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Linearization of Nonlinear Fractional Differential Systems with Riemann-Liouville and Hadamard Derivatives⋆

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Abstract: The present paper addresses the system of nonlinear fractional differential systems involving Riemann-Liouville and Hadamard derivatives with different types of initial value conditions. However, these initial value conditions are not equivalent with each other. We construct the new linearization theorems for nonlinear fractional differential systems defined by fractional differential equations with Riemann-Liouville and Hadamard derivatives which have never been explored before.

Keywords: Fractional differential system, Riemann-Liouville derivative, Hadamard derivative, Linearization theorem.

1 Introduction

Fractional calculus, including fractional integral and fractional derivative, has recently become a topic of interest because of its wide applications in various areas of science and engineering. These phenomena in science and engineering problems can be effectively described by models using mathematical tools from fractional calculus [1,2,3,4,6,7,8,9]. It has been shown that the behaviors of many systems can be described using fractional differential systems [11,12,13,14] for instance, modeling anomalous diffusion [15], time dependent materials and process with long range dependence [16], dielectric relaxation phenomena in polymeric materials [17], transport of passive tracers carried by fluid flow in a porous medium in groundwater hydrology [18], viscoelastic behavior [19], transport dynamics in systems governed by anomalous diffusion [20], self-similar processes such as protein dynamics [21], long-time memory in financial time series [22] using fractional Langevin equations [23] etc. Recently, fractional order models of happiness [24] and love [25] have been derived. The authors claim that these models provide a better representation than the integer-order dynamical systems. In recent years, the study of fractional derivatives has gained a significant development, but the development of the theory of fractional dynamics is still poor because fractional derivative has weak singularity and does not obey the semigroup property. Thus the well-established results for ordinary dynamical system cannot always be applied in the same way [26,27,28,29].

The solution to a fractional differential system cannot define a dynamical system in the sense of semigroup property because of the history memory induced by the weakly singular kernel. However, we can still explore it in a similar manner. For example, we can define the Lyapunov exponents for the fractional differential system though borrowing ideas from the ordinary differential system [27]. Recently, Li. et al. [30] addressed fractional dynamical system with Caputo derivative and established some results. Motivated by that work, we pose the following question: Can we establish some results of fractional dynamical system with Riemann-Liouville and/or Hadamard derivatives with different initial conditions? The present paper presents an appropriate answer.

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The paper is outlined as follows: Section 2, comprises some definitions and previous results that will be used later on. In Section 3, linearization theorems of the nonlinear fractional differential systems are constructed. Conclusion is presented in the last section.

2 Preliminaries

In this section, we recall some definitions and results from the theory of ordinary dynamical system [31] and fractional calculus [1,2,3,4,6,7,8,9,10] which will be frequently used in our main analysis.

First, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ represents the set of non negative real numbers, $\mathbb{R}^n$ is the real $n-$dimensional Euclidean space, $\mathbb{Z}$ indicates the set of integer numbers, $\mathbb{Z}^+$ denotes the set of non negative integer numbers, $\mathbb{N}$ stands for the set of natural numbers, and $\mathbb{C}$ is the set of complex numbers.

Second, we recall the relationship between a vector field and a flow of diffeomorphisms [32]. We restrict the attention to Euclidean space $\Omega \subset \mathbb{R}^n$.

There are several definitions of fractional integrals and derivatives, such as Riemann-Liouville and Hadamard integrals; Grünwald-Letnikov, Riemann-Liouville, Caputo, Riesz, and Hadamard derivatives, etc. However, they are not equivalent with each other. In this paper, we only focus on two definitions i.e. Riemann-Liouville and Hadamard derivatives, which are mostly used in our analysis. Since Riesz derivative is a linear combination of the left Riemann-Liouville derivative and the right one, it is unnecessary to deal with the Riesz case.

Definition 1. *The Riemann-Liouville integral of function $f(t)$ with order $\alpha > 0$ is defined as*

$$ {}_{RL}\mathrm{D}_{t_0,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) \mathrm{d}s, \; t > t_0. \tag{1} $$

Definition 2. *The Riemann-Liouville derivative of function $f(t)$ with order $\alpha > 0$ is defined as*

$$ {}_{RL}\mathrm{D}_{t_0,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (t-s)^{n-\alpha-1} f(s) \mathrm{d}s, \tag{2} $$

where $t > t_0$, and $n-1 \leq \alpha < n \in \mathbb{Z}^+$.

Definition 3. *The Caputo derivative of function $f(t)$ with order $\alpha > 0$ is defined as*

$$ {}_{C}\mathrm{D}_{t_0,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f^{(n)}(s) \mathrm{d}s, \; t > t_0, \tag{3} $$

where $n-1 < \alpha \leq n \in \mathbb{Z}^+$.

Proposition 1. *From the above-mentioned definition and integration by parts, we obtain*

$$ {}_{RL}\mathrm{D}_{t_0,t}^{p} \left({}_{RL}\mathrm{D}_{t_0,t}^{q} x(t) \right) = {}_{RL}\mathrm{D}_{t_0,t}^{p+q} x(t) - \sum_{j=1}^{m} \left[{}_{RL}\mathrm{D}_{t_0,t}^{q-j} \right]_{t=t_0} \frac{t^{-p-j}}{\Gamma(1-p-j)}, $$

$$ {}_{RL}\mathrm{D}_{t_0,t}^{q} \left({}_{RL}\mathrm{D}_{t_0,t}^{p} x(t) \right) = {}_{RL}\mathrm{D}_{t_0,t}^{p+q} x(t) - \sum_{j=1}^{n} \left[{}_{RL}\mathrm{D}_{t_0,t}^{p-j} \right]_{t=t_0} \frac{t^{-q-j}}{\Gamma(1-q-j)}, $$

where $n-1 \leq p < n$, $m-1 \leq q < m$, $m, n \in \mathbb{N}$, so ${}_{RL}\mathrm{D}_{t_0,t}^{p}\left({}_{RL}\mathrm{D}_{t_0,t}^{q} x(t)\right)$, ${}_{RL}\mathrm{D}_{t_0,t}^{q}\left({}_{RL}\mathrm{D}_{t_0,t}^{p} x(t)\right)$, and ${}_{RL}\mathrm{D}_{t_0,t}^{p+q} x(t)$ are not generally equal to each other.

