$ and $t-s\leq t\leq T$. Then we obtain that $g_{\beta,\nu}(t)\geq c_5\exp(c_6t)$, where $c_5=c(t_0)(2t_0)^{\beta+\nu}(T+\eta)^{-1/\alpha}$, and $c_6=\frac{\lambda \mathrm{CL}_\sigma}{\Gamma(\beta+\nu+1)}(2t_0)^{\beta+\nu}c_1T^{-1/\alpha}$.

The proofs of Theorem 3 and Theorem 4 follow the above–mentioned proofs.

6 Conclusion

We observed an interesting shift from the usual exponential energy growth bounds for a multiplicative noise perturbation to a class of heat equation. We also showed that energy growth of the solution is bounded by a product of algebraic and exponential functions given by $t^{-(\beta+a\nu)}\exp(ct)$, though the exponential function dominates the time interval $[t_0,T]$, $t_0>1$, which causes the solution to behave exponentially.

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