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On Dynamics of a Fractional-Order SIRS Epidemic Model with Standard Incidence Rate and its Discretization

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Abstract: In this work, we study the dynamics of fractional-order SIRS epidemic model with standard incidence rate and its discretization. It is shown that the discretized system exhibits much richer dynamical behaviors than its corresponding fractional-order form. Local stability of the fractional-order model is studied. Also, many types of bifurcation have been obtained. Numerical simulations are carried out to verify theoretical results obtained.

Keywords: SIRS epidemic model, fractional-order, discretization, standard incidence rate, fixed points, dynamical behaviors of model, bifurcations.

1 Introduction

The theory of epidemics showed that, the mathematical models divided into two types: the first type, the discrete-time models described by difference equations and the second type, the continuous-time models described by differential equations. A lot of authors investigated the continuous-time epidemic models in many manuscripts [1,2,3,4,5,6,7,8,9,10]. In the last years, we interested to the discrete-time models because of the statical data showed that the discrete-time epidemic models more appropriate and accurate than the continuous-time models,

the numerical simulations obtained by the discrete-time models more accurate than the numerical simulations which are obtained by continuous-time models.

It is also known that, there are many status to get discrete-time epidemic models: firstly, by using the property of the epidemic disease directly [11,12]. Secondly, by transforming the continuous-time model into discrete- time model by using the center manifold theorem and the forward Euler scheme [13].

Moreover, a discrete-time models have more rich dynamical behaviors than its counterparts from a continuous-time models.

In recent years, mathematical models are the main objective to understand the epidemiological patterns and diseases control for along time. There are different approximation to investigate epidemic models, such as ordinary differential equations, difference equations, fractional-order differential equations and so on.

In [14,15], the fractional-order differential equations (FOD) have been used to study the modeling of memory and and genetical effects.

So, the fractional-order differential equations have been used in many mathematical biology [16,17] and other fields [18,19,20,21]. A lot of models in interdisciplinary fields can be investigated by the fractional-order differential equations such as nonlinear oscillation of earthquakes [21], hydrologic models [22], viscoelastic material models [23], diffusion waves [24], and other fields (see [25,26,27,28,29]). Moreover, the fractional-order differential equations can be used in the study of mathematical biology [30]. So, fractional-order equations widely used utilized in mathematical biology [31,32,33,34].

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There exist a lot of definitions for the fractional derivatives. The most famous definitions is Caputo definition of fractional derivatives [35] is given as follows

$$D^{\alpha} f(t) = I^{l-\alpha} f^{(l)}(t),\ \alpha > 0, \tag{1}$$

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where $f^{(l)}$ is l-derivative of f, $l = [\alpha]$ is the value of α and I^{θ} is the Riemann-Liouville integral operator of order θ given as follow

$$I^{\theta} h(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-\tau)^{\theta-1} h(\tau) d\tau,\ \theta > 0, \tag{2}$$

where $\Gamma(\theta)$ is the Euler's Gamma function. The operator D^{α} denoted to Caputo differential operator of order α.
Usually, in the theory of disease spreading, the population is divided into three kinds of individuals: susceptible (S), infective (I), and recovered (R). In the last years, A lot of authors investigated the qualitative properties of fractional-order population models [36,34,37,38,39,40]. The work pioneers Kermack and Mckendrick [41] introduced the first mathematical model of an epidemic. Many authors studied different types of epidemic models to recognize to diseases transmission [42,43,44,45,46,47]. Also, the authors [48,49,50] investigated the local stability and global stability of the disease free and endemic equilibrium for some kinds of diseases such as SI, SIS and SIR types.The aim of this manuscript to study one type of the basic epidemic models in details and compare between the theoretical results and its numerical simulations of the model.
The rest of this paper is organized as follows: in section 2, we study the fractional-order of SIRS epidemic model. Stability of the fractional differential equations are investigated in section 3. In section 4, we investigate stability analysis of the fractional-order SIRS epidemic model. We study dynamics of the fractional-order SIRS epidemic model and its discretization in section 5. In section 6, stability of the equilibrium points of the discretized system are studied. In section 7, the numerical simulations are carried out to confirm the theoretical results. Finally, the concluding remarks are given in section 8.

