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Point and Interval Estimation for Frechet Distribution Based on Progressive First Failure Censored Data

Fathy H. Riad [1,2] *and E. H. Hafez* [3,*]

[1] Department of Mathematics, College of Science Jouf University P. O .Box 2014, Sakaka, Saudi Arabia
[2]Department of Mathematics, Faculty of Science , Minia University, Egypt
[3]Mathematics Department, Faculty of Science, Helwan University, Helwan, Cairo, Egypt.

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Abstract: In this paper, we obtain the point and interval estimation for Frechetdistribution FD based on progressive first failure PFF censored data using two methods: Maximum likelihood estimation (MLE) and Bayesian estimation. A comparison of Bayesian estimation under symmetric and asymmetric loss functions are obtained. The highest posterior density (HPD) interval and approximate confidence interval CI was made.

Keywords: Frechet distribution, Progressive first failure censoring, Maximum likelihood estimation, Bayesian estimation, Interval estimation.

1 Introduction

One of the most common censoring schemes is Type-II censoring.which saves time and money. However, when the lifetimes of products are very high, the experimental time of a Type-II censoring life test can be still too long. A generalization of Type-II censoring is the progressive Type-II censoring. [1] described a life test in which the experimenter might decide to group the test units into several sets, each as an assembly of test units . Then, run all the test units simultaneously until the first failure in each group. Such a censoring scheme is called first-failure censoring, [2] and [3] obtained MLE, exact confidence intervals and exact confidence regions for the parameters of the Gompertz and Burr Type-XII distributions based on first failure-censored sampling, respectively. For more reading one can refer to [4] and [5]. The first-failure censoring does not allow for sets to be removed from the test at the points other than the final termination point. However, this allowance will be desirable in practice. This leads us to the area of progressive censoring. [6] combined the concepts of first-failure censoring and progressive censoring to develop a new life test plan called a PFF censoring scheme. [7] studied the coefficient of variation of Gompertz distribution under PFF censoring. [8] and [9] introduced MLE, Bayesian estimates, exact confidence intervals and exact confidence regions for the parameters of Gompertz and Burr Type-XII distributions under PFF censored sampling.

2 Model desciption

Suppose that number (n) of independent groups with k items within each group are put on a life test. R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure $Y^R_{1;m,n,k}$ has occurred, R_2 groups and the group in which the second failure is observed are randomly removed from the test as soon as the second failure occurred $Y^R_{2;m,n,k}$, and finally when the m-th failure $Y^R_{m;m,n,k}$ is observed, the remaining groups $R_m, (m \leq n)$ are removed from the test. Then $Y^R_{1;m,n,k} < ... < Y^R_{m;m,n,k}$ are called PFF censored order statistics with the progressive censored scheme $R = (R_1, R_2, ..., R_m)$, where $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $n \times k$ items originally

* Corresponding author e-mail: eslmhussam1986@gmail.com

in the test are from a continuous population with distribution function $F(y)$ and probability density function $f(y)$, the joint probability density function for $Y^R_{1;m,n,k}, Y^R_{2;m,n,k}, \ldots, Y^R_{m;m,n,k}$ is defined as follows:

$$f_{1,2,\ldots,m}(Y^R_{1;m,n,k}, Y^R_{2;m,n,k}, \ldots, Y^R_{m;m,n,k}) = A(n,m-1)k^m \prod_{i=1}^{m} f(Y^R_{i;m,n,k}) \left[1-F(Y^R_{i;m,n,k})\right]^{k(R_i+1)-1}, \tag{1}$$

$0 < y_1 < y_2 < \ldots < y_m < \infty$, where $A(n,m-1) = n(n-R_1-1)\ldots(n-R_1-R_2-\ldots-R_{m-1}-(m-1))$

Special cases:

1.Putting $k=1$ gives the progressively Type-II censored order statistics.
2.The complete sample case when $R=\{0,0,\ldots,0\}$ and $k=1$.

3 Frechet Distribution

The probability density function (PDF) and the cumulative distribution (CDF) respectively of the FD as follows:

$$f(x) = \left(\frac{c}{\beta}\right)\left(\frac{x-\theta}{\beta}\right)^{-(c+1)} \exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c}\right], \quad x>\theta, c>0, \beta>0. \tag{2}$$

and

$$F(x) = \exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c}\right], \quad x>\theta, c>0, \beta>0. \tag{3}$$

In the FD, the shape parameter c governs the shape of the PDF, the hazard function and the general properties of the FD.

4 Maximum Likelihood Estimation

This section discussed the process of obtaining point and interval estimations of the parameters of FD based on PFF censored data.

