Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/060106

Using Finite Volume-Element Method for Solving Space Fractional Advection-Dispersion Equation

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Received: 2 Sep. 2018, Revised: 27 May. 2019, Accepted: 27 Jul. 2019
Published online: 1 Jan. 2020

Abstract: In this paper, the numerical solution for space fractional advection-dispersion problem in one-dimension is proposed by B-spline finite volume element method. The fractional derivative is Grunwald-Letnikov in the proposed scheme. The stability and convergence of the proposed numerical method are studied, and the numerical results support the exact results.

Keywords: Finite volume-element method, Advection-dispersion equation, Grunwald-Letnikov derivative, space fractional, fractional calculus.

1 Introduction

Fractional calculus (FC) has been applied in different fields of engineering and science, including electro-magnetics, visco-elasticity, optics, electro-chemistry, fluid mechanics, and signals processing [1–9]. This method has been used in modeling contaminant flow as well [2, 3, 10–13]. Moreover, a wide range of physical phenomena can be modelled by FC to be described more precisely. Furthermore, the fractional derivative-based models are perfect in analysing the damping systems. Because of the wide range of FC applications, most of the analytical and numerical methods, which are recently proposed, are inapplicable [14–23].

Many people have recently worked on solving fractional partial differential equations (FPDEs). Some have addressed the analytical solution of FPDE [2, 24–32], others have explored numerical solutions [24, 33–79]. To solve the differential equations, the three following approaches are adopted: Finite Difference Methods (FDM), Finite Volume Methods (FVM), and Finite Element Methods (FEM) [80, 81].

Finite element volume methods (FEVM) are linked to finite element methods. Precisely, FVEMs are the Petrov-Galerkin form of FEMs, which are developed using two types of partitions; a primal partition and its dual, on a domain Ω. The primal mesh approximates the exact solution, and the equations are discretized over the control volumes by its dual. The two main advantages of FVEM are the accuracy of the method, dependent only on the degree of the approximation polynomial, and flexibility of the control volumes. It is advantageous to handle complicated domains. Badr et al. [82] investigated FVEM for solving a time-fractional advection-diffusion equation and proved the stability of this method.

Transport activity enclosed by complex and non-homogenous conditions sometimes leads to non-classical diffusion that is not completely matched by Fick's law or pedesis theory [2, 3, 10–13, 83]. Fractional calculus helps overcome such a challenge. If a random walk model takes place as continuous time, it results in a fractional advection-dispersion equation [6]. Space advection-dispersion equation is obtained by putting the fractional derivative term in classical diffusion equation [12]. In the present paper, we will work on space fractional homogenous advection-dispersion equation.

The present paper is outlined as follows. In section two, we apply the FVEM to approximate the numerical solution of the initial value fractional advection-dispersion equation. Stability and convergence of this method are discussed in section three. Some numerical results are illustrated in section four. Section five is devoted to some conclusion.

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2 Statement of the problem and method of solution

Definition 1. *Right and left Riemann-Lioville fractional derivative respectively on Ω is defined as follows [44]:*

$$\frac{\partial^\gamma f(x,t)}{\partial x^\gamma} = \frac{1}{\Gamma(n-\gamma)} \frac{\partial^n}{\partial x^n} \int_a^x \frac{f(\xi,t)}{(x-\xi)^{\gamma-n+1}} d\xi, \tag{1}$$

$$\frac{\partial^\gamma f(x,t)}{\partial (-x)^\gamma} = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{\partial^n}{\partial x^n} \int_x^b \frac{f(\xi,t)}{(\xi-x)^{\gamma-n+1}} d\xi, \tag{2}$$

where n is the ceil of γ.

Suppose that we discrete the domain $[a,b]$ to $N+1$ equal parts, so we can define $h=(b-a)/N$ and $x_i = a+ih$, $i=0,\dots,N$.

Definition 2. *Shifted Grunwald formulas on $[a,b]$ for $\forall 0 \le p \le 1$ are defined as follows [44]:*

$$\frac{\partial^\alpha f(x,t)}{\partial x^\alpha} = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[\frac{x-a}{h}+p]} (-1)^\alpha \binom{\alpha}{j} f(x-(j-p)h,t), \tag{3}$$

$$\frac{\partial^\alpha f(x,t)}{\partial (-x)^\alpha} = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[\frac{b-x}{h}+p]} (-1)^\alpha \binom{\alpha}{j} f(x+(j-p)h,t). \tag{4}$$

For smooth function f, two equations (3) and (4) are equivalent with two equations (1) and (2) respectively [22]. Thus, we can write equations (3) and (4) as:

$$\frac{\partial^\alpha f(x,t)}{\partial x^\alpha} \approx \frac{1}{h^\alpha} \sum_{j=0}^{[\frac{x-a}{h}]+p} (-1)^\alpha \binom{\alpha}{j} f(x-(j-p)h,t), \tag{5}$$

$$\frac{\partial^\alpha f(x,t)}{\partial (-x)^\alpha} \approx \frac{1}{h^\alpha} \sum_{j=0}^{[\frac{b-x}{h}]+p} (-1)^\alpha \binom{\alpha}{j} f(x+(j-p)h,t). \tag{6}$$

In this part, we consider the homogeneous space fractional advection-dispersion equation (7) with initial condition and boundary conditions as,

$$\begin{cases} \dfrac{\partial C(x,t)}{\partial t} + \dfrac{\partial (VC(x,t))}{\partial x} = K\left(\beta \dfrac{\partial^\gamma C(x,t)}{\partial x^\gamma} + (1-\beta)\dfrac{\partial^\gamma C(x,t)}{\partial(-x)^\gamma}\right) + S(x,t) & in \quad \Omega \times [0,T], \\ C(x,t) = 0 & on \quad \partial\Omega \times [0,T], \\ C(x,0) = f(x) & in \quad \Omega, \end{cases} \tag{7}$$

that considered $V>0$, $K>0$, $0 \le \beta \le 1$ and $1<\gamma \le 2$ are constants and $\Omega = [a,b]$.
Putting $\alpha = \gamma - 1$ in equation (7) we will have [83, 84]:

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x}(VC) = \frac{\partial}{\partial x}[K(\beta \frac{\partial^\alpha C(x,t)}{\partial x^\alpha} - (1-\beta)\frac{\partial^\alpha C(x,t)}{\partial(-x)^\alpha})] + S(x,t). \tag{8}$$

According to this agreement, some introductions are obtained around FVEM and Sobolev spaces. Let $\Omega = (a,b) \subset \mathbb{R}$ be a domain, we define the Sobolev space $H^m(\Omega), m \ge 0$, to be the space of all functions v such that weak derivative $D^\alpha v \in L_2(\Omega)$ for all $|\alpha| \le 1$ and equipped with the norm and seminorm

$$\|v\|_k = (\sum_{|\alpha|\le k} \|D^\alpha v\|^2)^{\frac{1}{2}}, \; |v|_k = (\sum_{|\alpha|=k} \|D^\alpha v\|^2)^{\frac{1}{2}}.$$