2 The fractional-order SIRS epidemic model

In the following section, the continuous-time SIRS epidemic model given by the equations.

$$\begin{aligned} \frac{dS}{dt} &= A - d_1 S(t) - \beta S(t) I(t) + \sigma R(t), \\ \frac{dI}{dt} &= \beta S(t) I(t) - d_2 I(t) - \gamma I(t), \\ \frac{dR}{dt} &= \gamma I(t) - d_3 R(t) - \sigma R(t), \end{aligned} \tag{3}$$

where $S(t), I(t)$ and $R(t)$ denote the numbers of susceptible, infective and recovered individuals at time t, respectively, A is the recruitment rate of the population, $d_i (i = 1,2,3)$ is the death rate of $S(t), I(t)$ and $R(t)$, respectively, γ is the recovery rate of the infective individuals, βSI is the bilinear incidence rate and σ is the rate by which the recovered individuals become susceptible again. If we considered $d_i (i = 1,2,3) = m_1$.
The model (3) becomes:

$$\begin{aligned} \frac{dS}{dt} &= A - m_1 S(t) - \beta S(t) I(t) + \sigma R(t), \\ \frac{dI}{dt} &= \beta S(t) I(t) - (m_1 + \gamma) I(t), \\ \frac{dR}{dt} &= \gamma I(t) - (m_1 + \sigma) R(t), \end{aligned} \tag{4}$$

by letting $\bar{t} = m_1 t$, $\frac{A}{m_1} = \bar{a}$, $\frac{\beta}{m_1} = \bar{b}$, $\frac{\sigma}{m_1} = \bar{c}$ and $\frac{\gamma}{m_1} = \bar{m}$,
The model (4) transformed to:

$$\begin{aligned} \frac{dS}{dt} &= a - S - bSI + cR, \\ \frac{dI}{dt} &= bSI - (1+m)I, \\ \frac{dR}{dt} &= mI - (1+c)R. \end{aligned} \tag{5}$$

The fractional-order of SIRS model (5) is given as follows

$$\begin{aligned} \frac{d^\alpha S}{dt^\alpha} &= a - S - bSI + cR, \\ \frac{d^\alpha I}{dt^\alpha} &= bSI - (1+m)I, \\ \frac{d^\alpha R}{dt^\alpha} &= mI - (1+c)R, \end{aligned} \tag{6}$$

with initial conditions

$$S(0) = S_0 \geq 0,\ I(0) = I_0 \geq 0,\ R(0) = R_0 \geq 0$$

.
Where α denote the fractional order verifying $0 < \alpha \leq 1$. In (6), we study the fractional-order SIRS epidemic model to deduce the memory-effect on its complex nonlinear dynamics.

3 Stability of fractional-differential equations and applications to dynamical systems

In [51,52,53], we get the stability conditions of fractional-order systems. the following equation denote to the nonlinear fractional-order model

$$D^\alpha S(t) = f(S(t)),\ S(0) = S_0, \tag{7}$$

where $\alpha \in (0,1]$, $S(t) \in R^3$. In [51] the Matignons conditions showed that the stability of the equilibrium points.

$$|arg(\lambda_i)| > \alpha\pi/2,\ (i = 1,2,3), \tag{8}$$

where λ_1, λ_2 and λ_3 are the eigenvalues $J = \frac{\partial f}{\partial S}$.

4 Stability of the fractional-order SIRS epidemic model

In order to calculate the equilibrium points of model (6), we put $\frac{d^\alpha S}{dt^\alpha} = 0$, $\frac{d^\alpha I}{dt^\alpha} = 0$, $\frac{d^\alpha R}{dt^\alpha} = 0$.
We deduced that the system has two equilibrium points:

1. $E_1 = (a,0,0)$ is the disease-free equilibrium.
2. $E_2 = (S^*, I^*, R^*) = (\frac{m+1}{b}, \frac{(ab-m-1)(1+c)}{b(m+c+1)}, \frac{abm-m^2-m}{b(m+c+1)})$ is an endemic equilibrium.