To find the point estimation, let $\underline{X} = x^{R_1}_{1:m,n,k}, x^{R_2}_{2:m,n,k}, \cdots, x^{R_m}_{m:m,n,k}$ be the PFF censored order statistics from FD with censored scheme R. From equations (1), (2) and (3) the likelihood function is given by

$$L(c,\beta,\theta|\underline{X}) = Ak^m \prod_{i=1}^{m} \left[\left(\frac{c}{\beta}\right)\left(\frac{x_i-\theta}{\beta}\right)^{-(c+1)} \exp\left[-\left(\frac{x_i-\theta}{\beta}\right)^{-c}\right]\right]\left[1-\exp\left[-\left(\frac{x_i-\theta}{\beta}\right)^{-c}\right]\right]^{k(R_i+1)-1}, \tag{4}$$

The log likelihood function is :

$$\ell(c,\beta,\theta|\underline{X}) = \ln A + m\ln k + m\ln c + mc\ln\beta - (1+c)\sum_{i=1}^{m}\ln(x_i-\theta) - \sum_{i=1}^{m}\left(\frac{x_i-\theta}{\beta}\right)^{-c} + \sum_{i=1}^{m}(k(R_i+1)-1)\ln\left[1-e^{-\left(\frac{x_i-\theta}{\beta}\right)^{-c}}\right], \tag{5}$$

we obtain the the first partial derivatives of the parameters with respect to c, θ, β and equating to zero.

$$\frac{\partial\ell}{\partial c} = \frac{m}{c} + m\ln\beta - \sum_{i=1}^{m}\ln(x_i-\theta) + \sum_{i=1}^{m}\left(\frac{x_i-\theta}{\beta}\right)^{-c}\ln\left(\frac{x_i-\theta}{\beta}\right) - \sum_{i=1}^{m}(k(R_i+1)-1)\frac{e^{-\left(\frac{x_i-\theta}{\beta}\right)^{-c}}\left(\frac{x_i-\theta}{\beta}\right)^{-c}\ln\left(\frac{x_i-\theta}{\beta}\right)}{1-e^{-\left(\frac{x_i-\theta}{\beta}\right)^{-c}}} = 0, \tag{6}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{mc}{\beta} - \frac{c}{\beta}\sum_{i=1}^{m}\left(\frac{x_i-\theta}{\beta}\right)^{-c} + \frac{c}{\beta}\sum_{i=1}^{m}(k(R_i+1)-1)\frac{\left(\frac{x_i-\theta}{\beta}\right)^{-c}e^{-\left(\frac{x_i-\theta}{\beta}\right)^{-c}}}{1-e^{-\left(\frac{x_i-\theta}{\beta}\right)^{-c}}} = 0, \tag{7}$$

$$\frac{\partial\ell}{\partial\theta} = (1+c)\sum_{i=1}^{m}\left(\frac{1}{x_i-\theta}\right) + \frac{c}{\beta}\sum_{i=1}^{m}(k(R_i+1)-1)\frac{\left(\frac{x_i-\theta}{\beta}\right)^{-c-1}e^{-\left(\frac{x_i-\theta}{\beta}\right)^{-c}}}{1-e^{-\left(\frac{x_i-\theta}{\beta}\right)^{-c}}} - \frac{c}{\beta}\sum_{i=1}^{m}(\frac{x_i-\theta}{\beta})^{-c-1} = 0 \tag{8}$$

Equations (6), (7), (8) cannot be solved analytically, so we used Newton-Raphson method to solve them.

5 Interval Estimation

In this section we obtain the approximate CI of the parameters θ, c and β.

5.1 Approximate confidence intervals ACI

In this subsection we obtain approximate confidence intervals of the parameters based on asymptotic distribution of the MLEs of the unknown parameters $\Theta = (\beta, \theta, c)$. The asymptotic variances and covariances of the MLE for parameters θ, c and β are given by elements of the inverse of the Fisher information matrix Unfortunately, it is difficult to obtain the exact mathematical expressions for the above-mentioned equations. Therefore, we give the approximate (observed) asymptotic variance-covariance matrix for the MLE, which is obtained by dropping the expectation operator E.

$$I_{ij}^{-1}(c,\beta,\theta) = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial c} & \frac{\partial^2 \ell}{\partial \beta \partial \theta} \\ \frac{\partial^2 \ell}{\partial c \partial \beta} & -\frac{\partial^2 \ell}{\partial c^2} & \frac{\partial^2 \ell}{\partial c \partial \theta} \\ \frac{\partial^2 \ell}{\partial \theta \partial \beta} & \frac{\partial^2 \ell}{\partial \theta \partial c} & -\frac{\partial^2 \ell}{\partial \theta^2} \end{pmatrix}^{-1} = \begin{pmatrix} var(\widehat{\beta}) & cov(\widehat{\beta},\widehat{c}) & cov(\widehat{\beta},\widehat{\theta}) \\ cov(\widehat{c}\widehat{\beta}) & var(\widehat{c}) & cov(\widehat{c},\widehat{\theta}) \\ cov(\widehat{\theta},\widehat{\beta}) & cov(\widehat{\theta},\widehat{c}) & var(\widehat{\theta}) \end{pmatrix} \tag{9}$$

