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# Certain Results of (p, q)-Analogue of Aleph-Function with (p, q)-Derivative

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**Abstract:** In this paper, the authors have derived the $(p,q)$-analogue of Aleph-Function with $(p,q)$-derivative by using the generalization of the Gamma and Beta functions. Some particular cases of these results, in terms of $(p,q)$-analogue of G-Function, were established earlier by Swati et al. and I, H-functions and derived earlier by Altaf et al.

**Keywords:** (p,q)-analogue of Aleph-function, (p,q)-analogue of I-function, (p,q)-analogue of G-function, (p,q)-derivatives, (p,q)-Gamma function

## 1 Introduction

The $(p,q)$-shifted factorial is based on the concept of twin-basic number $[n]_{(p,q)} = \frac{(p^n-q^n)}{(p-q)}$. The basic number occurs in the theory of two parameter quantum algebras and has also been introduced in combinatorics by Jagannathan et al. [1]. Several properties of this number were studied briefly in [2]. Around the same time as [2], Brodimas et al. [3] and Arik et al. [4] also, independently, introduced $(p,q)$-number in the physics literature, but in a very much less detailed manner. (p,q)-identities are, thus, derived, by doubling the number of parameters, which offers more choices for applications. It has been observed that many of q-results can be generalized directly to $(p,q)$-results. If we have $(p,q)$-results, the q-results can be obtained more easily by mere substitutions for the parameters instead of any limiting process, as required in the usual q-theory [5]. This also provides a new look for q-identities.
The q-deformed algebra [6,7] and their generalization to $(p,q)$-analogue [2,8] have attracted much attention of the researchers to increase the accessibility of different dimensions of $(p,q)$-analogue algebra. The main reason is that these topics stand for real life problems in mathematics and physics, discussed later in the theory of quantum calculus.
In the present paper, the authors attention is towards the $(p,q)$-analogue of Aleph-Function with $(p,q)$-derivative by using the generalization of the Gamma and Beta functions, namely $(p,q)$-Gamma and Beta functions.
From the theory of basic hypergeometric series [9], some basic definitions are given below:
The q-shifted factorial is given by

$$(a,q)_n = \left[ \begin{array}{r} 1, \quad n=0 \\ (1-a)(1-aq)...(1-aq^{n-1}), n=1,2,3,... \end{array} \right. \tag{1}$$

with $(a_1,a_2,...,a_k;q)_n = (a_1;q)_n(a_2;q)_n...(a_k;q)_n$.
The $q$-gamma function was first introduced by Thomae and later by Jackson. The $q$-analogue of gamma function, which is defined by F.H. Jackson [10], is given by

$$\Gamma_q(x) = \frac{(q;q)_\infty(1-q)^{1-x}}{(q^x;q)_\infty}, 0<q<1.$$

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Jackson gave the general definition that is given below:

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt,$$

*where*
$$\int_0^a f(t)d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n.$$

Jackson also defined an integral, i.e.

$$\int_0^{\infty} f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(aq^n)q^n$$

P. Njionou Sadjang [11] introduced, the so-called, the shifted factorial as follows:

$$(x\ominus a)_{p,q}^n = (x-a)(px-aq)(p^2x-aq^2)...(xp^{n-1}-aq^{n-1}) \tag{2}$$

$$(x\oplus a)_{p,q}^n = (x+a)(px+aq)(p^2x+aq^2)...(xp^{n-1}+aq^{n-1}) \tag{3}$$

These definitions are extended as follows:

$$(a\ominus b)_{p,q}^n = \prod_{k=0}^{\infty}(ap^k-bq^k) \tag{4}$$

$$(a\oplus b)_{p,q}^n = \prod_{k=0}^{\infty}(ap^k+bq^k) \tag{5}$$

Let x be a complex number, the $(p,q)$-Gamma function is defined by P. Njionou Sadjang [12], where

$$\varGamma_{p,q}(x) = \frac{(p\ominus q)_{p,q}^{\infty}}{(p^x\ominus q^x)_{p,q}^{\infty}}(p-q)^{1-x}\ ,0<q<p \tag{6}$$