Interrelated to the Sobolev space $H^m(\Omega)$, we defined the space $H^1_0(\Omega) \in H^1(\Omega)$ including functions, vanished on the boundary of Ω, i.e.,

$$H^1_0 = \{v \in H^1(\Omega), v|_{\partial\Omega} = 0\}.$$

To solve the equation (8) using our proposed numerical method, the domain Ω is divided into finite elements $\xi_i = [x_{i-1}, x_i]$ and h is set equal to $x_i - x_{i-1}$, where the nodes are represented by x_i and $i = 0, 1, 2, \ldots, M$. In addition, $\Omega_h = \{\xi_i\}$ represents the primal partition of Ω with elements ξ_i. For each node, the domain $V_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ is identified as the control volume. The Ω_h^* is defined as the set of control volumes, $\{V_i\}$. Using piecewise space S_h^* on Ω_h^*, which was first introduced in [85], we established a variational FVE form for equation (8) as follows:

$$(\tfrac{\partial C}{\partial t}, v_h^*) + (\tfrac{\partial}{\partial t}(VC), v_h^*) = K\beta \left(\tfrac{\partial}{\partial x}(\tfrac{\partial^\alpha C}{\partial x^\alpha}), v_h^*\right) - K(1-\beta)\left(\tfrac{\partial}{\partial x}(\tfrac{\partial^\alpha C}{\partial(-x)^\alpha}), v_h^*\right) + (S, v_h^*) \; v_h^* \in S_h^* \tag{9}$$

that $(.,.)$ is inner product in $L_2(\Omega)$. An estimation of $C \in H^1(\Omega \times (0,T))$ in S_h for (9) is taken by the discrete FVEM. S_h is defined on Ω_h as

$$S_h(x) = \{v_h(x) \in C(\Omega) : v_h\,|_{\xi_i} \text{ is linear and } v_h\,|_{\partial\Omega} = 0\}.$$

Different elections for solution spaces, test spaces, and dual partitions cause different FVEMs [85]. In this paper, we have shown $S_h = \mathrm{span}\{\phi_i : 0 \le i \le M\}$ and $S_h^* = \mathrm{span}\{\chi_i : 0 \le i \le M\}$ where ϕ_is (linear B-spline functions) defined by

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x_{i-1} \le x \le x_i, \\ \frac{x_{i+1} - x}{h}, & x_i \le x \le x_{i+1}, \\ 0, & otherwise, \end{cases} \tag{10}$$

and χ_is (characteristic functions) related by the control volume V_i defined by

$$\chi_i(x_j) = \begin{cases} 1, & x_j \in V_i, \\ 0, & otherwise. \end{cases} \tag{11}$$

Each $C_h(x,t) \in S_h$ may be written as

$$C_h(x,t) = \sum_{i=0}^{M} \delta_i(t)\phi_i(x), \tag{12}$$

where the coefficients $\delta_i(t)$ should be calculated from the initial conditions and boundary conditions using the FVEM.The discrete FVEM is defined as: Find $C_h(t) = C_h(.,t)$ depending on S_h, for each $t > 0$, in order that, for any $V_i, i = 1, 2, \ldots, M - 1$

$$\begin{aligned} &(\frac{\partial C_h}{\partial t}, v_h^*) + (\frac{\partial}{\partial t}(VC_h), v_h^*) \\ &= K\beta(\frac{\partial}{\partial x}(\frac{\partial^\alpha C_h}{\partial x^\alpha}), v_h^*) - K(1-\beta)(\frac{\partial}{\partial x}(\frac{\partial^\alpha C_h}{\partial(-x)^\alpha}), v_h^*) + (S, v_h^*) \end{aligned} \quad \forall v_h^* \in S_h^*, t > 0, \tag{13}$$

to statement of the FVEM for (8), it is verified that

$$\begin{aligned} &\int_\Omega \frac{\partial C_h}{\partial t}\chi_i dx + \int_\Omega \frac{\partial C_h}{\partial x}\chi_i dx \\ &= K\beta \frac{\partial}{\partial x}\int_\Omega \frac{\partial^\alpha C_h}{\partial x^\alpha}\chi_i dx - K(1-\beta)\frac{\partial}{\partial x}\int_\Omega \frac{\partial^\alpha C_h}{\partial(-x)^\alpha}\chi_i dx + \int_\Omega S(x,t)\chi_i dx, \end{aligned}$$

According to definition of $\chi_i, i = 1, 2, \ldots, M-1$, we have

$$\begin{aligned} &\int_{V_i} \frac{\partial C_h}{\partial t} dx + \int_{V_i} \frac{\partial C_h}{\partial x} dx \\ &= K\beta \frac{\partial}{\partial x} V \int_{V_i} \frac{\partial^\alpha C_h}{\partial x^\alpha} dx - K(1-\beta)\frac{\partial}{\partial x}\int_{V_i} \frac{\partial^\alpha C_h}{\partial(-x)^\alpha} dx + \int_{V_i} S(x,t) dx. \end{aligned} \tag{14}$$

Let $\delta_i = \delta_i(t)$, $w_j^\alpha = (-1)^j \binom{\alpha}{j}$ and $\delta_i^j = \delta_i(t_j)$ where $t_j = j * \tau$.

Using shifted and simple Grunwald-Letnikov fractional derivatives on $[a,b]$ results in:

$$
\begin{aligned}
&\frac{\partial^\alpha}{\partial x^\alpha}C(x_{i+\frac{1}{2}},t) \approx \frac{1}{h^\alpha}\sum_{j=0}^{i+1} w_j^\alpha C(x_{i-j+1},t),\\
&\frac{\partial^\alpha}{\partial(-x)^\alpha}C(x_{i+\frac{1}{2}},t) \approx \frac{1}{h^\alpha}\sum_{j=0}^{M-i} w_j^\alpha C(x_{i+j},t),\\
&\frac{\partial^\alpha}{\partial x^\alpha}C(x_{i-\frac{1}{2}},t) \approx \frac{1}{h^\alpha}\sum_{j=0}^{i} w_j^\alpha C(x_{i-j},t),\\
&\frac{\partial^\alpha}{\partial(-x)^\alpha}C(x_{i-\frac{1}{2}},t) \approx \frac{1}{h^\alpha}\sum_{j=0}^{M-i+1} w_j^\alpha C(x_{i+j-1},t),\\
&\frac{\partial^\alpha}{\partial x^\alpha}C(x_i,t) \approx \frac{1}{h^\alpha}\sum_{j=0}^{i} w_j^\alpha C(x_{i-j},t),\\
&\frac{\partial^\alpha}{\partial(-x)^\alpha}C(x_i,t) \approx \frac{1}{h^\alpha}\sum_{j=0}^{M-i} w_j^\alpha C(x_{i+j},t).
\end{aligned} \tag{15}
$$

Putting relation (2) in equation (13) results in:

$$
\begin{aligned}
\frac{d}{dt}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\sum_{i=0}^{M}\delta_i\phi_i(x)dx = &-V\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\frac{\partial}{\partial x}(\sum_{i=0}^{M}\delta_i\phi_i(x))dx\\
&+K\beta\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\frac{\partial}{\partial x}(\frac{\partial^\alpha}{\partial x^\alpha}\sum_{i=0}^{M}\delta_i\phi_i(x))dx\\
&-K(1-\beta)\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\frac{\partial}{\partial x}(\frac{\partial^\alpha}{\partial(-x)^\alpha}\sum_{i=0}^{M}\delta_i\phi_i(x))dx\\
&+\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}S(x,t)dx.
\end{aligned} \tag{16}
$$

We need to calculate each term of relation (16).