Approximate confidence intervals for c, θ and β can be obtained Thus, the $(1-\gamma)100\%$ confidence intervals for parameters c, θ and β become,

$$\widehat{c} \pm Z_{\frac{\gamma}{2}} \sqrt{var(\widehat{c})} \qquad \widehat{\beta} \pm Z_{\frac{\gamma}{2}} \sqrt{var(\widehat{\beta})} \qquad \widehat{\theta} \pm Z_{\frac{\gamma}{2}} \sqrt{var(\widehat{\theta})} \tag{10}$$

where $Z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right-tail probability.

5.2 Bootstrap confidence Intervals

In this section, we construct the percentile bootstrap (Boot-p) (see, [19]) The bootstrap is a resampling method for statistical inference. It is commonly used to estimate confidence intervals as well as be used to estimate bias and variance of an estimator or calibrate hypothesis tests. The following steps are adopted to obtain PFF censoring bootstrap sample from FD with parameters c, θ and β based on simulated PFF censored data set.

Algorithm

1. From an original data set $X = (x^R_{1,m,n,k}, x^R_{2,m,n,k}, x^R_{3,m,n,k}, \cdots, x^R_{m,m,n,k})$, compute the MLE of the parameters c and β and θ say $\widehat{c}$, $\widehat{\theta}$ and $\widehat{\beta}$ from equations (6), (7) and (8).

2. Use $\widehat{c}$, $\widehat{\theta}$ and $\widehat{\beta}$ to generate a bootstrap sample $\underline{X}^*$ with the same values of $R_i, (i = 1.2.\cdots, m)$ using the algorithm made by [20] .

3. Similar to step 1 based on $\underline{X}^*$ compute the bootstrap sample estimates of c, θ and β say $\widehat{c}^*$, $\widehat{\theta}^*$ and $\widehat{\beta}^*$.

4. Repeat steps 2 and 3 N times representing N bootstrap MLE of c, θ and β based on N different bootstrap samples.

5. Arrange $\widehat{c}^{*i}$, $\widehat{\theta}^{*i}$ and $\widehat{\beta}^{*i}$ in an ascending order to obtain bootstrap sample $(\psi_t^{(1)}, \psi_t^{(2)}, \cdots, \psi_t^{(N)}), t = 1,2,3$ where $\psi_1 = \widehat{c}^*, \psi_2 = \widehat{\theta}^*, \psi_3 = \widehat{\beta}^*$ Let $G(x) = P(\psi_t \le x)$ be cumulative distribution function of ψ_t. Define $\psi_{tboot} = G^{-1}(x)$ for given x. The approximate bootstrap $100(1-\gamma)\%$ confidence interval of ψ_t given by:

$$\left[\psi_{tboot}\left(\frac{\gamma}{2}\right), \psi_{tboot}\left(1-\frac{\gamma}{2}\right)\right] \tag{11}$$

6 Bayes estimators under symmetric and asymmetric loss function

This section covers with obtaining the Bayesian estimation for the FD parameters under different loss functions. [10] had studied the Bayesian estimation using squared error loss function SELF for lindely distribution using important sampling technique and Metropolis-Hasting algorithm MHA . In practical works the parameters cannot be treated as a constant during the life testing time. Thus, it would be a fact to assume the parameters used in the life time model as random variables. We have also conducted a Bayesian study assuming the following independent gamma priors for θ, β and c:

$$g_1(\theta) \propto \theta^{\mu-1} e^{-\lambda\theta}, \theta > 0, \tag{12}$$

$$g_2(\beta) \propto \beta^{\mu_1-1} e^{-\lambda_1\beta}, \beta > 0 \tag{13}$$

$$g_3(c) \propto c^{\mu_2-1} e^{-\lambda_2 c}, c > 0 \tag{14}$$

Where $\mu_1, \mu_2, \mu, \lambda, \lambda_2, \lambda_1$ are hyperparameters and $\mu_1, \mu_2, \mu, \lambda, \lambda_2, \lambda_1 > 0$

Assume that the model parameters θ, c and β are independent, then the joint prior PDF of θ, β and c is given by

$$\pi(\theta, c, \beta) \propto c^{\mu_2-1} \beta^{\mu_1-1} \theta^{\mu-1} e^{-(\theta\lambda+\beta\lambda_1+c\lambda_2)}, \quad \theta, \beta, c > 0. \tag{15}$$