If we put $p=1$, then $\varGamma_{p,q}$ reduces to $\varGamma_q$.
The (p, q)-Gamma function fulfils the following fundamental relation

$$\varGamma_{p,q}(x+1) = [x]_{p,q}\varGamma_{p,q}(x)$$

If n is a nonnegative integer, it follows from above that

$$\varGamma_{p,q}(x+1) = [x]_{p,q}!$$

It can be also easily seen from the definition that

$$\varGamma_{p,q}(n+1) = \frac{(p\ominus q)_{p,q}^n}{(p-q)_{p,q}^n}$$

P. Njionou Sadjang [12] also defined the $(p,q)$-Beta function as

$$B_{p,q}(x,y) = \frac{\varGamma_{p,q}(x)\varGamma_{p,q}(y)}{\varGamma_{p,q}(x+y)}$$

The (p,q)-derivative of the function f(x) is defined as follows [2,6]:

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, x\neq 0 \tag{7}$$

Where $D_{p,q}f(0) = f'(0)$, provided that $f(x)$ is differentiable at $x=0$. The $(p,q)$-numbers $[n]_{p,q}$ and $(p,q)$ factorials $[n]_{p,q}!$ are defined [11] as:

$$[n]_{p,q} = \frac{p^n-q^n}{(p-q)}$$

and

$$[n]_{p,q}! = [1]_{p,q}[2]_{p,q}...[n]_{p,q},$$

respectively. Also it happens that $D_{p,q}(x^n) = [n]_{p,q}x^{n-1}$.
**Remark 1:** $D_{p,q}(x)$ reduces to Hahn Derivative $d_qf(x)$ iff $p\to 1$.

**Remark 2:** $[n]_{p,q} = [n]_q$ (Hahn Basic Number) iff $p \to 1$. where $[n]_q = \frac{1-q^n}{1-q}, q \neq 1$.
**Note:**

$$D^n_{p,q}(x^\mu) = \frac{\Gamma_{p,q}(\mu+1)}{\Gamma_{p,q}(\mu-n+1)} x^{\mu-n}, Re(\mu)+1>0 \tag{8}$$

$$I^n_{p,q}(x^\mu) = \frac{\Gamma_{p,q}(\mu+1)}{\Gamma_{p,q}(\mu+n+1)} x^{\mu+n}, Re(\mu)+1>0 \tag{9}$$

B.K. Dutta et. al. [13] defined the q-analogue of Aleph Function in terms of Mellin-Barnes type contour integrals in the following manner:

$$\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} G(q^{b_j+B_js}) \prod_{j=1}^{n} G(q^{1-a_j-A_js})\pi z^{-s}}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i} G(q^{1-b_{ji}-B_{ji}s}) \prod_{j=n+1}^{p_i} G(q^{a_{ji}+A_{ji}s})G(q^s)G(q^{1-s})sin\pi s]} ds \tag{10}$$

where $A_j, B_j, A_{ji}, andB_{ji}$ are real and positive, $\tau_i > 0$, for all $i$, $a_j, b_j, a_{ji}, andb_{ji}$ are complex numbers and

$$G(q^\alpha) = \prod_{n=0}^{\infty}(1-q^{\alpha+n})^{-1} = \frac{1}{(q^\alpha;q)_\infty},$$

where L is the contour of integration, running from $-i\infty$ to $+i\infty$ in such a manner that all poles of $G(q^{b_j+B_js}); 1 \le j \le m$ are on the right of the path and those of $G(q^{1-a_j-A_js}); 1 \le j \le n$, are on the left. The integral converges if $Re[slog(z) - logsin\pi s] < 0$, for large values of $|s|$ are on contour L.
Setting $\tau_i = 1$, in (10) we get basic analogue of I-Function in terms of the Mellin-Barnes type basic contour integral introduced by Saxena et al. [14] as:

$$I(z) = I^{m,n}_{p_i,q_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} G(q^{b_j+B_js}) \prod_{j=1}^{n} G(q^{1-a_j-A_js})\pi z^{-s}}{\sum_{i=1}^{r}[\prod_{j=m+1}^{q_i} G(q^{1-b_{ji}-B_{ji}s}) \prod_{j=n+1}^{p_i} G(q^{a_{ji}+A_{ji}s})G(q^s)G(q^{1-s})sin\pi s]} ds \tag{11}$$

Setting $r = 1, p_i = p, q_i = q$, in equation (11) we get q-analogue of H-Function defined by Saxena et al. [14] as follows:

$$H^{m,n}_{p,q}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right.\right)\right] =$$

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} G(q^{b_j+B_js}) \prod_{j=1}^{n} G(q^{1-a_j-A_js})\pi z^{-s}}{\prod_{j=m+1}^{q} G(q^{1-b_j-B_js}) \prod_{j=n+1}^{p} G(q^{a_j+A_js})G(q^s)G(q^{1-s})sin\pi s} ds \tag{12}$$

Further, if we put $A_j = B_j = 1$, equation (12) is simplified to the basic analogue of Meijer's G-Function given by Saxena et al. [14].

$$G^{m,n}_{p,q}\left[\left(z;q\left|\begin{matrix}a_1,a_2,\ldots,a_p\\b_1,b_2,\ldots,b_q\end{matrix}\right.\right)\right] =$$

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} G(q^{b_j+s}) \prod_{j=1}^{n} G(q^{1-a_j-s})\pi z^{-s}}{\prod_{j=m+1}^{q} G(q^{1-b_j-s}) \prod_{j=n+1}^{p} G(q^{a_j+s})G(q^s)G(q^{1-s})sin\pi s} ds \tag{13}$$

Altaf et al. [15] gave the alternative definition of q-analogue of Aleph-function by using q-Gamma function as:

$$\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j+B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q(a_{ji}+A_{ji}s)\Gamma_q(s)\Gamma_q(1-s)sin\pi s]}ds$$

Where L is the contour of integration running from $-i\infty$ to $+i\infty$ in such a manner so that all poles of $\Gamma_q(b_j+B_js); 1\leq j\leq m$ are to right of the path and those of $\Gamma_q(1-a_j-A_js); 1\leq j\leq n$, are on the left. The integral converges if $Re[slog(z)-logsin\pi s]<0$, for large values of $|s|$ are on contour L.

## 2 Main Results

**(A)** In this section we will find the (p,q)-analogue of Aleph-function.
Swati Pathak and Renu Jain [16] defined the (p,q)-Gamma function as follows:

$$\Gamma_{p,q}(x)=\frac{((p,q);(p,q))_\infty(p-q)^{1-x}}{((p^x,q^x);(p,q))_\infty},(0<p<q). \tag{14}$$

When $x=n+1$ with a non-negative integer, this definition is simplified to

$$\Gamma_{p,q}(n+1)=[n]_{p,q}! \tag{15}$$

And

$$\Gamma_{p,q}(x+1)=[x]_{p,q}\Gamma_{p,q}(x),$$

we can deduce that $\Gamma_{p,q}(1)=1$.
Now we shall make use of $(p,q)$-Gamma function for defining $(p,q)$-analogue of Aleph-function, which is as follows:

$$\aleph^{m,n}_{p_i,q_i+1;\tau_ir}\left[\left(z(p-q)^{\sum_{t=1}^m\beta_t-\sum_{t=1}^n\alpha_t+\sum_{i=1}^r\tau_i[\sum_{t=m+1}^{q_i}\beta_{ti}-\sum_{t=n+1}^{p_i}\alpha_{ti}]};(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n},...,[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}(1,1)\end{matrix}\right.\right)\right]$$