At the endemic point (S^*, I^*, R^*) the Jacobian matrix of model (6) take the form

$$J(S^*, I^*, R^*) = \begin{pmatrix} -1 - bI^* & -bS^* & c \\ bI^* & bS^* - (1+m) & 0 \\ 0 & m & -(1+c) \end{pmatrix}. \tag{9}$$

Theorem 1. *The disease-free equilibrium $E_1(a,0,0)$ has the following topological properties*

1. E_1 is sink if $ba < 1+m, c > 0$.

2.E_1 is saddle if $ba > 1+m, c < 0$.

*Proof.*The Jacobian matrix for E_1 is given by

$$J(E_1) = \begin{pmatrix} -1 & -ba & c \\ 0 & ba-(1+m) & 0 \\ 0 & m & -(1+c) \end{pmatrix}. \tag{10}$$

The eigenvalues corresponding to E_1 are $\lambda_1 = -1, \lambda_2 = ba-(1+m)$ and $\lambda_3 = -1-c$. It is clear that $|\lambda_1| < 0, |\lambda_2| < 0$ if $ba < 1+m$ and $|\lambda_3| < 0$ if $c > 0$ which implies that E_1 of the system (5.1.4) is a sink, so the sink is locally asymptotically stable, E_1 is saddle if $|\lambda_1| < 0, |\lambda_2| < 0$ if $ba > 1+m$ and $|\lambda_3| < 0$ if $c < 0$.

5 Dynamical behaviors of the fractional-order SIRS epidemic model and its discretization

We study dynamical behaviors of the fractional-order SIRS epidemic model and its discretization for equation (6). We used a discretization method to discretize fractional-order differential equations (see [54],[55]).
the following steps to get the discretization method is:
when $S(0) = S_0,\ I(0) = I_0,\ R(0) = R_0$ are the initial conditions of system (6). The discretization process of model (6) with piecewise constant arguments as follow

$$\begin{aligned} D^\alpha S(t) &= a - S([t/s]s) - bS([t/s]s)I([t/s]s) - cR([t/s]s), \\ D^\alpha I(t) &= bS([t/s]s)I([t/s]s) - (1+m)I([t/s]s), \\ D^\alpha R(t) &= mI([t/s]s) - (1+c)R([t/s]s), \end{aligned} \tag{11}$$

we put t verify the condition $0 \le t < s$, which implies that $0 \le \frac{t}{s} < 1$. Therefore, the Eq.(11) becomes

$$\begin{aligned} D^\alpha S(t) &= a - S_0 - bS_0I_0 - cR_0, \\ D^\alpha I(t) &= bS_0I_0 - (1+m)I_0, \\ D^\alpha R(t) &= mI_0 - (1+c)R_0, \end{aligned} \tag{12}$$

and the solution of (12) becomes

$$\begin{aligned} S_1(t) &= S_0 + I^\alpha[a - S_0 - bS_0I_0 - cR_0] = S_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)}[a - S_0 - bS_0I_0 - cR_0], \\ I_1(t) &= I_0 + I^\alpha[bS_0I_0 - (1+m)I_0] = I_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)}[bS_0I_0 - (1+m)I_0], \\ R_1(t) &= R_0 + I^\alpha[mI_0 - (1+c)R_0] = I_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)}[mI_0 - (1+c)R_0], \end{aligned}$$

then, we take $t \in [s, 2s)$, which makes $1 \le \frac{t}{s} < 2$. Hence, we get

$$\begin{aligned} D^\alpha S(t) &= a - S_1 - bS_1I_1 - cR_1, \\ D^\alpha I(t) &= bS_1I_1 - (1+m)I_1, \\ D^\alpha R(t) &= mI_1 - (1+c)R_1, \end{aligned}$$