6.1 Symmetric loss function

In this subsection, we conducted Bayesian estimation using SELF. The likelihood function has the form :

$$L(\theta) = Ak^m \prod_{i=1}^{m} \left[\left(\frac{c}{\beta}\right) \left(\frac{x-\theta}{\beta}\right)^{-(c+1)} \exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c} \right] \right] \left[1 - \exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c} \right] \right]^{k(R_i+1)-1} \tag{16}$$

The joint posterior density function of the parameters θ, β and c can be written from (4) and (15) as follows:

$$\pi^*(\theta, \beta, c \mid x) \propto L(\theta, \beta, c)\pi(\theta, \beta, c) \tag{17}$$

$$\begin{aligned} \pi^*(\theta, \beta, c \mid x) \propto L(\theta) = Ak^m \prod_{i=1}^{m} \left[\left(\frac{c}{\beta}\right) \left(\frac{x-\theta}{\beta}\right)^{-(c+1)} \exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c} \right] \right] \\ \left[1 - \exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c} \right] \right]^{k(R_i+1)-1} * c^{\mu_2-1} \beta^{\mu_1-1} \theta^{\mu-1} e^{-(\theta\lambda+\beta\lambda_1+c\lambda_2)}. \end{aligned} \tag{18}$$

The posterior density function is:

$$\pi^*(\theta, \beta, c \mid x) = \frac{L(\theta)\pi(\theta, \beta, c \mid x)}{\int_0^\infty L(\theta)\pi(\theta, \beta, c \mid x) d\theta dc d\beta} \tag{19}$$

Based on SE loss function, the Bayes estimator of the function of parameters $U = U(\Theta)$, $\Theta = (\theta, \beta, c)$ is given by

$$\hat{U}_{SE} = \int_\Theta U\, \pi^*(\Theta)\, d\Theta, \tag{20}$$

where $\pi^*(\Theta)$ is given by equation (19).

Accordingly, the Bayes estimate of any function of θ say $h(\theta)$ under Squared Error loss function is

$$\hat{\theta}_{SE} = E_{(\theta|data)}[h(\theta)] = \frac{\int_0^\infty h(\theta)L(\theta)g(\theta \mid x)d\theta}{\int_0^\infty L(\theta)g(\theta \mid x)d\theta.} \tag{21}$$

It is impossible to compute equation (21) analytically. The posterior density function cannot be reduced analytically to well known distributions. However, its plot shows that it is similar to normal distribution. Hence, to calculate the integral that we cannot calculate exactly, we use the Metropolis-Hasting algorithm MHA with normal as a proposal distribution.

6.2 Asymmetric loss function

Asymmetric loss function may be more appropriate in some fields. Recently, many authors considered asymmetric loss functions in reliability and life testing. One of the most popular asymmetric loss functions is linear-exponential loss function (LINEX) which was introduced by [11]. It used in several papers, for example, [12], [13], [14] and [15]. This function is approximately linearly on one side and rises approximately to zero on the other side. Under the assumption that the minimal loss occurs at $\hat{\theta} = \theta$, the LINEX loss function can be expressed as:

$$L_1(\delta) \propto \exp(w\delta) - w\delta - 1, \tag{22}$$

where $\delta = \hat{\theta} - \theta, \hat{\theta}$ is the estimate of $\theta, w \neq 0$. The magnitude of w represents the direction, and degree of symmetry. Where $w > 0$ means overestimation is more serious than underestimation, and $w < 0$ means the opposite. For w close to zero the LINEX loss function is approximately the Squared Error Loss SEL function. The posterior expectation of the LINEX loss function is :

$$E_\theta(L1(\hat{\theta} - \theta)) \propto \exp(w\theta)E_\theta[\exp(-w\theta)] - w((\hat{\theta} - E_\theta[\theta]) - 1. \tag{23}$$

The Bayes estimator under the LINEX loss function is the value of

$$\hat{\theta}_{LINEX} = \frac{-1}{w}\log(E_\theta[\exp(-w\theta)]), \tag{24}$$

such that $E_\theta[\exp(-w\theta)]$ exists. Another asymmetric loss function called a General Entropy loss function GELF was proposed by [16] which can be expressed as:

$$L_2(\hat{\theta}, \theta) \propto (\frac{\hat{\theta}}{\theta})^q - q\log(\frac{\hat{\theta}}{\theta}) - 1. \tag{25}$$

The weighted SELF results from $q = -1$. The Bayes estimate $\hat{\theta}_{GE}$ under GEL function is

$$\hat{\theta}_{GE} = (E_\theta[\theta]^{-q})^{\frac{-1}{q}}, \tag{26}$$

such that $E_\theta[\theta]^{-q}$ exists. Since it is impossible to compute equation (24) and 26 analytically, we used the Markov chain Mont-Carlo (MCMC) method such as Metropolis-Hastings algorithm. to draw samples from the posterior density function and then to compute the Bayes estimate.