$$=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}G(p^{b_j+B_js},q^{b_j+B_js})\prod_{j=1}^{n}G(p^{1-a_j-A_js},q^{1-a_j-A_js})(p-q)^{s[\sum_{t=1}^mB_t-\sum_{t=1}^nA_t+\sum_{i=1}^r\tau_i[\sum_{t=m+1}^{q_i}B_{ti}-\sum_{t=n+1}^{p_i}A_{ti}]]}\pi z^{-s}}{\sum_{i=1}^r\tau_i[\prod_{j=m+1}^{q_i}G(p^{1-b_{ji}-B_{ji}s},q^{1-b_{ji}-B_{ji}s})\prod_{j=n+1}^{p_i}G(p^{a_{ji}+A_{ji}s},q^{a_{ji}+A_{ji}s})G(p^s,q^s)G(p^{1-s},q^{1-s})sin\pi s]} \tag{16}$$

On multiplying equation (16) by

$$(p-q)^{\sum_{t=1}^na_t-\sum_{t=1}^mb_t+m+n-1+\sum_{i=1}^r\tau_i[\sum_{t=n+1}^{p_i}A_{ti}-\sum_{t=m+1}^{q_i}B_{ti}-A_i]}G(p,q)^{\sum_{t=1}np_t+q_t-(m+n-1)}$$

and then making use of identity

$$\Gamma_{p,q}(x)=\frac{G(p^x,q^x)}{(p-q)^{x-1}G(p,q)},|\frac{q}{p}|<1 \tag{17}$$

where

$$G(p^\alpha,q^\alpha)=\prod_{n=0}^{\infty}(p^{\alpha+n},q^{\alpha+n})^{-1} \tag{18}$$

$$=\frac{1}{((p^\alpha,q^\alpha);(p,q))_\infty}$$

The R.H.S. of equation (10) becomes

$$=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^r\tau_i[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}ds$$

**Special Cases:**
Taking $\tau_i = 1$, in above equation we will get the (p,q)-analogue of I-Function defined by Altaf et al. [17] as follows:

$$I^{m,n}_{p_i,q_i;r}\left[\left(z;(p,q)\middle|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right)\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}}{\sum_{i=1}^{r}[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}ds \tag{19}$$

Taking $r = 1$, in equation (19) we will get the (p,q)-analogue of Fox's H-Function defined by Altaf et al. [17] as follows:

$$H^{m,n}_{P,Q}\left[\left(z;(p,q)\middle|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right)\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}ds}{\prod_{j=m+1}^{Q}\Gamma_{p,q}(1-b_j-B_js)\prod_{j=n+1}^{P}\Gamma_{p,q}(a_j+A_js)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s} \tag{20}$$

If we take $A_j = B_j = 1$, in equation (20) we will get the (p,q)-analogue of Meijer's G-Function defined by Swati Pathak et al. [16] as follows:

$$G^{m,n}_{P,Q}\left[\left(z;(p,q)\middle|\begin{matrix}a_1,a_2,...,a_P\\b_1,b_2,...,b_Q\end{matrix}\right)\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+s)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-s)\pi z^{-s}ds}{\prod_{j=m+1}^{Q}\Gamma_{p,q}(1-b_j-s)\prod_{j=n+1}^{P}\Gamma_{p,q}(a_j+s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s} \tag{21}$$

Put $p = 1$, the above results of (p,q)-analogue change into well known results of basic analogues of Aleph-function, I-function, H-function and G-function [15].