by discretizing the previous equation the solution become

$$\begin{aligned} S_2(t) &= S_1 + \frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)}[a - S_1 - bS_1I_1 - cR_1], \\ I_2(t) &= I_1 + \frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)}[bS_1I_1 - (1+m)I_1], \\ R_2(t) &= R_1 + \frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)}[mI_1 - (1+c)R_1], \end{aligned}$$

by repeating the discretization method n times, we get

$$\begin{aligned} S_{n+1}(t) &= S_n(ns) + \frac{(t-ns)^\alpha}{\alpha\Gamma(\alpha)}[a - S_n(ns) - bS_n(ns)I_n(ns) - cR_n(ns)], \\ I_{n+1}(t) &= I_n(ns) + \frac{(t-ns)^\alpha}{\alpha\Gamma(\alpha)}[bS_n(ns)I_n(ns) - (1+m)I_n(ns)], \\ R_{n+1}(t) &= R_n(ns) + \frac{(t-ns)^\alpha}{\alpha\Gamma(\alpha)}[mI_n(ns) - (1+c)R_n(ns)], \end{aligned} \tag{13}$$

where $ns \leq t < (n+1)s$. As t approaches to $(n+1)s$, the system (13) is reduced to

$$\begin{aligned} S_{n+1} &= S_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)}[a - S_n - bS_nI_n - cR_n], \\ I_{n+1} &= I_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)}[bS_nI_n - (1+m)I_n], \\ R_{n+1} &= R_n + \frac{s^\alpha}{\alpha\Gamma(\alpha)}[mI_n - (1+c)R_n]. \end{aligned} \tag{14}$$

Remark 2 In Eq. (14) if the parameter α tends to one, we get the forward Euler discretization of model (6).

6 Stability of the equilibrium points of the discretized system

In this section, we study the stability of the equilibrium points of model (14) at the same equilibrium points of model (6). The stability of model (14) can be get by calculating the Jacobian matrices at its equilibrium points. The Jacobian matrix of model (14) take the form

$$J(S_n,I_n,R_n) = \begin{pmatrix} 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}(-1-bI_n) & -\frac{s^\alpha}{\alpha\Gamma(\alpha)}bS_n & \frac{s^\alpha}{\alpha\Gamma(\alpha)}c \\ \frac{s^\alpha}{\alpha\Gamma(\alpha)}bI_n & 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}(bS_n-(1+m)) & 0 \\ 0 & \frac{s^\alpha}{\alpha\Gamma(\alpha)}m & 1-\frac{s^\alpha}{\alpha\Gamma(\alpha)}(1+c) \end{pmatrix}. \tag{15}$$

In order to study the stability analysis of the equilibrium points of system (14), we recall the following lemma.

Lemma 1.*[56] Let λ_1,λ_2 and λ_3 are the three roots of matrix $J(E)$, we have the following definitions*
(i) If $|\lambda_1|<1, |\lambda_2|<1$ and $|\lambda_3|<1$, then the equilibrium point $E(S^*,I^*,R^*)$ is locally asymptotically stable (sink).
(ii) If $|\lambda_1|>1, |\lambda_2|>1$ and $|\lambda_3|>1$, then the equilibrium point $E(S^*,I^*,R^*)$ is unstable (source).
(iii) If $|\lambda_1|>1, |\lambda_2|>1$ and $|\lambda_3|<1$ (or $|\lambda_1|<1, |\lambda_2|<1$ and $|\lambda_3|>1$), then the equilibrium point $E(S^*,I^*,R^*)$ is locally unstable (saddle).
(iv) If $|\lambda_1|=1$ or $|\lambda_2|=1$ or $|\lambda_3|=1$, then the equilibrium point $E(S^*,I^*,R^*)$ is called non-hyperbolic.

Then, we study the local stability of the equilibrium points of discrete-time model (14).