6.3 Bayesian estimation using MCMC method

In this subsection, MCMC method is considered to generate samples from the posterior distribution and then compute the BEs of θ, c and β From the joint posterior density function in equation (19), the conditional posterior distributions of θ, c and β are given respectively by:

$$\pi^*(\theta|c,\beta) \propto \theta^{\mu_2-1}e^{-\lambda\theta}\prod_{j=1}^{m_i}\left[\exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c}\right]\right]\left[1-\exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c}\right]\right]^{(k(R_i+1)-1)}, \tag{27}$$

$$\pi^*(\beta|\theta,c) \propto \beta^{m*c+\mu_1-1}e^{-\lambda_1\beta}\prod_{j=1}^{m_i}\left[\exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c}\right]\right]\left[1-\exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c}\right]\right]^{(k(R_i+1)-1)}, \tag{28}$$

$$\pi^*(c|\beta,\theta) \propto c^{\mu+m-1}e^{-c\lambda_2}\prod_{j=1}^{m_i}\left[\exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c}\right]\right]\left[1-\exp\left[-\left(\frac{x-\theta}{\beta}\right)^{-c}\right]\right]^{(k(R_i+1)-1)}. \tag{29}$$

The conditional posterior distributions of θ, c and β in equations (27), (28), and (29) cannot be reduced analytically to well known distribution. Consequently, MHA is used to generate random samples from these distributions see [17]. The following algorithm is proposed to compute BE of $U = U(\theta, c, \beta)$ under SELF, GELF and LINEX loss function.

6.4 MHA

The MHA was originally introduced by [17]. Suppose that our goal was to draw samples from some distributions $f(x|\theta) = \nu g(\theta)$, where ν is the normalizing constant which may be unknown or very difficult to compute. The MHA provided a way of sampling from $f(x|\theta)$ with no need us to know ν. Let $g\left(\theta^{(b)}|\theta^{(a)}\right)$ be an arbitrary transition kernel: that is the probability of moving, or jumping, from current state $\theta^{(a)}$ to $\theta^{(b)}$. This is sometimes called the proposal distribution. The following algorithm generated a sequence of values $\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)}$ which forms a Markov chain with stationary distribution given by $f(x|\theta)$.

6.5 MHA (Algorithm 1)

1.Start with initial point $\theta^{(0)} = \hat{\theta}_{MLE}$, $c^{(0)} = \hat{c}_{MLE}$, $\beta^{(0)} = \hat{\beta}_{MLE}$.
2.Set i=1.
3.Generate $\theta^{(*)}$ from the proposal distribution $N(\theta^{(i-1)}, var\theta^{(i-1)})$.
4.Calculate the acceptance probability $r(\theta^{(i-1)}, \theta^{(*)}) = \min\left[1, \dfrac{\pi(\theta^{(*)})}{\pi(\theta^{(i-1)})}\right]$.
5.Generate U from uniform on (0, 1).
6.If $U < r(\theta^{(i-1)}, \theta^{(*)})$ accepts the proposal distribution and set $\theta^{(i)} = \theta^{(*)}$. Otherwise, reject the proposal distribution and set $\theta^{(i)} = \theta^{(i-1)}$.
7.Set $i = i+1$.
8.Repeat Steps (1-7) for the other parameters c, β N times.
9.The approximate estimates of SE , LINEX, GELF are given respectively by

$$\hat{\theta}_{SE} = \frac{1}{N-M} \sum_{i=M+1}^{N} U(c^{(i)}, \theta^{(i)}, \beta^{(i)}), \tag{30}$$

$$\hat{\theta}_{LINEX} = \frac{-1}{c} \log\left[\frac{1}{N-M} \sum_{i=M+1}^{N} \exp\{-cU(c^{(i)}, \theta^{(i)}, \beta^{(i)})\}\right] \tag{31}$$

$$\hat{\theta}_{GE} = \left(\frac{1}{N-M} \sum_{i=M+1}^{N} \exp\{-cU(c^{(i)}, \theta^{(i)}, \beta^{(i)})\}\right)^{\frac{-1}{q}}, \tag{32}$$

we can compute c and β where M is nburn units and N is the number of mcmc iterations.
10.Do the same steps for the other two parameters c and β.