**(B)** In this section, we will evaluate the $(p,q)$-derivative operator involving $(p,q)$-analogue of Aleph-function.
**Theorem 1**: Let the parameters $p_i, q_i$ be non-negative integers satisfying the inequality $0 \le n \le p_i, 0 \le m \le q_i$ and $\tau_i > 0; i = 1,2,3,...,r$ is finite and $a_j, b_j, a_{ji} and b_{ji}$ are complex numbers, then

$$zD_{p,q}\left[z^{1-a_1}\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[\left(z;(p,q)\middle|\begin{matrix}(a_j,1)_{1,n},...,[\tau_i(a_{ji},1)]_{n+1,p_i}\\(b_j,1)_{1,m},...,[\tau_i(b_{ji},1)]_{m+1,q_i}\end{matrix}\right)\right]\right]$$
$$=z^{1-a_1}\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[\left(z;(p,q)\middle|\begin{matrix}(a_1-1,1)(a_j,1)_{2,n},...,[\tau_i(a_{ji},1)]_{n+1,p_i}\\(b_j,1)_{1,m},...,[\tau_i(b_{ji},1)]_{m+1,q_i}\end{matrix}\right)\right] \tag{22}$$

Where $z \neq 0, 0 < q < p$ and $\omega = \sqrt{-1}$.
**Proof:** To prove theorem (1), when $a_1 > 0$, we apply equation (8)

$$L.H.S. = zD_{p,q}\left[z^{1-a_1}\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+s)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-s)\pi z^{-s}ds}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}\right]$$

$$=z\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+s)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-s)\pi D_{p,q}[z^{1-a_1-s}]ds}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+s)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-s)\pi[1-a_1-s]_{p,q}z^{1-a_1-s}ds}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}$$

Since

$$\Gamma_{p,q}(1+a) = \frac{p^a-q^a}{p-q}\Gamma_{p,q}(a) = [a]_{p,q}\Gamma_{p,q}(a)$$

$$\therefore [1-a_1-s]_{p,q}\Gamma_{p,q}(1-a_1-s) = \Gamma_{p,q}(1-(a_1-1)-s)$$

Thus

$$L.H.S. = \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+s)\Gamma_{p,q}(1-(a_1-1)-s)\prod_{j=2}^{n}\Gamma_{p,q}(1-a_j-s)\pi z^{-s}z^{1-a_1}ds}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}$$

$$= z^{1-a_1}\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[\left(z;(p,q)\left|\begin{matrix}(a_1-1,1)(a_j,1)_{2,n},\dots,[\tau_i(a_{ji},1)]_{n+1,p_i}\\(b_j,1)_{1,m},\dots,[\tau_i(b_{ji},1)]_{m+1,q_i}\end{matrix}\right.\right)\right]$$

Hence the result.

**Theorem 2:** Let parameters $p_i$ and $q_i$ be non-negative integers satisfying the inequality $0 \le n \le p_i, 0 \le m \le q_i$ and $\tau_i > 0; i = 1,2,3,\frac{1}{2},r$ is finite, and $A_j, B_j, B_{ji}$ and $B_{ji}$ are positive real numbers, and $a_j, b_j, a_{ji}$ and $b_{ji}$ are complex numbers, then

$$D^{\mu}_{p,q}\left[\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right]\right]$$
$$= \frac{(-1)^{\mu}x^{-\mu}}{(pq)^{\mu(\mu-1)/2}}\aleph^{m+1,n}_{p_i,q_i+1;r}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(\mu,\lambda)(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}(0,\lambda)\end{matrix}\right.\right)\right] \tag{23}$$

Where $x \neq 0, 0 < q < p$, and $\omega = \sqrt{-1}$.

**Proof:** For $\lambda \geq 0$, the left side of above equation becomes

$$L.H.S. = D^{\mu}_{p,q}\left[\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right]\right]$$

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi D_{p,q}[(px^{\lambda})^{-s}]ds}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}$$