Proposition 2. *Suppose that $x(t)$ satisfies the definitions of Riemann -Liouville derivative and Caputo derivative with order α, $n-1 < \alpha < n \in \mathbb{Z}^+$, then they have the following connection*

$$ {}_{C}\mathrm{D}_{t_0,t}^{\alpha} x(t) = {}_{RL}\mathrm{D}_{t_0,t}^{\alpha} x(t) - \sum_{j=0}^{n-1} \frac{x^{(j)}(t_0)}{\Gamma(j-\alpha+1)} (t-t_0)^{j-\alpha}, \tag{4} $$

${}_{C}\mathrm{D}_{t_0,t}^{\alpha} x(t) = {}_{RL}\mathrm{D}_{t_0,t}^{\alpha} x(t)$ *holds if and only if* $x'(t_0) = x''(t_0) = \cdots = x^{(n-1)}(t_0) = 0$.

Remark. Because of Propositions 1 and 2, the Riemann-Liouville and the Caputo fractional differential operators do not satisfy the classical semigroup property. In most situations, t_0 is always set to 0. We will not specifically state this if no confusion appears.

Definition 4. *The Hadamard fractional integral of order $\alpha \in \mathbb{R}^n$ of a function $f(x)$, for all $x > a$ is defined as*

$$ {}_H\mathrm{D}_{a,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t) \frac{\mathrm{d}t}{t}, \; x > a \geq 0. \tag{5}$$

Definition 5. *The Hadamard derivative of order $\alpha \in [n-1, n)$, $n \in \mathbb{Z}^+$, of function $f(x)$ is given as follows*

$$ {}_H\mathrm{D}_{a^+}^{\alpha} f(x) = \delta^n \left({}_H\mathrm{D}_{a^+}^{-(n-\alpha)} f(x) \right), \tag{6}$$

where, $x > a$, $\delta = x\frac{\mathrm{d}}{\mathrm{d}x}$, $n-1 \leq \alpha < n \in \mathbb{Z}^+$.

Remark. The kernel in Riemann-Liouville integral has the form $(x-t)$ whenever Hadamard integral has the form of $\ln \frac{x}{t}$. Second, the Riemann-Liouville derivative has the operator $\frac{\mathrm{d}^n}{\mathrm{d}t^n}$, while the Hadamard derivative has $\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n$ operator whose construction is well suited to the case of the half-axis.

Theorem 1. *If $n-1 < \alpha < n$, $n \in \mathbb{N}$, ${}_C\mathrm{D}_{0,t}^{\alpha} x(t) \geq {}_C\mathrm{D}_{0,t}^{\alpha} y(t)$, and $x^{(k)}(0) \geq y^{(k)}(0)$, $(k = 0, 1, \cdots, n-1)$, then $x(t) \geq y(t)$. Parallel , if $n-1 < \alpha < n$, $n \in \mathbb{N}$, ${}_{RL}\mathrm{D}_{0,t}^{\alpha} x(t) \geq {}_{RL}\mathrm{D}_{0,t}^{\alpha} y(t)$, and ${}_{RL}\mathrm{D}_{0,t}^{\alpha-k-1} x(t)|_{t=0} \geq {}_{RL}\mathrm{D}_{0,t}^{\alpha-k-1} y(t)|_{t=0}$, $(k = 0, 1, \cdots, n-1)$, then $x(t) \geq y(t)$.*

Proof. The proof of this theorem can be referred to [33].

Definition 6. *The Mittag -Leffler function of two parameters is defined by*

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \; \alpha, \beta > 0. \tag{7}$$

Now, we consider the initial value problems (IVPs) of FDEs are in Riemann-Liouville derivative sense

$$ \begin{cases} {}_{RL}\mathrm{D}_{0,t}^{\alpha} y(t) & = f(y), \; t > 0, \\ {}_{RL}\mathrm{D}_{0,t}^{\alpha-1} y(t)\Big|_{t=0} & = y_0, \end{cases} \tag{8}$$

or equivalently,

$$ \begin{cases} {}_{RL}\mathrm{D}_{0,t}^{\alpha} y(t) = f(y), \; t > 0, \\ \lim_{t\to 0^+} \left[t^{1-\alpha} y(t)\right]_{t=0} = \frac{y_0}{\Gamma(\alpha)}, \end{cases} \tag{9}$$

and in Hadamard derivative sense

$$ \begin{cases} {}_H\mathrm{D}_{a^+}^{\alpha} y(t) & = f(y), \; t > a > 0, \\ {}_H\mathrm{D}_{a^+}^{\alpha-1} y(t)\Big|_{t=a} & = y_a, \end{cases} \tag{10}$$

or equivalently,

$$ \begin{cases} {}_H\mathrm{D}_{a^+}^{\alpha} y(t) & = f(y), \; t > a > 0, \\ \left(\ln \frac{t}{a}\right)^{1-\alpha} y(t)\Big|_{t=a} & = \frac{y_a}{\Gamma(\alpha)}, \end{cases} \tag{11}$$

respectively, where $0 < \alpha < 1$, $f(y) = (f_1(y), \cdots, f_n(y))^T$, $y \in \mathbb{R}^n$. We always assume that they have unique solutions respectively.

Lemma 1. *[4] If $f(y)$ is continuous, the IVP (8) is equivalent to the following nonlinear Volterra integral equation of the second kind*

$$ y(t) = \frac{y_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) \mathrm{d}\xi. \tag{12}$$

In other words, every solution of the Volterra integral equation (12) is also the solution of IVP (8) and vise versa.

Lemma 2. *[4,5] The initial value problem*

$$\begin{cases} {}_{RL}\mathrm{D}_{0,t}^{\alpha}u(t) &= f(t),\ t>0, \\ t^{1-\alpha}u(t)\big|_{t=0} &= u_0, \end{cases} \tag{13}$$

has following integral form

$$u(t) = u_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \tag{14}$$

where $0<\alpha<1$ *and* $q \in C([0,T]\times\mathbb{R})$.