6.6 Highest Posterior Density HPD interval algorithm

In Bayesian statistics, a credible interval is an interval in the domain of a posterior probability distribution used for interval estimation .The credible intervals are analogous to confidence intervals in frequentist statistics although they differ on a philosophical basis, Bayesian intervals treat their bounds as a fixed value and the estimated parameter as a random variable, whereas frequentist confidence intervals treat their bounds as random variables and the parameter as a fixed value. In this section, we described the algorithm for finding $(1-\gamma)$ HPD interval for θ, c, β (credible intervals). This algorithm proposed by [18]

6.7 (Algorithm 2)

1.Arrange the values of the estimates obtained from BE ($\theta^{(*)}$) in ascending order.
2.Find the position of the lower bound which is $N[(N-M)*\gamma/2]$, M is the nburn.
3.The lower bound of θ is the observed value has the number in arrangement $\theta^{(*)}_{low} = \theta^{(*)}_{(N-M)*(\gamma/2)}$.
4.Find the position of upper bound which is $N[(N-M)*(1-(\gamma/2))]$.

5.The upper bound of θ is the observed value has the number in arrangement $\theta^{(*)}_{upp} = \theta^{(*)}_{(N-M)*(1-(\gamma/2))}$.
6.Repeat the steps N times and find the average value of $\theta^{(*)}_{low}$ and $\theta^{(*)}_{upp}$.
7.Do the same steps for the other two parameters c, β.

7 Simulation studies

In this section, simulation studies are conducted to investigate the performances of the MLEs and BEs (under SELF, GELF and LINEX loss function) in terms of their mean square errors (MSEs) and for various choices of sample sizes (n, m, and censoring schemes (R_i, $i = 1,2,...,m$). Table (1) contains censoring schemes (C.S) used in the simulation, Table(2) and Table(3), Table(4) contains the results concluded from the simulation study.
Using the fact that the PFF censored sample with distribution function F(x), can be viewed as a progressive Type-II censored sample from a population with distribution function $1-(1-F(x))^k$, we generate a PFF censored samples from the continuous random variable using the algorithm described in [20]. The estimation procedure is performed according to the following algorithm, for mor reading about PFF see [21], for more reading about BE see [22] .

Algorithm (3)

1.Specify the values of k, n, m.
2.Generate random samples of size m using the algorithm made by [20] from Uniform$(0,1)$ distribution, $(U_1,U_2,...,U_m)$, $i = 1,2,...,m$.
3.Define the values of the censored schemes, R_i, $i = 1,2,...,m$, and such that $\sum_{i=1}^{m} R_i = n-m$.
4.Set $E_{ij} = U_i^{1/(i+\sum_{j=m-i+1}^{m} R_j)}$, and $i = 1,2,...,m$.
5.Obtain PFF censored samples $(U_1^*,U_2^*,...,U_m^*)$, where $U_i^* = 1-\prod_{j=m-i+1}^{m} R_j$, $i = 1,2,...,m$.
6.Use step (4) to generate random samples $(t_1,t_2,...,t_m)$, $i = 1,2,...,m$. from equation (3) as follows: $[t_i = (\theta+\beta(-Log[(1-(1-U_i^*)^{(1/k)})])^{(-1/c)})]$, $i = 1,2,...,m$.
7.Use the progressive censored data to compute the MLEs of the model parameters by solving the nonlinear system ((6),(7),(8)).
8.Compute the BEs of the distribution parameters according to algorithm (1), with $N = 11000$, $M = 2000$, where N is the number of mcmc iterations and M is the nburn.
9.Compute the CIs bounds with confidence level 95% for the 3 parameters θ, c and β.
10.Compute 95% credible CIs using algorithm (2) of the parameters θ, c and β.
11.Replicate the steps $((2)-(10))$, 1000 times.
12.Compute the average values of the MSEs associated with the MLEs and BEs of the parameters.
13.Compute the average values of the lengths of bootstrap CI, ACI, credible CI and the coverage probability of these CIs of the parameters.
14.Do steps ((1)-(13)) with different values of n, m, and R_i, $i = 1,2,...,m$.

Table (1) censoring schemes used in simulation.