Theorem 2.*The equilibrium point E_1 of model (14) has four various topological types of equilibrium points:*

(i) E_1 is a sink if one of the following conditions holds:
(i.1) $(2-\varepsilon(2+c))^2-4(1-\varepsilon(2+c)+\varepsilon^2(1+c))\geq 0$, $c>-1$, $\frac{-2+(1+m)\varepsilon}{\varepsilon}<ba<1+m$, *where* $\varepsilon=\frac{s^\alpha}{\alpha\Gamma(\alpha)}$,
(i.2) $(2-\varepsilon(2+c))^2-4(1-\varepsilon(2+c)+\varepsilon^2(1+c))<0$, $c>\frac{\varepsilon-2}{1-\varepsilon}$, $\frac{-2+(1+m)\varepsilon}{\varepsilon}<ba<1+m$.

(ii) E_1 is a source if one of the following conditions holds:
(ii.1) $(2-\varepsilon(2+c))^2-4(1-\varepsilon(2+c)+\varepsilon^2(1+c))\geq 0$, $c<-1$, $\frac{-2+(1+m)\varepsilon}{\varepsilon}>ba>1+m$,
(ii.2) $(2-\varepsilon(2+c))^2-4(1-\varepsilon(2+c)+\varepsilon^2(1+c))<0$, $c<\frac{\varepsilon-2}{1-\varepsilon}$, $\frac{-2+(1+m)\varepsilon}{\varepsilon}>ba>1+m$.

(iii) E_1 is non-hyperbolic if one of the following conditions holds:
(iii.1) $(2-\varepsilon(2+c))^2-4(1-\varepsilon(2+c)+\varepsilon^2(1+c))\geq 0$, $c=-1$, $ba=\frac{-2+(1+m)\varepsilon}{\varepsilon}$ *or* $ba=1+m$,
(iii.2) $(2-\varepsilon(2+c))^2-4(1-\varepsilon(2+c)+\varepsilon^2(1+c))<0$, $c=\frac{\varepsilon-2}{1-\varepsilon}$, $ba=\frac{-2+(1+m)\varepsilon}{\varepsilon}$ *or* $ba=1+m$.

(iv) E_1 is a saddle for the other values of parameters except those values in (i)-(iii).

Proof. At the free-disease point E_1 the Jacobian matrix (11)take the form

$$J(E_1)=\begin{pmatrix}1-\varepsilon & -\varepsilon ba & \varepsilon c\\ 0 & 1+\varepsilon(ba-(1+m)) & 0\\ 0 & \varepsilon m & 1-\varepsilon(1+c)\end{pmatrix} \tag{16}$$

where the eigenvalues are $\lambda_1=1+\varepsilon(ba-(1+m))$ and

$\lambda_{2,3}=\frac{(2-\varepsilon(2+c))\pm\sqrt{(2-\varepsilon(2+c))^2-4(1+\varepsilon(2+c)+\varepsilon^2(1+c))}}{2}$.

This showed that the characteristic equation $\lambda^2-TrM\lambda+DetM=0$ is verified at $\lambda_{2,3}$, where

$$J(M)=\begin{pmatrix}1-\varepsilon & \varepsilon c\\ 0 & 1-\varepsilon(1+c)\end{pmatrix},$$

$TrM=2-\varepsilon(2+c)$, $DetM=1-\varepsilon(2+c)+\varepsilon^2(1+c)$. Applying Jurys criterion [57] and using lemma (11), we achieve the results (i)-(iv).

Theorem 3. *The disease-free equilibrium point $E_1=(a,0,0)$ loses its stability:*
(i) Through a flip point when $c\neq-1$, $ba=\frac{-2+(1+m)\varepsilon}{\varepsilon}$ and $ba\neq1+m$.
(ii) Through a Neimark-Sacker point at $c=\frac{\varepsilon-2}{1-\varepsilon}$ and $ba\neq\frac{-2+(1+m)\varepsilon}{\varepsilon}$ or $ba\neq1+m$.

