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Bayesian Estimation and Prediction for the Inverse Weibull Distribution Based on Lower Record Values

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Abstract: In this paper, the author have attempted to obtained Bayes and maximum likelihood estimators for the two unknown parameters of the inverse Weibull distribution using lower record values. Also, Bayes estimators under the squared error loss with a bivariate prior distribution have been derived. Prediction for future lower record values is presented from a Bayesian view point, as well. Numerical computations are presented using R software to illustrate the results.

Keywords: Bayes estimator, Maximum likelihood estimator, Bayes prediction, Lower record values.

1 Introduction

Chandler [1] was the first to study record theory and define a mathematical model for record values. Record values arise naturally in many applications involving data related to weather, sports, economics and life testing studies. Interested readers may refer to [2,3] and [4] for a review of various developments concerning record values.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (*iid*) random variables (*rv's*) with a cumulative distribution function (*cdf*) $F(x)$ and a probability density function (*pdf*) $f(x)$. Let $Y_n = min\{X_1, X_2, ..., X_n\}$ for $n \geq 1$. The value X_j is said to be a lower record value of $\{X_n, n \geq 1\}$ if $Y_j < Y_{j-1}, j > 1$. The sequence $\{L_{(n)}, n \geq 1\}$ is called the lower record times where $L_{(n)} = 1$ and $L_{(n)} = min\{j | j > L_{(n-1)}, X_j < X_{L(n-1)}, n \geq 1\}$. We will denote $L_{(n)}$ as the indices where the lower record values occur. The $n-th$ lower record value will be denoted by $X_{L_{(n)}}$.

If $F_{L_{(n)}}(x)$ is the *cdf* of $X_{L_{(n)}}$ for $n \geq 1$, then we have,

$$F_{L_{(n)}}(x) = \int_{-\infty}^{x} \frac{[H(u)]^{n-1}}{\Gamma n} dF(u), \qquad -\infty < x < \infty. \tag{1}$$

The *pdf* of $X_{L_{(n)}}$ is given by

$$f_{L_{(n)}}(x) = \frac{[H(x)]^{n-1}}{\Gamma n} f(x), \qquad -\infty < x < \infty. \tag{2}$$

The conditional *pdf* of $X_{L_{(s)}}$ given $X_{L_{(m)}} = x_m, \quad 1 \leq m < s$, is

$$f(y|x_m) = \frac{[H(y) - H(x_m)]^{s-m-1}}{\Gamma(s-m)} \frac{f(y)}{F(x_m)}, \qquad 0 < y < x_m < \infty, \tag{3}$$

where $H(x) = -lnF(x)$.

The likelihood function associated with record data is given by [4]

$$L(\mu, \sigma | \underline{x}) = f(x_m; \mu, \sigma) \prod_{i=1}^{m-1} h(x_i; \mu, \sigma) \tag{4}$$

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where $\underline{x} = (x_1, x_2, ..., x_m)$ and $h(x_i;\mu,\sigma) = \frac{f(x_i;\mu,\sigma)}{F(x_i;\mu,\sigma)}$.

The two-parameter inverse Weibull(*IW*) distribution was first introduced in literature by [5] as a reasonable model to describe degradation phenomena of mechanical components of diesel engines. [6] showed that the *IW* distribution provides a good fit to several real data sets.

The *cdf* and *pdf* of the inverse Weibull distribution, $IW(\mu,\sigma)$, respectively, are given by

$$F(x,\mu,\sigma) = exp(-\mu x^{-\sigma}), \qquad x > 0 \tag{5}$$

and

$$f(x,\mu,\sigma) = \mu\sigma x^{-(\sigma+1)} exp(-\mu x^{-\sigma}), \qquad x > 0, \tag{6}$$

where $\mu > 0$ is the scale parameter and $\sigma > 0$ is the shape parameter.

The *IW* distribution can be used to model a variety of failure characteristics, such as infant mortality, useful life and wear-out periods. It can be also used to determine the cost effectiveness, maintenance periods of reliability centered maintenance activities and applications in medicine, reliability and ecology. The *IW* distribution has initiated a large volume of research. [7] have discussed the maximum likelihood and least square estimations of its parameters and [8] have considered Bayes 2-sample prediction of the distribution. Some work has been done on statistical inference based on record values, for instance [9,10,11,12,13,14,15,16,17].