First term is:

$$
\begin{aligned}
\frac{d}{dt}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\sum_{i=0}^{M}\delta_i\phi_i(x)dx &= \frac{d}{dt}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}(\delta_{i-1}\phi_{i-1}(x)+\delta_i\phi_i(x)+\delta_{i+1}\phi_{i+1}(x))dx,\\
&= \frac{d}{dt}(\delta_{i-1}\times\frac{h}{8}+\delta_i\times\frac{3h}{4}+\delta_{i+1}\times\frac{h}{8}),\\
&= \frac{h}{8}\frac{\delta_{i-1}^{j+1}-\delta_{i-1}^{j}}{\tau}+\frac{3h}{4}\frac{\delta_{i}^{j+1}-\delta_{i}^{j}}{\tau}+\frac{h}{8}\frac{\delta_{i+1}^{j+1}-\delta_{i+1}^{j}}{\tau}.
\end{aligned} \tag{17}
$$

Second term is:

$$
\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\frac{\partial}{\partial x}\sum_{i=0}^{M}(\delta_i\phi_i(x))dx &= (\delta_{i-1}\phi_{i-1}(x)+\delta_i\phi_i(x))\Big|_{x_{i-\frac{1}{2}}}^{x_i}+(\delta_i\phi_i(x)+\delta_{i+1}\phi_{i+1}(x))\Big|_{x_i}^{x_{i+\frac{1}{2}}},\\
&= \delta_{i-1}(0-\frac{1}{2})+\delta_i(1-\frac{1}{2})+\delta_i(\frac{1}{2}-1)+\delta_{i+1}(\frac{1}{2}-0),\\
&= -\frac{1}{2}\delta_{i-1}+\frac{1}{2}\delta_{i+1}.
\end{aligned} \tag{18}
$$

Third term is:

$$
\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x}(\frac{\partial^\alpha}{\partial x^\alpha}(\sum_{i=0}^{M}\delta_i\phi_i(x)))dx &= \frac{\partial^\alpha}{\partial x^\alpha}\sum_{i=0}^{M}\delta_i\phi_i(x)|_{x_{i-\frac{1}{2}}}^{x_i},\\
&+\frac{\partial^\alpha}{\partial x^\alpha}\sum_{i=0}^{M}\delta_i\phi_i(x)|_{x_i}^{x_{i+\frac{1}{2}}},\\
&=\frac{\partial^\alpha}{\partial x^\alpha}(\delta_{i-1}\phi_{i-1}(x)+\delta_i\phi_i(x))|_{x_{i-\frac{1}{2}}}^{x_i}+\frac{\partial^\alpha}{\partial x^\alpha}(\delta_i\phi_i(x)+\delta_{i+1}\phi_{i+1}(x))|_{x_i}^{x_{i+\frac{1}{2}}},\\
&=\frac{1}{h^\alpha}[\delta_{i-1}(\sum_{j=0}^{i}(w_j^\alpha\phi_{i-1}(x_{i-j}))-\sum_{j=0}^{i}(w_j^\alpha\phi_{i-1}(x_{i-j}))),\\
&+\delta_i(\sum_{j=0}^{i}(w_j^\alpha\phi_i(x_{i-j}))-\sum_{j=0}^{i}(w_j^\alpha\phi_i(x_{i-j})))],\\
&+\frac{1}{h^\alpha}[\delta_i(\sum_{j=0}^{i+1}(w_j^\alpha\phi_i(x_{i-j+1}))-\sum_{j=0}^{i}(w_j^\alpha\phi_i(x_{i-j}))),\\
&+\delta_{i+1}(\sum_{j=0}^{i+1}(w_j^\alpha\phi_{i+1}(x_{i-j+1}))-\sum_{j=0}^{i}(w_j^\alpha\phi_{i+1}(x_{i-j})))],\\
&=0+\frac{1}{h^\alpha}[\delta_i(w_1^\alpha-w_0^\alpha)+\delta_{i+1}(w_0^\alpha-0)],\\
&=\frac{1}{h^\alpha}[\delta_i(-\alpha-1)+\delta_{i+1}(1)],
\end{aligned}
\tag{19}
$$

and fourth term is:

$$
\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x}(\frac{\partial^\alpha}{\partial(-x)^\alpha}(\sum_{i=0}^{M}\delta_i\phi_i(x)))dx &= \frac{\partial^\alpha}{\partial(-x)^\alpha}\sum_{i=0}^{M}\delta_i\phi_i(x)|_{x_{i-\frac{1}{2}}}^{x_i}+\frac{\partial^\alpha}{\partial(-x)^\alpha}\sum_{i=0}^{M}\delta_i\phi_i(x)|_{x_i}^{x_{i+\frac{1}{2}}},\\
&=\frac{\partial^\alpha}{\partial(-x)^\alpha}(\delta_{i-1}\phi_{i-1}(x)+\delta_i\phi_i(x))|_{x_{i-\frac{1}{2}}}^{x_i},\\
&+\frac{\partial^\alpha}{\partial(-x)^\alpha}(\delta_i\phi_i(x)+\delta_{i+1}\phi_{i+1}(x))|_{x_i}^{x_{i+\frac{1}{2}}},\\
&=\frac{1}{h^\alpha}[\delta_{i-1}(\sum_{j=0}^{M-i}(w_j^\alpha\phi_{i-1}(x_{i+j}))-\sum_{j=0}^{M-i+1}(w_j^\alpha\phi_{i-1}(x_{i+j-1}))),\\
&+\delta_i(\sum_{j=0}^{M-i}(w_j^\alpha\phi_i(x_{i+j}))-\sum_{j=0}^{M-i+1}(w_j^\alpha\phi_i(x_{i+j-1})))],\\
&+\frac{1}{h^\alpha}[\delta_i(\sum_{j=0}^{M-i}(w_j^\alpha\phi_i(x_{i+j}))-\sum_{j=0}^{M-i}(w_j^\alpha\phi_i(x_{i+j}))),\\
&+\delta_{i+1}(\sum_{j=0}^{M-i}(w_j^\alpha\phi_{i+1}(x_{i+j}))-\sum_{j=0}^{M-i}(w_j^\alpha\phi_{i+1}(x_{i+j})))],\\
&=\frac{1}{h^\alpha}[\delta_{i-1}(0-w_0^\alpha)+\delta_i(w_0^\alpha-w_1^\alpha)],\\
&=\frac{1}{h^\alpha}[\delta_{i-1}(-1)+\delta_{i+1}(1+\alpha)].
\end{aligned}
\tag{20}
$$

Let

$$
\overline{S}_i(t)=\frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(x,t)dx. \tag{21}
$$

We know $\bar{S}_i(t)\approx S(x_i,t)$ [44].