Proof. At one of the eigenvalues equals to -1 and the other eigenvalues not equal to 1 nor -1 in this case the a flip bifurcation occurs in three-dimensional [57]. When the condition (iii.1) of theorem (2) holds, the model (14) give a flip bifurcation at the equilibrium point $E_1=(a,0,0)$ at small neighborhood of FB_{E_1} where
FB_{E_1}= {(a, b, c, m): $c\neq-1$, $ba=\frac{-2+(1+m)\varepsilon}{\varepsilon}$ and $ba\neq1+m$ and $(2-\varepsilon(2+c))^2-4(1-\varepsilon(2+c)+\varepsilon^2(1+c))\geq0$}.
Also, when two eigenvalues are a pair of complex conjugate and the others eigenvalues not equal to 1 nor -1 in this case a Neimark-Sacker bifurcation occurs. When the condition (iii.2) of theorem (2) holds, the equilibrium point $E_1=(a,0,0)$ undergoes a Neimark-Sacker bifurcation at c change in a small neighborhood of NSB_{E_1} where
NSB_{E_1}= {(a, b, c, m): $c=\frac{\varepsilon-2}{1-\varepsilon}$, $ba\neq\frac{-2+(1+m)\varepsilon}{\varepsilon}$ and $ba\neq1+m$ and $(2-\varepsilon(2+c))^2-4(1-\varepsilon(2+c)+\varepsilon^2(1+c))<0$}.

The Jacobian matrix at an endemic equilibrium point E_2 of system (14) given by the form

$$J(S^*,I^*,R^*)=\begin{pmatrix}1+\varepsilon(-1-bI^*) & -\varepsilon bS^* & \varepsilon c\\ \varepsilon bI^* & 1+\varepsilon(bS^*-(1+m)) & 0\\ 0 & \varepsilon m & 1-\varepsilon(1+c)\end{pmatrix}, \tag{17}$$

where
$E_2=(S^*,I^*,R^*)=(\frac{m+1}{b},\frac{(ab-m-1)(1+c)}{b(m+c+1)},\frac{abm-m^2-m}{b(m+c+1)})$.
The characteristic equation of the Jacobian matrix $J(E_2)$ at the endemic equilibrium $E_2(S^*,I^*,R^*)$ given by the equation

$$F(\lambda)=\lambda^3+q_1(\varepsilon)\lambda^2+q_2(\varepsilon)\lambda+q_3(\varepsilon)=0, \tag{18}$$

where
$q_1(\varepsilon)=-3+\varepsilon(2+c+bI^*)$,
$q_2(\varepsilon)=3-\varepsilon(2+2bI^*)+\varepsilon^2(1+c+bI^*+bcI^*+b^2S^*I^*)$,
$q_3(\varepsilon)=1-\varepsilon(2+c+bI^*)+\varepsilon^2(1+c+bI^*+bcI^*+b^2S^*I^*)+\varepsilon^3(bcmI^*+b^2S^*I^*+b^2cS^*I^*)$.
The equation (18) transformed to

$$\lambda^3+p\lambda+q=0, \tag{19}$$

according to the Cardano formula. We choose the parameters b,c,m, and ε satisfying $\Delta>0$ and the conjugate complex roots $\lambda_{2,3}$ equal to one according to the condition (iv) of lemma (15) in [56]. At $E_2(S^*,I^*,R^*)$ of system (14) appears a Neimark-Sacker bifurcation. Equation (19) has one real root $\lambda_1=\frac{-p}{3}+(\frac{-q}{2}+\sqrt{\Delta})^{\frac{1}{3}}+(\frac{-q}{2}-\sqrt{\Delta})^{\frac{1}{3}}$, and a pair of conjugate complex roots $\lambda_{2,3}=\alpha\pm\beta i$, where $\alpha=\frac{-p}{3}-\frac{1}{2}[(\frac{-q}{2}+\sqrt{\Delta})^{\frac{1}{3}}+(\frac{-q}{2}-\sqrt{\Delta})^{\frac{1}{3}}]$, $\beta=\frac{\sqrt{3}}{2}[(\frac{-q}{2}+\sqrt{\Delta})^{\frac{1}{3}}+(\frac{-q}{2}-\sqrt{\Delta})^{\frac{1}{3}}]$, $\Delta=(\frac{q}{2})^2+(\frac{p}{3})^3$, $p=\frac{q_2(\varepsilon)-\frac{1}{3}(q_1(\varepsilon))^2}{3}$, $q=\frac{\frac{2}{27}(q_1(\varepsilon))^3-\frac{1}{3}q_1(\varepsilon)q_2(\varepsilon)+q_3(\varepsilon)}{27}$.