Lemma 3. *[4,5] The initial value problem*

$$\begin{cases} {}_{RL}\mathrm{D}_{0,t}^{\alpha}u(t) = f(t,u), \\ u(a) \quad\quad\ = b,\ t>a>0, \end{cases} \tag{15}$$

has unique solution in $C(\mathbb{R}^+)\cap L^1_{loc}(\mathbb{R}^+)$ *given by*

$$u(t) = \left(b - \frac{1}{\Gamma(\alpha)}\int_0^a (a-\tau)^{\alpha-1} f(\tau,u(\tau))\mathrm{d}\tau\right)\frac{t^{\alpha-1}}{a^{\alpha-1}} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,u(\tau))\mathrm{d}\tau, \tag{16}$$

where $0<\alpha<1,\ f(t,u)\in C(\mathbb{R}^+)\cap L^1_{loc}(\mathbb{R}^+)$ *for all* $(a,b)\in\mathbb{R}^+\times\mathbb{R}$.

Lemma 4. *[4,10] Let G be an open set in* $\mathbb{R}$ *and let* $f:(a,b]\times G\to\mathbb{R}$ *be a function such that* $f(y)\in C_{\gamma,\ln}[a,b],\ 0\le\gamma<1$, *for any* $y\in G,\ y(x)\in C_{1-\alpha,\ln}[a,b]$. *Then the IVP (10) is equivalent to the following nonlinear integral equation*

$$y(t) = \frac{y_a}{\Gamma(\alpha)}\left(\ln\frac{t}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{\omega}\right)^{\alpha-1} f(\omega)\frac{\mathrm{d}\omega}{\omega}. \tag{17}$$

Lemma 5. *[9] The initial value problem*

$$\begin{cases} {}_{H}\mathrm{D}_{a^+}^{\alpha}u(t) = f(t),\ a<t\le b \\ u(t_0) = u_0, \quad\quad\ a<t_0\le b, \end{cases} \tag{18}$$

has unique solution. Then

$$u(t) = \left(u_0 - \frac{1}{\Gamma(\alpha)}\int_a^{t_0}\left(\ln\frac{t_0}{s}\right)^{\alpha-1} f(s)\frac{\mathrm{d}s}{s}\right)\left(\ln\frac{t_0}{a}\right)^{1-\alpha}\left(\ln\frac{t}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1} f(s)\frac{\mathrm{d}s}{s}, \tag{19}$$

where $0<\alpha<1$ *and* $u(t)\in C_{1-\alpha,\ln}[a,b]$.

3 The linerarization theorems

Some authors [34,35,36,37] investigated the linearization theorems of dynamical systems with integer orders. However, this section addresses the linearization theorems of fractional dynamical system defined by fractional differential equations with Riemann-Liouville and Hadamard derivatives.

Consider the homogenous linear system of FDEs in Riemann-Liouville derivative sense

$$\begin{cases} {}_{RL}\mathrm{D}_{0,t}^{\alpha}y(t) &= Ay(t),\ t>0, \\ {}_{RL}\mathrm{D}_{0,t}^{\alpha-1}y(t)\Big|_{t=0} &= y_0, \end{cases} \tag{20}$$

and in Hadamard derivative sense

$$\begin{cases} {}_{H}\mathrm{D}_{a^+}^{\alpha}y(t) &= Ay(t),\ t>a>0, \\ {}_{H}\mathrm{D}_{a^+}^{\alpha-1}y(t)\big|_{t=a} &= y_a, \end{cases} \tag{21}$$

where A is an $n\times n$ constant matrix, $0<\alpha<1$ and $y(t)\in\mathbb{R}^n$.

Definition 7. *The autonomous systems (20) and (21) are said to be (i) stable if and only if for any y_0 , y_a. Then, there exists $\varepsilon > 0$ such that $\|y(t)\| \leq \varepsilon$ for $t \geq 0$ respectively and (ii) asymptotically stable if and only if* $\lim_{t\to\infty} \|y(t)\| = 0$.

Definition 8. *If all the eigenvalues $\lambda(A)$ of A satisfy: $|\lambda(A)| \neq 0$ and $|\arg(\lambda(A))| \neq \frac{\alpha\pi}{2}$, the origin O of the autonomous systems (20) and (21) are called a hyperbolic equilibrium point.*

Now we consider the autonomous nonlinear differential system with Riemann-Liouville derivative

$$\begin{cases} {}_{RL}\mathrm{D}^{\alpha}_{0,t} y(t) & = f(y(t)),\ t > 0, \\ {}_{RL}\mathrm{D}^{\alpha-1}_{0,t} y(t)\Big|_{t=0} & = y_0, \end{cases} \tag{22}$$

and Hadamard derivative

$$\begin{cases} {}_{H}\mathrm{D}^{\alpha}_{a^+} y(t) & = f(y(t)),\ t > a > 0, \\ {}_{H}\mathrm{D}^{\alpha-1}_{a^+} y(t)\big|_{t=a} & = y_a, \end{cases} \tag{23}$$

where $0 < \alpha < 1$ and $f(y)$ is continuous function.

Definition 9. *The $y_{eq} = 0$ is said to be equilibrium point of fractional differential systems (22) and (23) if and only if $f(y_{eq}) = 0$.*

Definition 10. *Suppose that $y_{eq} = 0$ is an equilibrium points of the systems (22) and (23) and all the eigenvalues $\lambda(Df(y_{eq}))$ of the linearized matrix $Df(y_{eq})$ at the equilibrium point y_{eq} satisfy: $\left|\lambda(Df(y_{eq}))\right| \neq 0$ and $\left|\lambda(Df(y_{eq}))\right| \neq \frac{\pi\alpha}{2}$, then we call a hyperbolic equilibrium point.*

Definition 11.

(1) The equilibrium points $y_{eq} = 0$ of systems (22) and (23) are said to be: (i) locally stable if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left\|y(t) - y_{eq}\right\| < \varepsilon$ holds for all $y_0 \in \{z : \left\|z - y_{eq}\right\| < \delta\}$ and for all $t > 0$ and $t > a$ respectively; (ii) locally asymptotically stable if the equilibrium point is locally stable and $\lim_{t\to+\infty} y(t) = y_{eq}$.

(2) Consider $y(t)$ and $\tilde{y}(t)$ are the solutions of systems (22) and (23) with initial values $y_0(t)$ and $\tilde{y}_0(t)$ respectively. The solution $y(t)$ is said to be: (i) locally stable if for all $\varepsilon > 0$, there exist a $\delta > 0$ such that $\|y(t) - \tilde{y}(t)\| < \varepsilon$ holds for all $\|y(t) - \tilde{y}(t)\| < \delta$ and for all $t \geq 0$ and $t \geq a$, respectively; (ii) locally asymptotically stable if the equilibrium point is locally stable and $\lim_{t\to+\infty}(y(t) - \tilde{y}(t)) = 0$.