Predicting the future observations on the basis of the known information is an important aspect in statistics. Bayesian approach is useful in predicting the future observations using the predictive distribution. Also, prediction of future observations is treated as an important problem in clinical, industrial and agricultural experiments. Several researchers have studied Bayesian prediction, among others [18,19,20,21,22,23].

In this paper, we have obtained the Bayesian estimators and maximum likelihood estimators (*MLE's*) of the parameters of *IW* distribution based on lower record values. Bayesian prediction of the $s-th$ lower record is also presented. Numerical computations using R software are provided to illustrate the results.

2 Estimation of the Parameter

In this section, we focus on the estimation of the two unknown parameters μ and σ of the *IW* distribution given in equation (5) based on record data. We have obtained the *MLE's* and the Bayes estimators for the two unknown parameters under squared error loss (*SEL*) function.

2.1 Maximum Likelihood Estimator

Suppose we observe m lower record values $X_{L_{(1)}} = x_1, X_{L_{(2)}} = x_2, ..., X_{L_{(m)}} = x_m$, from the $IW(\mu,\sigma)$ distribution. Using equation (4), (5) and (6), the likelihood function is given by

$$L(\mu,\sigma|\underline{x}) = \mu^m \sigma^m exp(-\mu x_m^{-\sigma}) \prod_{i=1}^{m} x_i^{-\sigma-1}. \tag{7}$$

The log-likelihood function is

$$logL(\mu,\sigma|\underline{x}) = m\ log\mu + m\ log\sigma - \mu x_m^{-\sigma} - (\sigma+1)\sum_{i=1}^{m} logx_i. \tag{8}$$

Taking derivatives with respect to μ and σ of equation (8) and equating them to zero, we obtain the likelihood equations for μ and σ as

$$\frac{\partial}{\partial\mu} logL(\mu,\sigma|\underline{x}) = \frac{m}{\mu} - x_m^{-\sigma} = 0 \tag{9}$$

and

$$\frac{\partial}{\partial\sigma} logL(\mu,\sigma|\underline{x}) = \frac{m}{\sigma} + \mu x_m^{-\sigma} logx_m - \sum_{i=1}^{m} logx_i = 0. \tag{10}$$

Equation (9) yields the *MLE* of μ to be

$$\hat{\mu} = m x_m^{\sigma}. \tag{11}$$

Substituting equation (11) into equation (10), the *MLE* of σ can be obtained by solving the equation

$$\frac{m}{\sigma}+m\log x_m-\sum_{i=1}^{m} log x_i=0$$

$$\hat{\sigma}=\frac{m}{\sum_{i=1}^{m} log x_i - m\log x_m}. \tag{12}$$

MLE's of the parameters of *IW* distribution may be obtained using R Software.

2.2 Bayes Estimator under Squared Error Loss Function

In this subsection, we use the bivariate prior distribution of μ and σ in the form [24]

$$g(\mu,\sigma)=\frac{d^c b^a}{\Gamma c\Gamma a}\mu^{c+a-1}\sigma^{a-1}exp[-\mu(d+b\sigma)], \tag{13}$$

where a, b, c and d are positive real numbers.

The posterior density function of (μ,σ) using Bayes Theorem is given by

$$\pi(\mu,\sigma|\underline{x})=CL(\mu,\sigma|\underline{x})g(\mu,\sigma), \tag{14}$$

where C is normalizing constant.

Using equation (7), (13) and (14), we have

$$\pi(\mu,\sigma|\underline{x})=\frac{1}{\Gamma m+c+a}\frac{\mu^{m+c+a-1}\sigma^{m+a-1}}{\phi_1(a,b,c,x_m)}\prod_{i=1}^{m}x_i^{-\sigma-1}exp[-\mu(d+b\sigma+x_m^{-\sigma})], \tag{15}$$

where $\phi_1(a,b,c,x_m)=\int_0^\infty \frac{\sigma^{m+a-1}\prod_{i=1}^{m}x_i^{-\sigma-1}}{(d+b\sigma+x_m^{-\sigma})^{m+c+a}}d\sigma$.