7 Numerical simulations

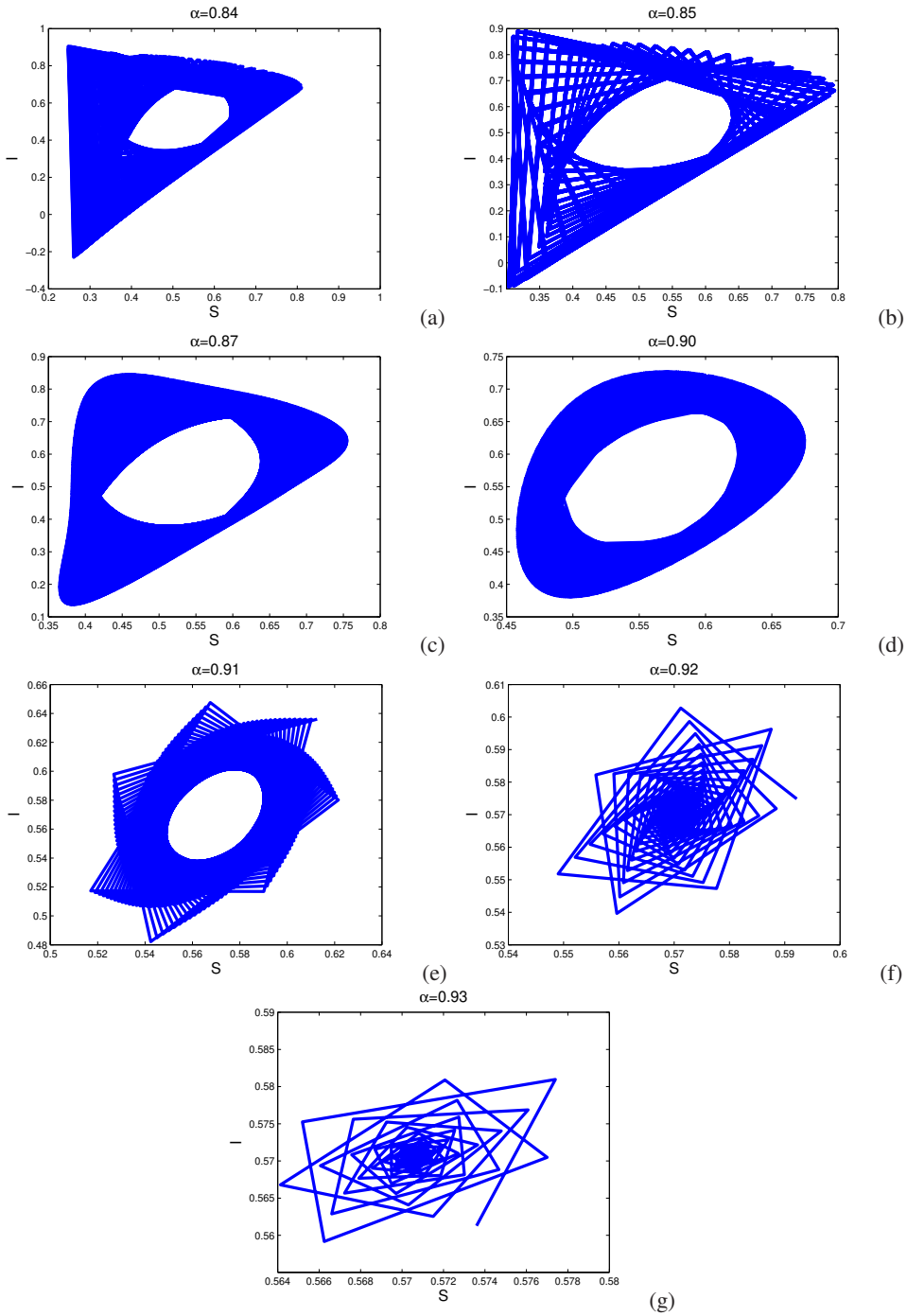

Fig. 1: Phase plane for system (14) with $a = 3.20, b = 1.7, c = 1.5, m = 2.5, s = 0.6$, and different α.