Suppose $f(x)$ and $g(y)$ are continuous vector fields defined on $U, V \subseteq \mathbb{R}^n$ and generate flows $\psi_{t,f} : U \to U, \psi_{t,g} : V \to V$, respectively.

Definition 12. *If there is a homeomorphism $h : U \to V$, satisfying: $h \circ \psi_{t,f}(x) = \psi_{t,g} \circ h(x)$, $x \in \delta(x_0, r) \subset U$, $x_0 \in U$, $f(x)$ and $g(y)$ are locally topologically equivalent. If the above relation holds in the whole space U, then they are globally topologically equivalent.*

Next, we give the linearization theorems of fractional differential equation with Riemann-Liouville and Hadamard derivatives. The equilibrium y_{eq} is always in the origin.

Theorem 2. *If the origin O is a hyperbolic equilibrium point of Riemann-Liouville fractional differential system (22), vector field $f(y)$ is topologically equivalent with its linearization vector field $Vf(0)y$ in the neighbourhood $\delta(0)$ of the origin O.*

Proof. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of $Vf(0)$, $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$, $i = 1, 2, \cdots, n_1$, $|\arg(\lambda_i)| < \frac{\alpha\pi}{2}$, $i = n_1 + 1, n_1 + 2, \cdots, n$. Let $n = n_1 + n_2$, then by non singular linear transformation $T : \mathbb{R}^n \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $y(t) \to g(t) = (g_1(t), g_2(t))$, $(g_1(t) \in \mathbb{R}^{n_1},\ g_2(t) \in \mathbb{R}^{n_2})$, fractional differential system (22) can be transformed into the following system

$$\begin{cases} {}_{RL}\mathrm{D}^{\alpha}_{0,t} g_1(t) = A_1 g_1(t) + F_1(g_1(t), g_2(t)), \\ {}_{RL}\mathrm{D}^{\alpha}_{0,t} g_2(t) = A_2 g_2(t) + F_2(g_1(t), g_2(t)), \end{cases} \tag{24}$$

where the eigenvalues of A_1, A_2 are $\lambda_1, \lambda_2, \cdots, \lambda_{n_1}$ and $\lambda_{n_1+1}, \lambda_{n_1+2}, \cdots, \lambda_n$, respectively. Moreover, $\|E_{\alpha,\alpha}(A_1)\| = a$, $(\|E_{\alpha,\alpha}(A_2)\|)^{-1} = b$. Without loss of generality, suppose $b < \frac{1}{a}$, $F_1, F_2 = o(\|g_1(t)\| + \|g_2(t)\|)$ as $(g_1(t), g_2(t)) \to 0$.

The solution $\psi_t(g) = (g_1(t), g_2(t))$ of (24) can be written as

$$
\begin{aligned}
g_1(t) &= g_1^0 t^{\alpha-1} E_{\alpha,\alpha}(A_1 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A_1(t-\tau)^\alpha)\, F_1(g_1(\tau), g_2(\tau))\, \mathrm{d}\tau \\
&= g_1^0 t^{\alpha-1} E_{\alpha,\alpha}(A_1 t^\alpha) + G_1(t, g_1^0, g_2^0),
\end{aligned}
$$

$$
\begin{aligned}
g_2(t) &= g_2^0 t^{\alpha-1} E_{\alpha,\alpha}(A_2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A_2(t-\tau)^\alpha)\, F_2(g_1(\tau), g_2(\tau))\, \mathrm{d}\tau \\
&= g_2^0 t^{\alpha-1} E_{\alpha,\alpha}(A_2 t^\alpha) + G_2(t, g_1^0, g_2^0).
\end{aligned}
$$

Our theorem refers only to the neighbourhood $\delta(0)$ of the origin O, when $(g_1^0, g_2^0) \notin \delta(0)$, we set $F_1(g_1^0, g_2^0) \equiv 0$, $F_2(g_1^0, g_2^0) \equiv 0$, consequently, $G_1, G_2 \equiv 0, (g_1^0, g_2^0) \notin \delta(0)$. Thus omit the case when $(g_1^0, g_2^0) \in \delta(0)$

Consider the homogenous linear system of (24)

$$
\begin{cases} {}_{RL}\mathrm{D}_{0,t}^{\alpha} w_1(t) = A_1 w_1(t), \\ {}_{RL}\mathrm{D}_{0,t}^{\alpha} w_2(t) = A_2 w_2(t), \end{cases} \tag{25}
$$

where $w(t) = (w_1(t), w_2(t)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $w_1(t) \in \mathbb{R}^{n_1}$, $w_2(t) \in \mathbb{R}^{n_2}$. The solution $\varphi_t(w)(t) = (w_1(t), w_2(t))$ of (25) can be expressed as

$$
\begin{cases} w_1(t) = w_1^0 t^{\alpha-1} E_{\alpha,\alpha}(A_1 t^\alpha), \\ w_2(t) = w_2^0 t^{\alpha-1} E_{\alpha,\alpha}(A_2 t^\alpha), \end{cases} \tag{26}
$$

If we can find a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$, satisfying $h \circ \psi_t = \varphi_t \circ h$, then the theorem is true. For this, we divide the proof into three steps.

Step 1: For $t = 1$, we find a continuous map $h_1 : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$
\theta_s \circ \varphi_t \circ h_1 = h_1 \circ \theta_s \circ \psi_1, \; s \in (0,1). \tag{27}
$$

Suppose that h_1 which satisfies (27) is expressed by the following coordinate transformation

$$
w_1^0 = U(g_1^0, g_2^0), \; w_2^0 = V(g_1^0, g_2^0). \tag{28}
$$

By (27) and (28), we have

$$
\theta_s E_{\alpha,\alpha}(A_1) U(g_1^0, g_2^0) = U(\theta_s(g_1^0 E_{\alpha,\alpha}(A_1) + G_1(1, g_1^0, g_2^0)), \theta_s(g_2^0 E_{\alpha,\alpha}(A_2) + G_2(1, g_1^0, g_2^0))), \tag{29}
$$

$$
\theta_s E_{\alpha,\alpha}(A_2) V(g_1^0, g_2^0) = V(\theta_s(g_1^0 E_{\alpha,\alpha}(A_1) + G_1(1, g_1^0, g_2^0)), \theta_s(g_2^0 E_{\alpha,\alpha}(A_2) + G_2(1, g_1^0, g_2^0))), \tag{30}
$$