Replacing (17),(18),(19), (20) and (21) in equation (16) and using backward method result in:

$$\begin{aligned}\frac{h}{8}\left(\frac{\delta_{i-1}^{j+1}-\delta_{i-1}^{j}}{\tau}\right)+\frac{3h}{4}\left(\frac{\delta_{i}^{j+1}-\delta_{i}^{j}}{\tau}\right)+\frac{h}{8}\left(\frac{\delta_{i+1}^{j+1}-\delta_{i+1}^{j}}{\tau}\right)=&-\frac{V}{2}(-\delta_{i-1}^{j+1}+\delta_{i+1}^{j+1})\\&+\frac{K\beta}{h^{\alpha}}[\delta_i^{j+1}(-\alpha-1)+\delta_{i+1}^{j+1}(1)]\\&-\frac{K(1-\beta)}{h^{\alpha}}[\delta_{i-1}^{j+1}(-1)+\delta_i^{j+1}(1+\alpha)]\\&+hS_i^{j+1}.\end{aligned}\tag{22}$$

We define matrix A,B,Δ^j and D^j as follows

$$A_{(M-1)\times(M-1)}=\begin{bmatrix}6&1&0&0&\cdots&\\1&6&1&0&\cdots&\\0&1&6&1&0&\\0&0&\ddots&\ddots&\ddots&\\&\cdots&0&1&6&1\\&&\cdots&0&1&6\end{bmatrix},$$

$$B_{(M-1)\times(M-1)}=\begin{bmatrix}a_M&b_M&0&0&\cdots&\\c_M&a_M&b_M&0&\cdots&\\0&c_M&a_M&b_M&0&\\0&0&\ddots&\ddots&\ddots&\\&\cdots&0&c_M&a_M&b_M\\&&\cdots&0&c_M&a_M\end{bmatrix},$$

$$\Delta^j=\left[\delta_1^j\ \delta_2^j\ \ldots\ \delta_{M-2}^j\ \delta_{M-1}^j\right]^T,$$

$$D^j=\left[S_1^j\ S_2^j\ \ldots\ S_{M-2}^j\ S_{M-1}^j{}^T\right],$$

where

$$c_M=\frac{-h}{8\tau}+\frac{V}{2}+\frac{K(1-\beta)}{h^{\alpha}},a_M=\frac{-3h}{4\tau}-\frac{K(1-\beta)(1+\alpha)}{h^{\alpha}}-\frac{K\beta}{h^{\alpha}}(1+\alpha),b_M=\frac{-h}{8\tau}-\frac{V}{2}+\frac{K\beta}{h^{\alpha}}.$$

Hence, we can rewrite equation (22) as matrix form:

$$-\frac{h}{8\tau}A\times\Delta^j=B\times\Delta^{j+1}+D^{j+1},\ \forall j\geq 0,\forall j\in\mathbb{Z}.\tag{23}$$

3 Stability and convergence analysis

Implementing the analysis of Von-Neumann stability [75, 86] and the mathematical inductions, the stability conditions of the proposed numerical method is investigated.

Theorem 1. *If h is small enough, then the method explained in equation (22) is stable.*

Proof. Let we define $C^j=\left[C(x_0,t_j)\ C(x_1,t_j)\ \ldots\ C(x_{M-2},t_j)\ C(x_{M-1},t_j)\right]$. We used Von-Neumann processes. We can write according to equation (23) :

$$\Delta^j=-\frac{h}{8\tau}(B^{-1}A)\Delta^{j-1}-B^{-1}D^j=(-\frac{h}{8\tau})^2(B^{-1}A)^2\Delta^{j-2}-\frac{h}{8\tau}B^{-1}ABD^{j-1}-B^{-1}D^j.\tag{24}$$

Proceeding with this method, we get the following result

$$\begin{aligned}\Delta^j=&(-\frac{h}{8\tau})^j(B^{-1}A)^j\Delta^0+(-\frac{h}{8\tau})^{j-1}(B^{-1}A)^{j-1}B^{-1}D^1\\&+(-\frac{h}{8\tau})^{j-2}(B^{-1}A)^{j-2}B^{-1}D^2+\cdots+B^{-1}D^j,\end{aligned}\tag{25}$$

and for exact solution

$$C^j = (-\frac{h}{8\tau})^j (B^{-1}A)^j C^0 + (-\frac{h}{8\tau})^{j-1} (B^{-1}A)^{j-1} B^{-1} D^1 + (-\frac{h}{8\tau})^{j-2} (B^{-1}A)^{j-2} B^{-1} D^2 + \cdots + B^{-1} D^j. \tag{26}$$

We define the error $e^j = \Delta^j - C^j$ so

$$e^j = (-\frac{h}{8\tau})^j (B^{-1}A)^j e^0. \tag{27}$$

For compatible matrix

$$\left\| e^j \right\| \leq \left\| (-\frac{h}{8\tau} B^{-1} A)^j \right\| \left\| e^0 \right\|. \tag{28}$$

If $M > 0$ such that $\left\| (-\frac{h}{8\tau} B^1 A)^j \right\| \leq M$ the difference scheme is stable [?]. We know

$$\left\| (-\frac{h}{8\tau} B^{-1} A)^j \right\| \leq \left\| (-\frac{h}{8\tau} B^{-1} A) \right\| \left\| (-\frac{h}{8\tau} B^{-1} A)^{j-1} \right\| \leq \cdots \leq \left\| (-\frac{h}{8\tau} B^{-1} A) \right\|^j. \tag{29}$$

As a result, Lax definition of stability is made certain if [87]

$$\left\| (-\frac{h}{8\tau} B^{-1} A) \right\| \leq 1. \tag{30}$$

We know h is small enough. Therefore,

$$\left\| -\frac{h}{8\tau} A \right\|_\infty = (\frac{h}{8\tau}(1+6+1)) = \frac{h}{\tau} < 1, \tag{31}$$

and again for small enough h

$$\begin{aligned} \|B\|_\infty &= \left| \frac{-h}{8\tau} + \frac{V}{2} + \frac{K(1-\beta)}{h^\alpha} \right| + \left| \frac{-3h}{4\tau} - \frac{K(1-\beta)(1+\alpha)}{h^\alpha} - \frac{K\beta}{h^\alpha}(1+\alpha) \right| + \left| \frac{-h}{8\tau} - \frac{V}{2} + \frac{K\beta}{h^\alpha} \right| \\ &= (-\frac{h}{8\tau} + \frac{V}{2} + \frac{K(1-\beta)}{h^\alpha}) + (\frac{3h}{4\tau} + \frac{K(1-\beta)(1+\alpha)}{h^\alpha} + \frac{K\beta(1+\alpha)}{h^\alpha}) + (\frac{-h}{8\tau} - \frac{V}{2} + \frac{K\beta}{h^\alpha}) \\ &= \frac{h}{2\tau} + \frac{K(2+\alpha)}{h^\alpha} > 1 \Rightarrow \left\| B^{-1} \right\| < 1. \end{aligned} \tag{32}$$

Using (31) and (32), the relation (30) is established and the method is stable for small enough h.