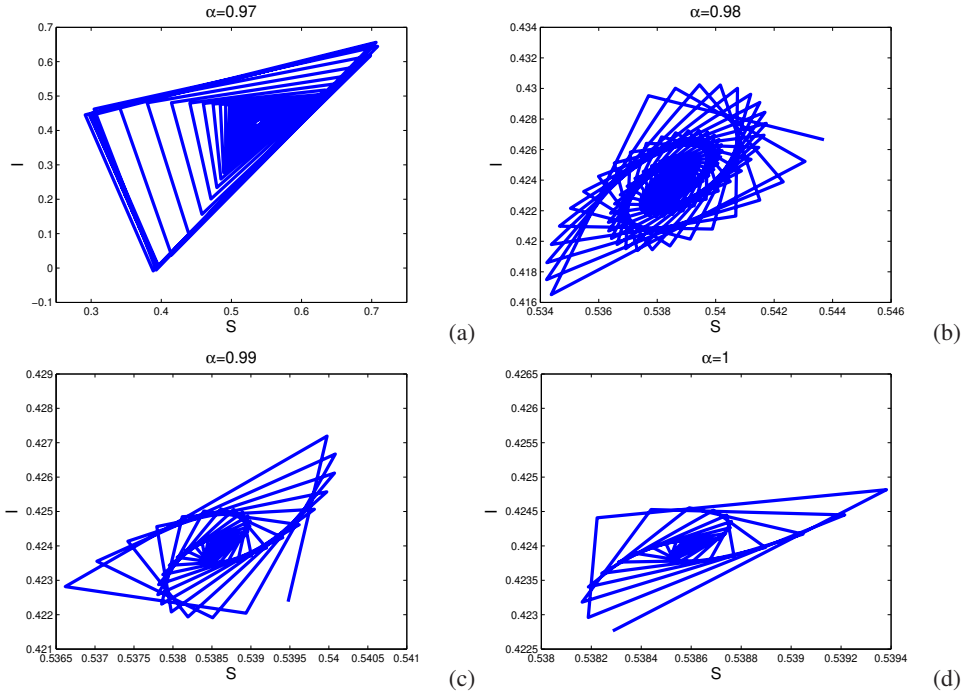

Fig. 2: Phase plane for system (14) with $a = 2.85, b = 1.51, c = 1.35, m = 1.85, s = 0.88$, and different α.

In this section, we study the phase portraits and chaotic attractor of system (14) to confirm the above theoretical results. By putting the parameters as $a = 3.20, b = 1.7, c = 1.5, m = 2.5$, and $h = 0.6$, we take the equilibrium points $(S, I, R) = (3.20, 0.1, 0.1)$ in all numerical simulation. Then we vary the parameters α. The phase portraits of model (14) are shown in Fig. 1 (a-d) and chaotic behaviors of system (14) are illustrated in Fig. 1(e-g). Also, chaotic behaviors of system (14) occurs when $a = 2.85, b = 1.51, c = 1.35, m = 1.85, h = 0.88$, and different α are shown in Fig. 2. At s is very small, we deduce that the discrete-time model (14) be more convenient of the fractional-order model(11).

8 Conclusion

In this manuscripts, we have investigated some nonlinear dynamics of the fractional-order SIRS epidemic model and its discretization . We have investigated the local stability of disease-free equilibrium and endemic equilibrium of the fractional-order system and its discretized counterpart with standard incidence rate. It is show that the numerical simulations are sufficient to prove the analytical analysis and to illustrate the difference the fractional-order SIRS epidemic model and its discretized counterpart. These results show that the SIRS epidemic model with standard incidence rate is very rich.

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