So V satisfies the following equation

$$
V(g_1^0, g_2^0) = (\theta_s)^{-1} (E_{\alpha,\alpha}(A_2))^{-1} V(\theta_s(g_1^0 E_{\alpha,\alpha}(A_1) + G_1(1, g_1^0, g_2^0)), \theta_s(g_2^0 E_{\alpha,\alpha}(A_2) + G_2(1, g_1^0, g_2^0))), \tag{31}
$$

Next, we use successive approximations to obtain solution to (31). Put

$$
\begin{cases} V_0(g_1^0, g_2^0) = g_2^0, \\ V_k(g_1^0, g_2^0) = (\theta_s)^{-1} (E_{\alpha,\alpha}(A_2))^{-1} V_{k-1}(\theta_s(g_1^0 E_{\alpha,\alpha}(A_1) + G_1(1, g_1^0, g_2^0)), \theta_s(g_2^0 E_{\alpha,\alpha}(A_2) + G_2(1, g_1^0, g_2^0))), \end{cases} \tag{32}
$$

for $k = 1, 2, \cdots$. We get

$$
\begin{aligned}
V_1(g_1^0, g_2^0) &= (\theta_s)^{-1} (E_{\alpha,\alpha}(A_2))^{-1} (\theta_s(g_2^0 E_{\alpha,\alpha}(A_2) + G_2(1, g_1^0, g_2^0))), \\
&= g_2^0 + (E_{\alpha,\alpha}(A_2))^{-1} G_2(1, g_1^0, g_2^0).
\end{aligned}
$$

Let $\delta > 0$ enough small, then it is easily known that

$$
r = b \left\| \theta \right\|^{-1} \left(2 \max\{ a \left\| \theta \right\|, 2c \left\| \theta \right\|, \| E_{\alpha,\alpha}(A_2) \| \left\| \theta \right\| \} \right)^{\delta} < 1. \tag{33}
$$

Since $G_2 = o(\|g_1^0\| + \|g_2^0\|)$ as $g_1^0, g_2^0 \to 0$, there exists a constant $L > 0$ satisfying

$$\|V_1(g_1^0,g_2^0) - V_0(g_1^0,g_2^0)\| < Lr\left(\|g_1^0\| + \|g_2^0\|\right)^{\delta}. \tag{34}$$

Suppose $\|V_k(g_1^0,g_2^0) - V_{k-1}(g_1^0,g_2^0)\| < Lr^k\left(\|g_1^0\| + \|g_2^0\|\right)^{\delta}$. One has

$$\begin{aligned}\|V_{k+1}(g_1^0,g_2^0) - V_k(g_1^0,g_2^0)\| = {} & \|(\theta_s)^{-1}(E_{\alpha,\alpha}(A_2))^{-1}V_k(\theta_s(g_1^0E_{\alpha,\alpha}(A_1)+G_1(1,g_1^0,g_2^0)),\theta_s(g_2^0E_{\alpha,\alpha}(A_2)\\ & +G_2(1,g_1^0,g_2^0))) - (\theta_s)^{-1}(E_{\alpha,\alpha}(A_2))^{-1}V_{k-1}(\theta_s\times(g_1^0E_{\alpha,\alpha}(A_1)\\ & +G_1(1,g_1^0,g_2^0)),\theta_s(g_2^0E_{\alpha,\alpha}(A_2)+G_2(1,g_1^0,g_2^0)))\|\end{aligned}$$

$$\begin{aligned}\|V_{k+1}(g_1^0,g_2^0) - V_k(g_1^0,g_2^0)\| \le {} & \|(E_{\alpha,\alpha}(A_2))\|^{-1}\|\theta\|^{-1}Lr^k(\|\theta_s(g_1^0E_{\alpha,\alpha}(A_1)+G_1(1,g_1^0,g_2^0))\|\\ & +\theta_s(g_2^0E_{\alpha,\alpha}(A_2)+G_2(1,g_1^0,g_2^0)))^{\delta}\\ \le {} & Lr^kb\|\theta\|^{-1}(\|\theta\|(a\|g_1^0\| + \|g_1^0E_{\alpha,\alpha}(A_1)\|\times\|g_2^0\|)+2c(\|g_1^0\|+\|g_2^0\|)\|\theta\|)^{\delta}\\ \le {} & Lr^kb\|\theta\|^{-1}(2\max\{a\|\theta\|,2c\|\theta\|,\|E_{\alpha,\alpha}(A_2)\|\|\theta\|\})^{\delta}\left(\|g_1^0\|+\|g_2^0\|\right)^{\delta}\\ \le {} & Lr^{k+1}\left(\|g_1^0\|+\|g_2^0\|\right)^{\delta}.\end{aligned}$$

where $b < \|\theta\| < \frac{1}{a}$.

So $V_k(g_1^0,g_2^0)$ uniformly converges to a continuous function $V(g_1^0,g_2^0)$ and we get

$$\begin{aligned}V(g_1^0,g_2^0) &= V_0(g_1^0,g_2^0) + \sum_{k=1}^{\infty}[V_k(g_1^0,g_2^0) - V_{k-1}(g_1^0,g_2^0)]\\ &= g_2^0 + V^*(g_1^0,g_2^0),\end{aligned}$$

where $V^*(g_1^0,g_2^0) = o(\|g_1^0\| + \|g_2^0\|)$.

Furthermore, U satisfies the following equation

$$\begin{aligned}\theta_sE_{\alpha,\alpha}(A_1)U(g_1^0,g_2^0) &= U(\theta_s(g_1^0E_{\alpha,\alpha}(A_1)+G_1(1,g_1^0,g_2^0)),\theta_s(g_2^0E_{\alpha,\alpha}(A_2)+G_2(1,g_1^0,g_2^0))),\\ &= U(u_1,u_2),\end{aligned} \tag{35}$$

and

$$\begin{cases} u_1 = \theta_s(g_1^0E_{\alpha,\alpha}(A_1)+G_1(1,g_1^0,g_2^0)),\\ u_2 = \theta_s(g_2^0E_{\alpha,\alpha}(A_2)+G_2(1,g_1^0,g_2^0)).\end{cases} \tag{36}$$