The Bayes estimator of μ under *SEL* function is the mean of the posterior density of $\mu|\underline{x}$ and is given by

$$E(\mu|\underline{x})=\int_0^\infty\int_0^\infty \mu\pi(\mu,\sigma|\underline{x})d\mu d\sigma$$

$$=\int_0^\infty\int_0^\infty \frac{1}{\Gamma(m+c+a)}\frac{\mu^{m+c+a}\sigma^{m+a-1}}{\phi_1(a,b,c,x_m)}\prod_{i=1}^{m}x_i^{-\sigma-1}exp[-\mu(d+b\sigma+x_m^{-\sigma})]d\mu d\sigma.$$

Thus,

$$E(\mu|\underline{x})=\hat{\mu}_B=\frac{\phi_1(a,b,c+1,x_m)}{\phi_1(a,b,c,x_m)}\frac{\Gamma(m+c+a+1)}{\Gamma(m+c+a)}. \tag{16}$$

Also, Bayes estimator for σ is

$$E(\sigma|\underline{x})=\int_0^\infty\int_0^\infty \sigma\pi(\mu,\sigma|\underline{x})d\mu d\sigma$$

$$=\int_0^\infty\int_0^\infty \frac{1}{\Gamma(m+c+a)}\frac{\mu^{m+c+a-1}\sigma^{m+a}}{\phi_1(a,b,c,x_m)}\prod_{i=1}^{m}x_i^{-\sigma-1}exp[-\mu(d+b\sigma+x_m^{-\sigma})]d\mu d\sigma.$$

Thus,

$$E(\sigma|\underline{x})=\hat{\sigma}_B=\frac{\phi_1(a+1,b,c-1,x_m)}{\phi_1(a,b,c,x_m)}. \tag{17}$$

Since these Bayes estimators involve integral function $\phi_1(a,b,c,x_m)$, we may use numerical integration technique in R Software.

3 Bayesian Prediction of Future Records

Based on such a sample, prediction, either point or interval, is needed for the $s-th$ lower records, $1 \leq m < s$.

The Bayes predictive density function of Y given $\underline{x}$ is

$$q(y|\underline{x}) = \int_{\Theta} f(y|x_m, \theta)\pi(\theta|\underline{x})d\theta. \tag{18}$$

Using equations (3), (5) and (6), we have

$$f(y|x_m, \mu, \sigma) = \frac{\mu^{s-m}}{\Gamma(s-m)}[y^{-\sigma} - x_m^{-\sigma}]^{s-m-1}\sigma y^{-\sigma-1}$$

$$exp[-\mu(y^{-\sigma} - x_m^{-\sigma})], \qquad y < x_m. \tag{19}$$

Combining the posterior density, given in equation (15), the conditional density, given in equation (19), and integrating out the parameters μ and σ, we may get the Bayesian predictive density function of $X_{L_{(s)}}$, given the past m records, in the form

$$q(y|\underline{x}) = \int_0^\infty \int_0^\infty f(y|x_m, \mu, \sigma)\pi(\mu, \sigma|\underline{x})d\mu d\sigma$$

$$= \frac{\phi_2(x_m, y)}{B(m+c+a, s-m)\,\phi_1(a, b, c, x_m)}, \quad 0 < y < x_m < \infty \tag{20}$$

and

$$\phi_2(x_m, y) = \int_0^\infty \frac{\prod_{i=1}^m x_i^{-\sigma-1}\sigma^{m+a}[y^{-\sigma} - x_m^{-\sigma}]^{s-m-1}y^{-\sigma-1}}{(d + b\sigma + y^{-\sigma})^{s+c+a}}d\sigma, \quad y \leq x_m.$$

The Bayesian predictive bounds of $Y = X_{L_{(s)}}$ are obtained by evaluating $P(Y > z|\underline{x})$ for some given value of z. It follows from equation (20) that

$$P(Y_s > z|\underline{x}) = \int_z^\infty \phi_2(x_m, y)dy \Big/ \int_{x_m}^\infty \phi_2(x_m, y)dy. \tag{21}$$