C.S	[1]	$m = 10, n = 40, R_1 = R_2 = R_3 = 10, R_4 = R_5 = R_6 = R_7 = ... = R_m = 0$
C.S	[2]	$m = 10, n = 40, R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = 5, R_7 = R_8 = ... = R_m = 0$
C.S	[3]	$m = 30, n = 70, R_1 = R_2 = R_3 = R_4 = 10, R_5 = ... = R_m = 0$
C.S	[4]	$m = 30, n = 70, R_1 = 20, R_2 = 10, R_3 = R_4 = ... = R_m = 0$
C.S	[5]	$m = 50, n = 70, R_1 = R_2 = R_3 = R_4 = R_5 = 10, R_6 = ... = R_m = 0$
C.S	[6]	$m = 50, n = 70, R_1 = R_2 = 20, R_3 = 10, R_4 = ... = R_m = 0$
C.S	[7]	$m = 70, n = 120, R_1 = R_2, ... = R_{14} = 5, R_{15} ... = R_m = 0$
C.S	[8]	$m = 70, n = 120, R_1 = R_2 = R_3 = R_4 = R_5 = R_6 =, R_7 = 10, R_8 = ... = R_m = 0$

Table (2) MSE, CI and HPD, with $n = 50, m = 10, k = 2$.

$Parameter, C.S[1]$	$[c]$	$[\theta]$	$[\beta]$
MSE_{MIE}	0.0486761	0.0136221	0.0890451
MSE_{SE}	0.00093363	0.000247784	0.00199665
MSE_{LINEX}	0.00091788	0.00024855	0.00199663
MSE_{GE}	0.00090892	0.0002491	0.0019984
$95\%CI$ Length	2.10274	1.0148	1.61856
$95\%CI forHPD$ Length	0.063618	0.0342	0.0544
$bootstrapCI$ Length	1.37561	0.71166	1.26723
COV_{MLE}	0.92	0.95	0.93
COV_{MCMC}	0.95	0.99	0.98
COV_{boot}	0.96	0.97	0.98
$Parameter, C.S[2]$	$[c]$	$[\theta]$	$[\beta]$
MSE_{MIE}	0.095757	0.0139325	0.0771662
MSE_{SE}	0.00154333	0.000303997	0.000849126
MSE_{LINEX}	0.00154525	0.00030456	0.00084941
MSE_{GE}	0.00154735	0.00030496	0.0008499
$95\%CI$ Length	2.21486	1.02571	1.65005
$95\%CI forHPD$ Length	0.06351	0.02677	0.04505
$bootstrap\ CI$ Length	1.34693	0.67021	1.24716
COV_{MLE}	0.92	0.95	0.93
COV_{MCMC}	0.95	0.99	0.98
COV_{boot}	0.94	0.96	0.97

Table (3) MSE, CI and HPD, with $n = 70, m = 30, k = 2$.

$Parameter, C.S[3]$	$[c]$	$[\theta]$	$[\beta]$
MSE_{MIE}	0.0579295	0.0150023	0.0425346
MSE_{SE}	0.00130812	0.000601582	0.000656316
MSE_{LINEX}	0.00130193	0.00060393	0.00066183
MSE_{GE}	0.0012983	0.0006056	0.0006659
$95\%CI$ Length	2.10274	1.0148	1.61856
$95\%CI forHPD$ Length	0.063618	0.0342	0.0544
$bootstrapCI$ Length	1.37561	0.71166	1.26723
COV_{MLE}	0.92	0.95	0.93
COV_{MCMC}	0.95	0.99	0.98
COV_{boot}	0.96	0.97	0.98
$Parameter, C.S[4]$	$[c]$	$[\theta]$	$[\beta]$
MSE_{MIE}	0.191946	0.0393184	0.0669236
MSE_{SE}	0.0019705	0.000810072	0.00108304
MSE_{LINEX}	0.00196987	0.00081577	0.00108934
MSE_{GE}	0.0019701	0.000819916	0.00109429999
$95\%CI$ Length	1.52	0.83	1.05146
$95\%CI forHPD$ Length	0.0422	0.02619	0.03346
$bootstrap\ CI$ Length	1.13916	0.62600	0.806306
COV_{MLE}	0.92	0.96	0.9
COV_{MCMC}	0.95	0.99	0.98
COV_{boot}	0.94	0.96	0.97

Table (4) MSE, CI and HPD, with $n = 70, m = 50, k = 2$.

$Parameter, C.S[5]$	$[c]$	$[\theta]$	$[\beta]$
MSE_{MIE}	0.029146	0.0103155	0.0288355
MSE_{SE}	0.000408453	0.000106225	0.000342256
MSE_{LINEX}	0.00040718	0.00010656	0.00034465
MSE_{GE}	0.00040645	0.00010679	0.0003464
$95\% CI$ Length	1.0679	0.56172	0.80877
$95\% CI for HPD$ Length	0.03537	0.01697	0.02697
$bootstrapCI$ Length	1.32561	0.71166	0.9284
COV_{MLE}	0.95	0.95	1
COV_{MCMC}	0.95	0.95	1
COV_{boot}	1	1	1
$Parameter, C.S[6]$	$[c]$	$[\theta]$	$[\beta]$
MSE_{MLE}	0.0765714	0.0252709	0.0569429
MSE_{SE}	0.0007307	0.00083689	0.001478952
MSE_{LINEX}	0.00070923	0.0008421	0.00148882
MSE_{GE}	0.00069704	0.00084597	0.0014969
$95\% CI$ Length	1.35658	0.75126	0.97617
$95\% CI for HPD$ Length	0.06451	0.04677	0.04505
$bootstrap\ CI$ Length	1.34693	0.67021	1.24716
COV_{MLE}	1	1	0.8
COV_{MCMC}	1	1	0.8
COV_{boot}	0.94	0.96	0.97