We can prove that there exists the inverse transformation of (36), namely

$$\begin{cases} g_1^0 = (\theta_s)^{-1}(E_{\alpha,\alpha}(A_1))^{-1}u_1 + P_1((\theta_s)^{-1}u_1,(\theta_s)^{-1}u_2),\\ g_2^0 = (\theta_s)^{-1}(E_{\alpha,\alpha}(A_2))^{-1}u_2 + P_2((\theta_s)^{-1}u_1,(\theta_s)^{-1}u_2).\end{cases} \tag{37}$$

So, function U satisfies

$$\begin{aligned}U(u_1,u_2) = {} & \theta_sE_{\alpha,\alpha}(A_1)U((E_{\alpha,\alpha}(A_1))^{-1}(\theta_s)^{-1}u_1 + P_1((\theta_s)^{-1}u_1,(\theta_s)^{-1}u_2),(E_{\alpha,\alpha}(A_2))^{-1}(\theta_s)^{-1}u_2\\ & +P_2((\theta_s)^{-1}u_1,(\theta_s)^{-1}u_2)).\end{aligned} \tag{38}$$

By successive approximation similar to function V, we obtain the solution of $U(g_1^0,g_2^0)$ satisfying

$$U(g_1^0,g_2^0) = g_1^0 + U^*(g_1^0,g_2^0), \tag{39}$$

where $U^*(g_1^0,g_2^0) = o(\|g_1^0\| + \|g_2^0\|)$.

If $b > \frac{1}{a}$, namely $\|(E_{\alpha,\alpha}(A_1))\| = \|(E_{\alpha,\alpha}(A_2))\|$, similar to (29), we can also use successive approximation to obtain the solution of (30). If $b > \frac{1}{a}$, the process is similar to $b < \frac{1}{a}$. As a result we get a continuous map h_1 satisfying $h_1(0,0) = (0,0)$, and when $(g_1^0,g_2^0) \notin \delta(0)$, $h_1(g_1^0,g_2^0) = (g_1^0,g_2^0)$. Moreover, the uniqueness is easily proved.

Step 2: h_1 is a homeomorphism, Based on step 1, there also exists a continuous map h_2 satisfying $h_2\circ\theta_s\circ\varphi_1 = \theta_s\circ\psi_1\circ h_2$.

$$h_1\circ h_2\circ\theta_s\circ\varphi_1 = h_1\circ\theta_s\circ\psi_1\circ h_2 = \theta_s\circ\psi_1\circ h_1\circ h_2, \tag{40}$$

$$\theta_s \circ \psi_1 \circ h_2 \circ h_1 = h_2 \circ \theta_s \circ \psi_1 \circ h_1 = h_2 \circ h_1 \circ \theta_s \circ \psi_1. \tag{41}$$

By the uniqueness of h_1 and h_2, so $(h_1)^{-1} = h_2$, and $(h_1)^{-1}$ is continuous. Therefore, h_1 is a homeomorphism.

Step 3: Let

$$h = \int_0^1 \varphi_s \circ h_1 \circ (\psi_1)^{-1} ds. \tag{42}$$

For $t \in \mathbb{R}^+$, similar to Step 2, we can prove h is a homeomorphism.

$$\begin{aligned}
\varphi_t \circ \theta_t \circ h &= \int_t^{1+t} \varphi_t \circ \theta_t \circ \varphi_{s-t} \circ h_1 \circ (\psi_{s-t})^{-1} \mathrm{d}s \\
&= \int_t^{1+t} \varphi_s \circ h_1 \circ (\psi_s)^{-1} \circ \psi_t \circ \theta_t \circ \psi_{s-t} \circ (\psi_{s-t})^{-1} \mathrm{d}s \\
&= \int_t^{1} \varphi_s \circ h_1 \circ (\psi_s)^{-1} \mathrm{d}s \circ \psi_t \circ \theta_t + \int_1^{1+t} \varphi_s \circ h_1 \circ (\psi_s)^{-1} \mathrm{d}s \circ \psi_t \circ \theta_t \\
&= \int_t^{1} \varphi_s \circ h_1 \circ (\psi_s)^{-1} \mathrm{d}s \circ \psi_t \circ \theta_t + \int_0^{t} \varphi_{s+1} \circ h_1 \circ (\psi_{s+1})^{-1} \mathrm{d}s \circ \psi_t \circ \theta_t \\
&= \int_t^{1} \varphi_s \circ h_1 \circ (\psi_s)^{-1} \mathrm{d}s \circ \psi_t \circ \theta_t + \int_0^{t} \varphi_s \circ \theta_s \circ \varphi_1 \circ h_1 \circ (\psi_1)^{-1} \circ (\theta_s)^{-1} \circ (\psi_s)^{-1} \mathrm{d}s \circ \psi_t \circ \theta_t \\
&= \int_t^{1} \varphi_s \circ h_1 \circ (\psi_s)^{-1} \mathrm{d}s \circ \psi_t \circ \theta_t + \int_0^{t} \varphi_s \circ h_1 \circ (\psi_s)^{-1} \mathrm{d}s \circ \psi_t \circ \theta_t \\
&= \int_0^{1} \varphi_s \circ h_1 \circ (\psi_s)^{-1} \mathrm{d}s \circ \psi_t \circ \theta_t = h \circ \psi_t \circ \theta_t.
\end{aligned}$$

Thus, the conclusion is true.

Remark.

(i) The above theorem is the fractional form of the Hartman theorem [34,35,36,37].
(ii) The condition hyperbolic equilibrium is necessary. If the origin O is not a hyperbolic equilibrium then the conclusion does not hold.

Lemma 6. *If $n-1 < \alpha < n \in \mathbb{N}$, ${}_H\mathrm{D}^{\alpha}_{a,t}x(t) \geq {}_H\mathrm{D}^{\alpha}_{a,t}y(t)$, and ${}_H\mathrm{D}^{\alpha-k-1}_{a,t}x(t)\big|_{t=a} \geq {}_H\mathrm{D}^{\alpha-k-1}_{a,t}y(t)\big|_{t=a}$, for $k = 0, 1, \cdots, n-1$, then $x(t) \geq y(t)$.*

Proof. Setting ${}_H\mathrm{D}^{\alpha}_{a,t}x(t) = \sigma(t) + {}_H\mathrm{D}^{\alpha}_{a,t}y(t)$, and taking the Mellin transform [4] on both sides, one has

$$(-s)^{\alpha} (\mathscr{M}x)(s) = (\mathscr{M}\sigma)(s) + (-s)^{\alpha}(\mathscr{M}y)(s)$$

dividing by $(-s)^{\alpha}$ taking the inverse Mellin transform in both sides, one can get

$$x(t) = y(t) + \mathscr{M}^{-1}((-s)^{\alpha}\mathscr{M}(\sigma)(s))$$

The right hand side of the above equality is positive. This completes the proof.