Theorem 2. *The method presented in (22) is convergent and the order of the scheme is one in space and time.*

Proof. Replacing exact solution with numerical solution in equation (22), we will have

$$\begin{aligned} &\frac{h^{\alpha+1}}{8\tau}(C_{i-1}^{j+1} - C_{i-1}^j) + \frac{3h^{\alpha+1}}{4\tau}(C_i^{j+1} - C_i^j) + \frac{h^{\alpha+1}}{8\tau}(C_{i+1}^{j+1} - C_{i+1}^j) \\ &\quad = -\frac{V}{2} h^\alpha (-C_{i-1}^{j+1} + C_{i+1}^{j+1}) + K\beta(C_i^{j+1}(-\alpha-1) + C_{i+1}^{j+1}) \\ &\quad - K(1-\beta)(-C_{i-1}^{j+1} + C_i^{j+1}(\alpha+1)) + h^{1+\alpha} S(x_i, t_{j+1}). \end{aligned} \tag{33}$$

Hence, local truncation error is calculated using Taylor series as follows:

$$\begin{aligned} T_{ij} = &- h^{1+\alpha} S(x_i,t_j) + K\alpha C(x_i,t_j) + h^{1+\alpha} C_t(x_i,t_j) + \frac{1}{2} h^{1+\alpha} \tau C_{tt}(x_i,t_j) \\ &+ \frac{1}{2} K\alpha \tau^2 C_{tt}(x_i,t_j) + hKC_x(x_i,t_j) + h^{1+\alpha} V C_x(x_i,t_j) - 2hK\beta C_x(x_i,t_j) \\ &+ hK\tau C_{xt}(x_i,t_j) + h^{1+\alpha} V\tau C_{xt}(x_i,t_j) - 2hK\beta\tau C_{xt}(x_i,t_j) + \frac{1}{2} hK\tau^2 C_{xtt}(x_i,t_j) \\ &+ \frac{1}{2} h^{1+\alpha} v\tau^2 C_{xtt}(x_i,t_j) + hK\beta\tau^2 C_{xtt}(x_i,t_j) - \frac{1}{2} h^2 K C_{xx}(x_i,t_j) \\ &+ \frac{1}{8} h^{3+\alpha} C_{xxt}(x_i,t_j) + \frac{1}{2} h^2 K\tau C_{xxt}(x_i,t_j) \\ &+ \frac{1}{16} h^{3+\alpha} \tau C_{xxtt}(x_i,t_j) - \frac{1}{4} h^2 K\tau^2 Cxxtt(x_i,t_j) + \cdots. \end{aligned} \tag{34}$$

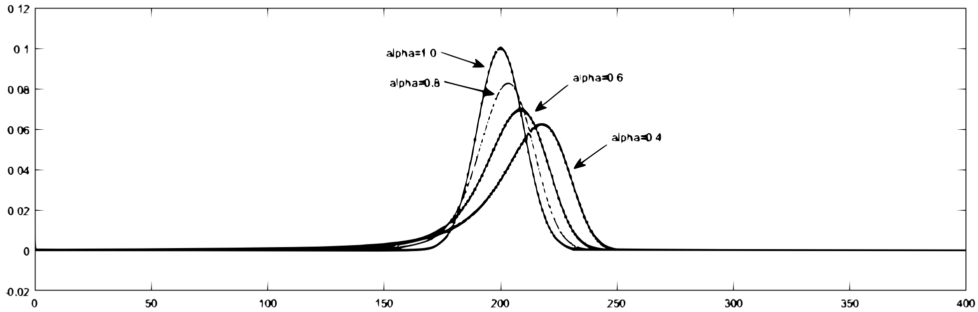

Fig. 1: Comparison numerical (symbols) and exact (solid) solution for several α

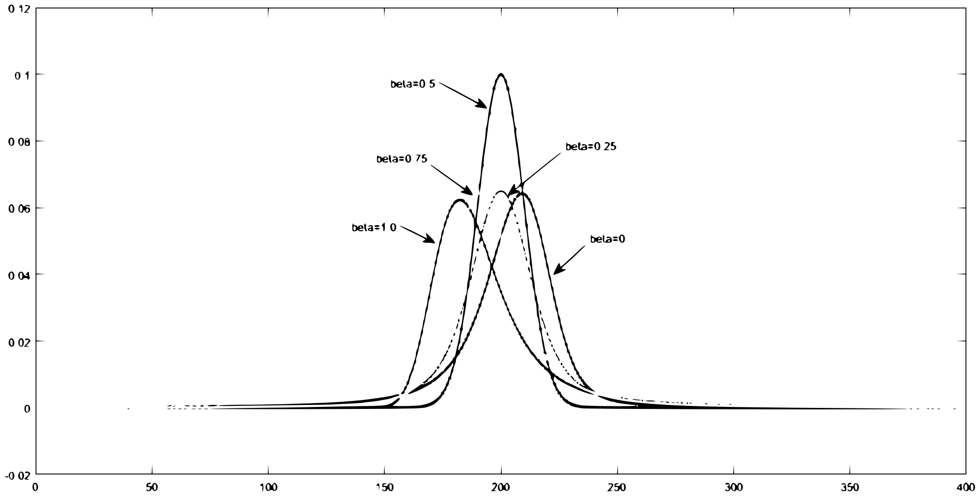

Fig. 2: Comparison numerical (symbols) and exact (solid) solution for several β

Therefore, it is noticeable that the local truncation error is of order $O(h)+O(\tau)$.

4 Numerical examples

In this section, we will compare the numerical solution resulted in FVEM with the exact solution the following example:

$$\begin{cases} \dfrac{\partial C(x,t)}{\partial t} + \dfrac{\partial (VC(x,t))}{\partial x} = K(\beta \dfrac{\partial^\gamma C(x,t)}{\partial x^\gamma} + (1-\beta)\dfrac{\partial^\gamma C(x,t)}{\partial(-x)^\gamma}), \qquad in \quad [0,400]\times[0,T], \\ C(0,t) = C(400,t) = 0, \\ C(x,0) = \delta(x-200). \end{cases} \tag{35}$$

From [14], the Fourier transform of the exact solution of (35) on infinite domain is:

$$\widehat{C}(k,t) = \exp[\frac{1}{2}(2-2\beta)(-ik)^\gamma Kt + \beta(ik)^\gamma - ikVt]. \tag{36}$$

Hejazi et al. [44] proved that domain $[0,400]$ is large enough to use the above-mentioned solution as exact solution of (35).