It can be shown that the two-sided $(1-\gamma)100\%$ prediction interval for $X_{L_{(s)}}$ is given by (L, U), where γ is the level of significance. The lower bound L and the upper bound U can be obtained by solving the following two equations:

$$1 - \frac{\gamma}{2} = P(Y_s > L)|\underline{x}) \quad and \quad \frac{\gamma}{2} = P(Y_s > U)|\underline{x}). \tag{22}$$

Thus, one may obtaine L and U by equating equation (21) to $1-\frac{\gamma}{2}$ and $\frac{\gamma}{2}$ respectively, and solving the resulting equations numerically.

In the special case, $s = m+1$, it can be shown that

$$P(Y_{m+1} > z|\underline{x}) = \frac{\phi_1(a, b, c, z)}{\phi_1(a, b, c, x_m)}. \tag{23}$$

4 Numerical Computations

To illustrate the usefulness of the inference procedures discussed in the previous sections, we generate ten sets of record values of sizes 10, 20, 30, 40, 50, 60, 70, 80, 90 and 100 from the *IW* distribution with parameter μ and σ. Taking prior parameters $a = 6, b = 3, c = 8$ and $d = 3$, the Bayes estimators of μ and σ are shown in the Table 1. The *MLE's* are also discussed in the same Table.

Moreover, using equation (23), the lower and upper 95% predictive bounds for Y_{m+1} (next lower record in the X_n sequence) are shown in the Table 2.

Table 1: *MLE's* and Bayes estimates based on generated record values, when the population parameters are $\mu = 2$ and $\sigma = 1$.

Sample size n	Number of Records m	Generated Records	MLE's $\hat{\mu}_M$	MLE's $\hat{\sigma}_M$	Bayes $\hat{\mu}_B$	Bayes $\hat{\sigma}_B$
10	5	3.3667322 2.5709933 2.2193163 0.8072700 0.7166880	2.1607484	0.884659	1.803357	0.9075674
20	6	10.595643 2.2662882 2.2416308 1.1903109 0.5979334 0.4858962	1.893992	1.268584	1.95686	0.7722093
30	6	2.1545767 1.7801679 1.1939821 0.8219102 0.5661028 0.4353150	1.834601	1.056860	1.730483	0.9360437
40	6	2.2731377 2.2386806 1.6887778 0.7277338 0.6860213 0.4476006	1.897458	1.074409	1.788978	0.8917656
50	6	5.7638814 3.9447318 1.8545482 0.7157506 0.6359043 0.5224726	2.1819959	0.995663	1.900031	0.8412047
60	6	25.713076 2.2597584 1.2191980 0.7062222 0.5166571 0.5036550	1.8470613	0.995521	1.925742	0.8080487
70	7	1.5328037 1.2326753 1.1785315 0.8804217 0.7984276 0.7304637 0.4358077	2.1958698	0.975241	1.738258	1.032918
80	7	2.8661752 1.4836633 1.3316177 0.7892711 0.7129092 0.5421006 0.3306418	2.013341	1.265962	1.859524	0.8264226
90	7	2.7874043 1.5686737 1.5074763 0.8919501 0.7990347 0.5670401 0.4571294	2.1614329	0.953109	1.811742	0.9827156
100	7	5.2221720 4.0800484 1.7230905 1.1425363 1.0291326 0.7503817	2.132089	1.042323	2.027445	0.8075958

Table 2: Lower and Upper 95% prediction bounds for Y_{m+1}.

Number of Records m	Interval prediction for the next record (Y_{m+1}) L	U	Length
5	0.704807	0.242647	0.462160
7	0.613979	0.233207	0.380772
9	0.755965	0.330072	0.425893
11	0.498799	0.232336	0.266463

5 Conclusion

In this paper, theoretical results of the study are explained numerically in section 4. We have explored the Bayesian estimator under squared error loss function and maximum likelihood estimation as well as prediction for the two parameter *IW* distribution based on generated lower record values. It might be noted that as sample size and number of record values increase, the *MLE's* and Bayes estimators of μ and σ increase and decrease in an irregular fashion.

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Conflict of interest There is no conflict of interest in this work.

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