Theorem 3. *If the origin O is a hyperbolic equilibrium point of Hadamard fractional differential system (23), then vector field $f(y)$ is topologically equivalent with its linearization vector field $Vf(y)$ in the neighbourhood $\delta(0)$ of the origin O.*

Proof. Fractional differential system (23) can be transformed into the following system

$$\begin{cases} {}_H\mathrm{D}^{\alpha}_{a^+}u_1(t) = A_1u_1(t) + F_1(u_1(t),u_2(t)), \\ {}_H\mathrm{D}^{\alpha}_{a^+}u_2(t) = A_2u_2(t) + F_2(u_1(t),u_2(t)), \end{cases} \tag{43}$$

where the eigenvalues of A_1,A_2 are $\lambda_1,\lambda_2,\cdots,\lambda_{n_1}$ and $\lambda_{n_1+1},\lambda_{n_1+2},\cdots,\lambda_n$, respectively. Let $n = n_1+n_2$, then by non singular linear transformation $T:\mathbb{R}^n \to \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, $y(t)\to u(t) = (u_1(t),u_2(t))$, $(u_1(t)\in\mathbb{R}^{n_1}, u_2(t)\in\mathbb{R}^{n_2})$. Moreover, $\left(\log\frac{1}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_1\left(\log\frac{1}{a}\right)^{\alpha}\right] = b$ and $\left(\left(\log\frac{1}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_2\left(\log\frac{1}{a}\right)^{\alpha}\right]\right)^{-1} = c$.

Without loss of generality, suppose $c<\frac{1}{b}, F_1,F_2 = o(\|u_1(t)\|+\|u_2(t)\|)$ as $u_1(t),u_2(t)\to 0$.

The solution $\psi_t(u) = (u_1(t),u_2(t))$ of (43) can be written as

$$\begin{aligned} u_1(t) &= u_1^0\left(\log\frac{t}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_1\left(\log\frac{t}{a}\right)^{\alpha}\right] + \int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_1\left(\log\frac{t}{\tau}\right)^{\alpha}\right]F_1(u_1(\tau),u_2(\tau))\mathrm{d}\tau \\ &= u_1^0\left(\log\frac{t}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_1\left(\log\frac{t}{a}\right)^{\alpha}\right] + S_1(t,u_1^0,u_2^0), \end{aligned}$$

$$\begin{aligned} u_2(t) &= u_2^0\left(\log\frac{t}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_2\left(\log\frac{t}{a}\right)^{\alpha}\right] + \int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_2\left(\log\frac{t}{\tau}\right)^{\alpha}\right]F_2(u_1(\tau),u_2(\tau))\mathrm{d}\tau \\ &= u_2^0\left(\log\frac{t}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_2\left(\log\frac{t}{a}\right)^{\alpha}\right] + S_2(t,u_1^0,u_2^0), \end{aligned}$$

Our this theorem refers only to the neighbourhood $\delta(0)$ of the origin O, when $(u_1^0,u_2^0)\notin\delta(0)$, we set $F_1(u_1^0,g_2^0)\equiv 0$, $F_2(u_1^0,u_2^0)\equiv 0$, consequently, $S_1,S_2\equiv 0, (u_1^0,u_2^0)\notin\delta(0)$. So we omit the case when $(u_1^0,u_2^0)\in\delta(0)$.

Consider the homogenous linear system of (43)

$$\begin{cases} {}_H\mathrm{D}^{\alpha}_{a^+}w_1(t) = A_1w_1(t), \\ {}_H\mathrm{D}^{\alpha}_{a^+}w_2(t) = A_2w_2(t), \end{cases} \tag{44}$$

where $w(t) = (w_1(t),w_2(t))\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, $w_1(t)\in\mathbb{R}^{n_1}$, $w_2(t)\in\mathbb{R}^{n_2}$. The solution $\varphi_t(w)(t) = (w_1(t),w_2(t))$ of (44) can be expressed as

$$\begin{cases} w_1(t) = w_1^0\left(\log\frac{t}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_1\left(\log\frac{t}{a}\right)^{\alpha}\right], \\ w_2(t) = w_2^0\left(\log\frac{t}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_2\left(\log\frac{t}{a}\right)^{\alpha}\right], \end{cases}$$

If we can find a homeomorphism $h:\mathbb{R}^n\to\mathbb{R}^n$, satisfying $h\circ\psi_t = \varphi_t\circ h$, then the theorem is true. For this, we divide the proof into three steps.

Step 1: For $t=1$, we find a continuous map $h_1:\mathbb{R}^n\to\mathbb{R}^n$ satisfying

$$\theta_s\circ\varphi_t\circ h_1 = h_1\circ\theta_s\circ\psi_1,\ s\in(0,1). \tag{45}$$

Suppose that h_1 which satisfies (45) is expressed by the following coordinate transformation

$$w_1^0 = U(u_1^0,u_2^0),\ w_2^0 = V(u_1^0,u_2^0). \tag{46}$$

By (45) and (46), we have

$$\begin{aligned} \theta_s(\log\frac{1}{a})^{\alpha-1}E_{\alpha,\alpha}[A_1(\log\frac{1}{a})^{\alpha}]U(u_1^0,u_2^0) = U(&\theta_s(u_1^0(\log\frac{1}{a})^{\alpha-1}E_{\alpha,\alpha}[A_1(\log\frac{1}{a})^{\alpha}] + S_1(1,u_1^0,u_2^0)), \theta_s(u_2^0(\log\frac{1}{a})^{\alpha-1} \\ &\times E_{\alpha,\alpha}[A_2(\log\frac{1}{a})^{\alpha}] + S_2(1,u_1^0,u_2^0))), \end{aligned} \tag{47}$$

$$\begin{aligned} \theta_s(\log\frac{1}{a})^{\alpha-1}E_{\alpha,\alpha}[A_2(\log\frac{1}{a})^{\alpha}]V(u_1^0,u_2^0) = V(&\theta_s(u_1^0(\log\frac{1}{a})^{\alpha-1}E_{\alpha,\alpha}[A_1(\log\frac{1}{a})^{\alpha}] + S_1(1,u_1^0,u_2^0)), \theta_s(u_2^0(\log\frac{1}{a})^{\alpha-1} \\ &\times E_{\alpha,\alpha}[A_2(\log\frac{1}{a})^{\alpha}] + S_2(1,u_1^0,u_2^0))). \end{aligned} \tag{48}$$