We will use $h=1, \tau=0.5$ for all examples. In Figure (1), we solved equation (35) for $V=0, K=1, \beta=0$ and $t=100$ for several amounts of α. In Figure (2) we solved equation (35) for $V=0, K=1, \alpha=0.4$ and $t=50$ for several different amount of β. In Figure (3), we solved equation (35) for $V=0.5, K=1, \beta=0.5$ and $\alpha=0.4$ numerous of t.

In Table (1) and Table (2), we calculate relationship between error with h and τ.

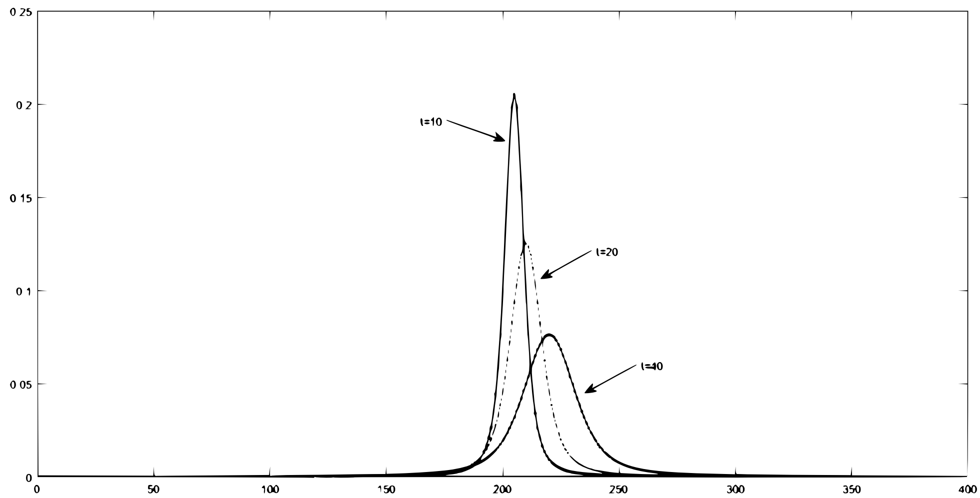

Fig. 3: Comparison numerical (symbols) and exact (solid) solution for several t

Table 1: calculating error for $V = 0.5, K = 1, \beta = 0.5, \alpha = 0.4$ **and** $\tau = 0.5$

h	Error	rate of convergence
1	8.01×10^{-4}	
0.5	3.08×10^{-4}	1.38
0.25	1.14×10^{-4}	1.43
0.125	4.48×10^{-5}	1.35
0.0625	1.73×10^{-5}	1.37

Table 2: calculating error for $V = 0.5, K = 1, \beta = 0.5, \alpha = 0.4$ **and** $h = 1$

τ	Error	rate of convergence
0.5	8.01×10^{-4}	
0.25	3.98×10^{-4}	1.01
0.125	2.03×10^{-4}	0.97
0.0625	1.02×10^{-4}	0.99

Note: Order of accuracy can be estimated as [81]

$$p_k := \frac{\log(\frac{\varepsilon_k}{\varepsilon_{k+1}})}{\log(2)}.$$

5 Conclusion

In the present study, a FVEM has been successfully used to introduce a solution for a space fractional advection-dispersion equation.The fractional derivative is considered in the Grunwald form. We proved that when the mesh grid size is small enough, the full-discretization is stable. Based on the presented results, FVEM is an accurate numerical solution for space fractional advection-dispersion equations. The accuracy of our model improves through increasing the degree of basis function ϕ_i. The above-mentioned example demonstrates that the results of our numerical method are compatible with those of the theoretical method.

References

[1] E. Barkai, R. Metzler and J. Klafter, From continuous time random walks to the fractional Fokker-Planck equation, *Phys. Rev. E*, **61**(1):132 (2000).