That is

$$= \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi\Gamma_{p,q}(\lambda s+\mu)p^{-s}x^{-\lambda s-\mu}ds}{\sum_{i=1}^{r}\tau_i[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(\lambda s+1)(pq)^{\mu\lambda s}\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}$$

$$= \frac{(-1)^{\mu}x^{-\mu}}{(pq)^{\mu(\mu-1)/2}}\frac{1}{2\pi\omega}\int_L \frac{\Gamma_{p,q}(\mu+\lambda s)\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi\frac{px^{\lambda}}{(pq)^{-\mu\lambda}}^{-s}ds}{\sum_{i=1}^{r}\tau_i[\Gamma_{p,q}(1-0+\lambda s)\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)sin\pi s]}$$

$$= \frac{(-1)^{\mu}x^{-\mu}}{(pq)^{\mu(\mu-1)/2}}\aleph^{m+1,n}_{p_i,q_i+1;r}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(\mu,\lambda)(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}(0,\lambda)\end{matrix}\right.\right)\right]$$

Hence the result.

**Special Cases:**

By putting $\tau_i = 1$ in (23) and (24), we get the (p,q)-analogue of I-function defined by Altaf et al.[17] as mentioned below:

$$zD_{p,q}\left[z^{1-a_1} I^{m,n}_{p_i,q_i;r}\left[\left(z;(p,q)\left|\begin{matrix}(a_j,1)_{1,n};(a_{ji},1)_{n+1,p_i}\\(b_j,1)_{1,m};(b_{ji},1)_{m+1,q_i}\end{matrix}\right.\right)\right]\right]$$

$$= z^{1-a_1} I^{m,n}_{p_i,q_i;r}\left[\left(z;(p,q)\left|\begin{matrix}(a_1-1,1)(a_j,1)_{2,n};(a_{ji},1)_{n+1,p_i}\\(b_j,1)_{1,m};(b_{ji},1)_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{24}$$

$$D^{\mu}_{p,q}\left[I^{m,n}_{p_i,q_i;r}\left[\left(px^{\lambda};(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right]\right]$$

$$= \frac{(-1)^{\mu}x^{-\mu}}{(pq)^{\mu(\mu-1)/2}} I^{m+1,n}_{p_i,q_i+1;r}\left[\left(\frac{px^{\lambda}}{(pq)^{-\mu\lambda}};(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(\mu,\lambda)(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}(0,\lambda)\end{matrix}\right.\right)\right] \tag{25}$$

By putting $p = 1$, the above results of $(p,q)$-analogue change into well known results of basic analogue of I-function [18] as mentioned below:

$$zD_{q}\left[z^{1-a_1} I^{m,n}_{p_i,q_i;r}\left[\left(z;q\left|\begin{matrix}(a_j,1)_{1,n};(a_ji,1)_{n+1,p_i}\\(b_j,1)_{1,m};(b_{ji},1)_{m+1,q_i}\end{matrix}\right.\right)\right]\right]$$

$$= z^{1-a_1} I^{m,n}_{p_i,q_i;r}\left[\left(z;q\left|\begin{matrix}(a_1-1,1)(a_j,1)_{2,n}(a_{ji},1)_{n+1,p_i}\\(b_j,1)_{1,m};(b_{ji},1)_{m+1,q_i}\end{matrix}\right.\right)\right] \tag{26}$$

$$D^{\mu}_{q}\left[I^{m,n}_{p_i,q_i;r}\left[\left(x^{\lambda};q\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)\right]\right]$$

$$= \frac{(-1)^{\mu}x^{-\mu}}{(q)^{\mu(\mu-1)/2}} I^{m+1,n}_{p_i,q_i+1;r}\left[\left(\frac{x^{\lambda}}{(q)^{-\mu\lambda}};q\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(\mu,\lambda)(b_j,1)_{1,m};(b_{ji},1)_{m+1,q_i}(0,\lambda)\end{matrix}\right.\right)\right] \tag{27}$$

## 3 Conclusion

The results proved in this paper give some contribution to the $(p,q)$-algebra and are believed to be new and fruitful for $(p,q)$ analogue and are likely to find certain applications to the solution of the $(p,q)$-integral equations involving various $(p,q)$-hypergeometric functions.This research is certainly not complete and should be a starting point of many other researches.


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## Conflict of Interest

The authors declare that they have no conflict of interest.

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