Next, we use successive approximations to obtain solution to (48). Put

$$\begin{cases} V_0(u_1^0,u_2^0)=u_2^0, \\ V_k(u_1^0,u_2^0)=(\theta_s)^{-1}\left(\log\frac{1}{a}\right)^{1-\alpha}(E_{\alpha,\alpha}(A_2\left(\log\frac{1}{a}\right)^{\alpha}))^{-1}V_{k-1}\left(\theta_s\left(u_1^0\left(\log\frac{1}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_1\left(\log\frac{1}{a}\right)^{\alpha}\right]\right.\right. \\ \left.\left.+S_1(1,u_1^0,u_2^0)\right),\theta_s\left(u_2^0\left(\log\frac{1}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_2\left(\log\frac{1}{a}\right)^{\alpha}\right]+S_2(1,u_1^0,u_2^0)\right)\right), \end{cases} \tag{49}$$

for $k=1$,we get

$$V_1(u_1^0,u_2^0)=u_2^0+(\log\frac{1}{a})^{1-\alpha}(E_{\alpha,\alpha}[A_2(\log\frac{1}{a})^{\alpha}])^{-1}S_2(1,u_1^0,u_2^0).$$

Let $\delta>0$ enough small, then it is easily known that

$$r=c\|\theta\|^{-1}(2\max\{b\|\theta\|,2d\|\theta\|,\left\|\left(\log\frac{1}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_2\left(\log\frac{1}{a}\right)^{\alpha}\right]\right\|\|\theta\|\})^{\delta}<1. \tag{50}$$

Since $S_2=o(\|u_1^0\|+\|u_2^0\|)$ as $u_1^0,u_2^0\to 0$, there exists a constant $L>0$ satisfying

$$\left\|V_1(u_1^0,u_2^0)-V_0(u_1^0,u_2^0)\right\|<Lr\left(\|u_1^0\|+\|u_2^0\|\right)^{\delta}. \tag{51}$$

Suppose $\left\|V_k(u_1^0,u_2^0)-V_{k-1}(u_1^0,u_2^0)\right\|<Lr^k\left(\|u_1^0\|+\|u_2^0\|\right)^{\delta}$. One has

$$\begin{aligned}\left\|V_{k+1}(u_1^0,u_2^0)-V_k(u_1^0,u_2^0)\right\| \le{}& \|\theta\|^{-1}\left\|\left(\log\frac{1}{a}\right)^{\alpha-1}E_{\alpha,\alpha}(A_2\left(\log\frac{1}{a}\right)^{\alpha})\right\|^{-1} \\ &\times Lr^k\left(\left\|\theta_s\left(u_1^0\left(\log\frac{1}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_1\left(\log\frac{1}{a}\right)^{\alpha}\right]+S_1(1,u_1^0,u_2^0)\right)\right\|\right. \\ &\left.+\left\|\theta_s\left(u_2^0\left(\log\frac{1}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_2\left(\log\frac{1}{a}\right)^{\alpha}\right]+S_2(1,u_1^0,u_2^0)\right)\right\|\right)^{\delta} \\ \le{}& Lr^kc\|\theta\|^{-1}\left(\|\theta\|\left(b\left\|u_1^0\right\|\right.\right. \\ &\left.\left.+\left\|\left(\log\frac{1}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_2\left(\log\frac{1}{a}\right)^{\alpha}\right]\right\|\|u_2^0\|\right)+2d(\|u_1^0\|+\|u_2^0\|)\|\theta\|\right)^{\delta} \\ \le{}& Lr^kc\|\theta\|^{-1}(2\max\{b\|\theta\|,2d\|\theta\|,\left\|\left(\log\frac{1}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left[A_2\left(\log\frac{1}{a}\right)^{\alpha}\right]\right\|\|\theta\|\})^{\delta} \\ &\times\left(\|u_1^0\|+\|u_2^0\|\right)^{\delta} \\ \le{}& Lr^{k+1}\left(\|u_1^0\|+\|u_2^0\|\right)^{\delta}.\end{aligned}$$

where $c<\|\theta\|<\frac{1}{b}$. So $V_k(u_1^0,u_2^0)$ uniformly converges to a continuous function $V(u_1^0,u_2^0)$ and we get

$$V(u_1^0,u_2^0)=V_0(u_1^0,u_2^0)+\sum_{k=1}^{\infty}[V_k(u_1^0,u_2^0)-V_{k-1}(u_1^0,u_2^0)]=u_2^0+V^*(u_1^0,u_2^0),$$

where $V^*(u_1^0,u_2^0)=o(\|u_1^0\|+\|u_2^0\|)$.

By successive approximation similar to function V, we obtain the solution of $U(u_1^0,u_2^0)$ satisfying

$$U(u_1^0,u_2^0)=u_1^0+U^*(u_1^0,u_2^0), \tag{52}$$

where $U^*(u_1^0,u_2^0)=o(\|u_1^0\|+\|u_2^0\|)$.

If $c>\frac{1}{b}$, namely, $\left\|(E_{\alpha,\alpha}(A_1\left(\log\frac{1}{a}\right)^{\alpha}))\right\|=\left\|(E_{\alpha,\alpha}(A_2\left(\log\frac{1}{a}\right)^{\alpha}))\right\|$, similarly to (47), we can also use successive approximation to obtain the solution of (48). For $c>\frac{1}{b}$, the process is similar to $c<\frac{1}{b}$. Thus we get a continuous map h_1 when $(u_1^0,u_2^0)\notin\delta(0)$, $h_1(u_1^0,u_2^0)=(u_1^0,u_2^0)$. Moreover, the uniqueness is easily proved. Then one can utilize the same arguments in steps 2 and 3 of Theorem 2 to end this proof.

4 Conclusion

The present paper has addressed non linear fractional differential systems involving Riemann-Liouville and Hadamard derivatives with different types of initial value conditions whenever such initial conditions are not equal with each other. We also have proved the new linearization theorems of those nonlinear fractional differential systems.

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