[2] D. A. Benson, S. W. Wheatcraft and M. M. Meerschaert, Application of a fractional advection-dispersion equation, *Water Resour. Res.* **36**(6), 1403–1412 (2000).
[3] D. A. Benson, S. W. Wheatcraft and M. M. Meerschaert, The fractional-order governing equation of Levy motion, *Water Resour. Res.* **36**(6), 1413–1423 (2000).
[4] R. Gorenflo, F. Mainardi, E. Scalas and M. Raberto, Fractional calculus and continuous-time finance iii: the diffusion limit. In Mathematical Finance, pp. 171–180, Springer, 2001.
[5] R. Metzler and J. Klafter, Boundary value problems for fractional diffusion equations, *Phys. A. 278*(1-2), 107–125 (2000).
[6] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* **339**(1), 1–77 (2000).
[7] M. Raberto, E. Scalas and F. Mainardi, Waiting-times and returns in high-frequency financial data: an empirical study, *Phys. A* **314**(1-4), 749–755 (2002).
[8] W. Wyss, The fractional Black-Scholes equation, *Fract. Calc. Appl. Anal.* **3**, 51–62 (2017).
[9] G. M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, *Phys. Rep.* **371**(6), 461—580 (2002).
[10] E. E. Adams and L. W. Gelhar, Field study of dispersion in a heterogeneous aquifer: 2. spatial moments analysis, *Water Resour. Res.* **28**(12), 3293–3307 (1992).
[11] F. Liu, P. Zhuang and K. Burrage, Numerical methods and analysis for a class of fractional advection–dispersion models, *Comput. Math. Appl.* **64**(10), 2990—3007 (2012).
[12] Y. Zhang, D. A. Benson, M. M. Meerschaert and E. M. LaBolle, Space-fractional advection-dispersion equations with variable parameters: Diverse formulas, numerical solutions, and application to the macrodispersion experiment site data, *Water Resour. Res.* **43**(5) (2007).
[13] Y. Zhang, D. A Benson and D. M. Reeves, Time and space nonlocalities underlying fractional-derivative models: Distinction and literature review of field applications, *Adv. Water Res.* **32**(4), 561—581 (2009).
[14] A. Atangana, On the stability and convergence of the time-fractional variable order telegraph equation, *J. Comput. Phys.* **293**, 104—114 (2015).
[15] A. Atangana, On the stability of iteration methods for special solution of time-fractional generalized nonlinear ZK-BBM equation, *J. Vibr. Contr.* **22**(7), 1769—1776 (2016).
[16] A. Atangana and E. F. Doungmo-Goufo, Solution of diffusion equation with local derivative with new parameter, *Therm. Sci.* **19**(1), 231—238 (2015).
[17] Z. Avazzadeh, V. R. Hosseini and W. Chen, Radial basis functions and FDM for solving fractional diffusion-wave equation, *Iranian J. Sci. Techn. (Sci.)* **38**(3), 205—212 (2014).
[18] V. Reza Hosseini,W. Chen and Z. Avazzadeh, Numerical solution of fractional telegraph equation by using radial basis functions, *Eng. Anal. Bound. Elem.* **38**, 31–39 (2014).
[19] V. Reza Hosseini, E. Shivanian and W. Chen, Local integration of 2-d fractional telegraph equation via local radial point interpolant approximation, *Eur. Phys. J. Plus*, **130**(2):33 (2015).
[20] H. Jafari, M. Dehghan and K. Sayevand, Solving a fourth-order fractional diffusion-wave equation in a bounded domain by decomposition method, *Numer. Meth. Part. Differ. Equ.* **24**(4), 1115—1126 (2008).
[21] S. Momani and Z. Odibat, Numerical solutions of the space-time fractional advection-dispersion equation, *Numer. Meth. Part. Differ. Equ.* **24**(6), 1416—1429 (2008).
[22] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, volume 198, Elsevier, 1998.
[23] X. Zhang, P. Huang, X. Feng and L. Wei, Finite element method for two-dimensional time-fractional Tricomi-type equations, *Numer. Meth. Part. Differ. Equ.*, **29**(4), 1081—1096 (2013).
[24] R. Almeida and M. Luisa Morgado, Analysis and numerical approximation of tempered fractional calculus of variations problems, *J. Comput. Appl. Math.* **361**, 1—12 (2019).
[25] S. Raja Balachandar, K. Krishnaveni, K. Kannan and S. G. Venkatesh, Analytical solution for fractional gas dynamics equation, *Nat. Acad. Sci. Lett.* 1—7 (2019).
[26] X. L. Ding and Y. L. Jiang, Analytical solutions for the multi-term time–space fractional advection–diffusion equations with mixed boundary conditions, *Nonlin. Anal. Real World Appl.* **14**(2), 1026—1033 (2013).
[27] S. Guo, L. Mei, Ying Li and Y. Sun, The improved fractional sub-equation method and its applications to the space–time fractional differential equations in fluid mechanics, *Phys. Lett. A* **376**(4), 407—411 (2012.
[28] H. Jafari, B. Mehdinejadiani and D. Baleanu, Fractional calculus for modeling unconfined groundwater, *Appl. Eng. Life Soc. Sci.* p. 119 2019.
[29] H. Jiang, F. Liu, I. Turner and K. Burrage, Analytical solutions for the multi-term time–space caputo–riesz fractional advection–diffusion equations on a finite domain, *J. Math. Anal. Appl.* **389**(2),1117—1127 (2012).
[30] L. J. Lv, J.B. Xiao, L. Zhang and L. Gao, Solutions for a generalized fractional anomalous diffusion equation, *J. Comput. Appl. Math.* **225**(1), 301—308 (2009).
[31] T. H. Ning and X. Y. Jiang, Analytical solution for the time-fractional heat conduction equation in spherical coordinate system by the method of variable separation, *Acta Mech. Sin.* **27**(6), 994—1000 (2011).
[32] Y. Povstenko, Neumann boundary-value problems for a time-fractional diffusion-wave equation in a half-plane, *Comput. Math. Appl.* **64**(10), 3183—3192 (2012).
[33] K. Al-Khaled and S. Momani An approximate solution for a fractional diffusion-wave equation using the decomposition method, *Appl. Math Comput.* **165**(2), 473—483 (2005).

[34] O. Abu Arqub and S. Momani, Numerical solutions of singular time-fractional PDES, *Appl. Eng. Life Soc. Sci.* p. 43, (2019)
[35] C. Celik and M. Duman, Crank–Nicolson method for the fractional diffusion equation with the Riesz fractional derivative, *J. Comput. Phys.* **231**(4), 1743—1750 (2012).
[36] S. Chen, F. Liu, I. Turner and V. Anh, An implicit numerical method for the two-dimensional fractional percolation equation, *Appl. Math. Comput.* **219**(9), 4322—4331 (2013).
[37] M. Cui, Compact finite difference method for the fractional diffusion equation, *J. Comput. Phys.* **228**(20), 7792-–7804 (2009).
[38] V. Daftardar-Gejji and H. Jafari, Solving a multi-order fractional differential equation using Adomian decomposition. *Appl. Math. Comput.* **189**(1), 541—548 (2007).
[39] K. Deng and W. Deng, Finite difference/predictor–corrector approximations for the space and time fractional Fokker– Planck equation, *Appl. Math. Lett.* **25**(11), 1815—1821 (2012).
[40] A. M. A. El-Sayed, S. H. Behiry and W. E. Raslan, Adomian's decomposition method for solving an intermediate fractional advection– dispersion equation, *Comput. Math. Appl.* **59**(5), 1759—1765, (2010).
[41] G. J. Fix and J. P. Roof, Least squares finite-element solution of a fractional order two-point boundary value problem, *Comput. Math. Appl.* **48**(7-8), 1017–1033 (2004).
[42] N. Ford, J. Xiao and Y. Yan, A finite element method for time fractional partial differential equations, *Fract. Calc. Appl. Anal.* **14**(3), 454—474 (2011).
[43] G. H. Gao, Z. Z. Sun and Y. Zhang, A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions, *J. Comput. Phys.* **231**(7), 2865—2879 (2012).
[44] H. Hejazi, T. Moroney and F. Liu, Stability and convergence of a finite volume method for the space fractional advection–dispersion equation, *J. Comput. Appl. Math.* **255**, 684—697 (2014).
[45] Q. Huang, G. Huang and H. Zhan, A finite element solution for the fractional advection–dispersion equation, *Adv. Water Res.* **31**(12), 1578—1589 (2008).
[46] M. Ilic, F. Liu, I. Turner and V. Anh, Numerical approximation of a fractional-in-space diffusion equation (ii)–with nonhomogeneous boundary conditions, *Fract. Calc. Appl. Anal.* **9**(4), 333—349 (2006).
[47] H. Jafari and V. Daftardar-Gejji, Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition, *Appl. Math. Comput.* **180**(2), 488-–497 (2006).
[48] [Y. Jiang and J. Ma. High-order finite element methods for time-fractional partial differential equations, *J. Comput. Appl. Math.* **235**(11), 3285—3290 (2011).
[49] Y. J. Jiang and J. T. Ma, Moving finite element methods for time fractional partial differential equations, *Scien.China Math.* **56**(6), 1287-–1300, (2013).
[50] T. A.M. Langlands and B. I. Henry, The accuracy and stability of an implicit solution method for the fractional diffusion equation, *J. Comput. Phys.* **205**(2), 719—736 (2005).
[51] C. Li, Z. Zhao and Y. Q. Chen, Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion, *Comput. Math. Appl.* **62**(3), 855—875 (2011).
[52] L. Li, D. Xu and M. Luo, Alternating direction implicit galerkin finite element method for the two-dimensional fractional diffusion-wave equation, *J. Comput. Phys.* **255**, 471—485 (2013).
[53] R. Lin, F. Liu, V. Anh and I Turner, Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation, *Appl. Math. Comput.* **212**(2), 435—445 (2009).
[54] F. Liu, C. Yang and K. Burrage, Numerical method and analytical technique of the modified anomalous subdiffusion equation with a nonlinear source term, *J. Comput. Appl. Math.* **231**(1), 160—176 (2009).
[55] J. Ma, J. Liu and Z. Zhou, Convergence analysis of moving finite element methods for space fractional differential equations, *J.Comput. Appl. Math.* **255**, 661—670 (2014).
[56] M. M. Meerschaert and C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations, *Appl. Numer. Math.* **56**(1), 80-–90 (2006).
[57] K. Moaddy, S. Momani and I. Hashim, The non-standard finite difference scheme for linear fractional PDEs in fluid mechanics, *Comput. Math. Appl.* **61**(4), 1209—1216 (2011).
[58] A. Mohebbi, M. Abbaszadeh and M. Dehghan, A high-order and unconditionally stable scheme for the modified anomalous fractional sub-diffusion equation with a nonlinear source term, *J. Comput. Phys.* **240**, 36-–48 (2013).
[59] S. Momani, A. Abu Rqayiq and D. Baleanu, A nonstandard finite difference scheme for two-sided space fractional partial differential equations, *Int. J. Bifur. Chaos* **22**(04):1250079 (2012).
[60] T. Moroney and Q. Yang, A banded preconditioner for the two-sided, nonlinear space-fractional diffusion equation, *Comput. Math. Appl.* **66**(5), 659—667 (2013).
[61] S. Saha Ray, Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method, *Commun. Nonlin. Sci. Numer. Simul.* **14**(4), 1295—1306 (2009).
[62] J. P. Roop, Computational aspects of fem approximation of fractional advection dispersion equations on bounded domains in r2, *J. Comput. Appl. Math.* **193**(1), 243—268 (2006).
[63] J. P. Roop, Numerical approximation of a one-dimensional space fractional advection–dispersion equation with boundary layer, *Comput. Math. Appl.* **56**(7), 1808-–1819 (2008).
[64] K. M. Saad, M. M. Khader, J. F. Gomez-Aguilar and D. Baleanu, Numerical solutions of the fractional fisher's type equations with Atangana-Baleanu fractional derivative by using spectral collocation methods, *Chaos* **29**(2):023116 (2019).