Table (5) MSE, CI and HPD, with $n = 120, m = 70, k = 2$,

$Parameter, C.S[7]$	$[c]$	$[\theta]$	$[\beta]$
MSE_{MIE}	0.00328937	0.00207793	0.00519712
MSE_{SE}	3.53×10^{-6}	8.65×10^{-7}	$2. \times 10^{-6}$
MSE_{LINEX}	3.53×10^{-6}	8.6×10^{-7}	$2. \times 10^{-6}$
MSE_{GE}	3.53×10^{-6}	8.6×10^{-7}	$2. \times 10^{-6}$
$95\% CI$ Length	0.91769	0.50269	0.70113
$95\% CI for HPD$ Length	0.00305	0.00157	0.00241
$bootstrapCI$ Length	1.03102	0.68414	0.73012
COV_{MLE}	1	1	1
COV_{MCMC}	1	1	1
COV_{boot}	1	1	1
$Parameter, C.S[8]$	$[c]$	$[\theta]$	$[\beta]$
MSE_{MLE}	0.0004307	0.00013689	0.005478952
MSE_{SE}	2.87×10^{-6}	8.22×10^{-7}	2.057×10^{-6}
MSE_{LINEX}	2.87×10^{-6}	8.2×10^{-7}	2.6×10^{-6}
MSE_{GE}	2.87×10^{-6}	8.2×10^{-7}	2.1×10^{-6}
$95\% CI$ Length	1.35658	0.75126	0.97617
$95\% CI for HPD$ Length	0.06451	0.04677	0.04505
$bootstrap\ CI$ Length	1.34693	0.67021	1.24716
COV_{MLE}	1	1	1
COV_{MCMC}	1	1	1
COV_{boot}	1	1	1

Table (6) MSE, CI and HPD, with $n = 130, m = 100, k = 2$.

$Parameter, C.S[9],$	$[c]$	$[\theta]$	$[\beta]$
MSE_{MIE}	0.00663903	0.00152868	0.00395928
MSE_{SE}	0	0	$6*10^{-9}$
MSE_{LINEX}	0	0	$1*10^{-9}$
MSE_{GE}	0	0	$1*10^{-9}$
$95\%CI$ Length	0.79028	0.46499	0.64017
$95\%CI forHPD$ Length	0.00003	0.00012	0.00002
$bootstrap\ CI$ Length	0.71166	0.68634	0.648
COV_{MLE}	0.98	0.95	0.93
COV_{MCMC}	1	1	1
COV_{boot}	0.96	0.97	0.98

8 Conclusion

Point and interval estimation using MLE, symmetric and asymmetric BE For FD parameter based on PFF samples are derived and computed. Asymmetric Bayesian estimation is always better than symmetric Bayesian Estimation. The MSE of $\hat{\theta}_{LINEX}$ is always smaller than MSE of $\hat{\theta}_{SE}$, the HPD interval length and the CI length of the parameter reduce as n, m increase, olso as the diffrence between n, m reduces. The results are defined, as follows:

1. For all censoring schemes and $k = 2$ as n, m increase the MSE of the FD parameters decreases.
2. For all censoring schemes and $k = 2$ as n, m increase the HPD interval length and the CI length of the parameter decreases.
3. The MSE of $\hat{\theta}_{SE}$, $\hat{\theta}_{LINEX}$, $\hat{\theta}_{GE}$ (bayesian estimators) is always smaller than MSE of $\hat{\theta}_{MLE}$.
4. When m is large the MSE of $\hat{\theta}_{SE}$, $\hat{\theta}_{LINEX}$ is almost the same as MSE of $\hat{\theta}_{GE}$.
5. When m is large and using BE the MSE of the parameters tends to zero.
6. When m is large the $CI, bootstrap\ CI, 95\%CI forHPD$ lenghts also decrease.

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Eslam hossam received the PhD degree in statistics at Helwan university 2018. His research interests are in the areas of applied statistics, he also was geaduated at the faculty of science Helwan university 2007 in major of statistics.

Fathi H. Reyad is doctor of statistics at Al Minia University 2014.His research interests are in the areas of applied statistics, he also was geaduated at the faculty of science Minia university in major of statistics.