[65] C. Shen and M. S. Phanikumar, An efficient space-fractional dispersion approximation for stream solute transport modeling, *Adv. Water Res.* **32**(10), 1482-–1494 (2009).
[66] E. Sousa, A second order explicit finite difference method for the fractional advection diffusion equation, *Comput. Math. Appl.* **64**(10), 3141-–3152 (2012).
[67] L. Su, W. Wang and Q. Xu, Finite difference methods for fractional dispersion equations, *Appl. Math. Comput.* **216**(11), 3329—3334 (2010).
[68] N. H. Sweilam, S. Mal-Mekhlafi and A. O. Albalawi, A novel variable-order fractional nonlinear klein gordon model: A numerical approach, *Numer. Meth. Part. Differ. Equ.* https://doi.org/10.1002/num.22367 (2019).
[69] C. Tadjeran and M. M. Meerschaert, A second-order accurate numerical method for the two-dimensional fractional diffusion equation, *J. Comput. Phys.* **220**(2), 813—823 (2007).
[70] H. Wang and N. Du, Fast solution methods for space-fractional diffusion equations, *J. Comput. Appl. Math.* **255**, 376-–383 (2014).
[71] L. Wei and Y. He, Analysis of a fully discrete local discontinuous Galerkin method for time-fractional fourth-order problems, *Appl. Math. Mod.* **38**(4), 1511—1522 (2014).
[72] Q. Yang, F. Liu and I. Turner, Numerical methods for fractional partial differential equations with riesz space fractional derivatives, *Appl. Math. Model.* **34**(1), 200—218 (2010).
[73] Q. Yang, I. Turner, T. Moroney and F. Liu, A finite volume scheme with preconditioned lanczos method for two-dimensional space-fractional reaction–diffusion equations, *Appl. Mathe. Model.* **38**(15-16), 3755—3762 (2014).
[74] Q. Yu, F. Liu, I. Turner and K. Burrage, A computationally effective alternating direction method for the space and time fractional Bloch–Torrey equation in 3-d, *Appl. Math. Comput.* **219**(8), 4082—4095 (2012).
[75] S. B. Yuste and L. Acedo, An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations, *SIAM J. Numer. Anal.* **42**(5), 1862—1874 (2005).
[76] Y. Zhang and H. Ding, Improved matrix transform method for the Riesz space fractional reaction dispersion equation, *J. Comput. Appl. Math.* **260**, 266—280 (2014).
[77] J. Zhao, J. Xiao and Y. Xu Stability and convergence of an effective finite element method for multiterm fractional partial differential equations, *Abstr. Appl. Anal.* vol. 2013 (2013).
[78] Y. Zheng, C. Li and Z. Zhao, A note on the finite element method for the space-fractional advection diffusion equation, *Comput. Math. Appl.* **59**(5), 1718-–1726 (2010).
[79] P. Zhuang, F. Liu, I. Turner and Y. T. Gu, Finite volume and finite element methods for solving a onedimensional space-fractional Boussinesq equation, *Appl. Mathe. Model.* **38**(15-16), 3860—3870 (2014).
[80] S. Larsson and V. Thomee, Partial differential equations with numerical methods, volume 45, Springer, 2008.
[81] M. E. Vazquez-Cendon, Solving hyperbolic equations with finite volume methods, volume 90. Springer, 2015.
[82] M. Badr, A. Yazdani and H. Jafari, Stability of a finite volume element method for the time-fractional advection-diffusion equation, Numerical Methods for Partial Differential Equations, 2018.
[83] H. Ahmad Hejazi, Finite volume methods for simulating anomalous transport, PhD thesis, Queensland University of Technology, 2015.
[84] X. Zhang, J. W. Crawford, L. K. Deeks, M. I. Stutter, A. Glyn Bengough and I. M. Young, A mass balance based numerical method for the fractional advection-dispersion equation: theory and application, *Water Res. Res.* **41**(7) 2005.
[85] Y. Lin, J. Liu and M. Yang, Finite volume element methods: an overview on recent developments, *Int. J. Numer. Anal. Model. Ser. B* **4**(1), 14—34 (2013).
[86] G. Evans, J. Blackledge and P. Yardley, Numerical methods for partial differential equations, Springer, 2012.
[87] R. D. Richtmyer and K. W. Morton, Difference methods for boundary-value problems, 1967.