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Approximate solutions for difference equations with non-instantaneous impulses

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Abstract: Difference equations with a special type of impulses is studied. This type of impulses are called non-instantaneous and they start their action abruptly at initially given points and then continue to act on given finite intervals. Since the studied type of equation could not be solved recursively it requires to be proved and applied approximate methods. One of the approximate methods providing a constructive approach to find solutions for the nonlinear problem via linear iterates is the monotone iterative technique. It uses the method of upper and lower solutions which generates the existence of solutions in a closed sector. In this paper the approximate solution of the studied problem is obtained as a limit of successive approximations which are solutions of linear difference equation with constant coefficients and their explicit formula is given. Each term of the sequences is a lower/upper solution of the studied nonlinear problem.

Keywords: Difference equations, non-instantaneous impulses, lower and upper solutions, monotone-iterative technique.

1. Introduction

In modeling one of the main problems is connected with time. Some researchers see the real systems as continuous-time systems and therefore use differential equation-based tools to simulate a system. In the contrary, other researchers consider the real systems as discrete time systems. Therefore, they select discrete-time simulation tools. In [17] the author emphasizes that instead of first order systems in continuous time modeling which can not generate oscillated behavior, first order systems in discrete-time approach can oscillate or even generate chaotic behavior. In [3] a discrete-time version of epidemic model is developed. In [13] the authors study pattern formation of the discrete-time predator prey model. Recently, the study of difference equations has caused a greater interest, for example, see [11], [8].

In the real world life there are many processes and phenomena that are characterized by rapid changes in their state. In the literature these pro- cesses are modeled by two types of impulses: instantaneous impulses (whose duration is changing relatively short compared to the overall duration of the whole process, see, for example, [6], [9], [16], [7], [10])) and non-instantaneous im- pulses (which starts at a point and remains active on a finite time interval) ([5]).

Often, the solving of difference equations in a closed form is difficult ([1],[4]). Specially in the case when the unknown function in the present time is involved on both parts of the equation non-linearly. One of the approximate method is based on the method of upper and lower solutions, combined by a monotone-iterative technique. It is used to construct two monotonous sequences of upper and lower solutions of the nonlinear non-instantaneous impulsive difference equation. This method is applied for difference equations in [14], [15] and for impulsive difference equations in [2].

The main purpose of this paper is the application of monotone iterative technique to the initial value problem for difference equations with non-instantaneous impulses and providing an applicable algorithm for obtaining successive approximations to the solution.

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2. Statement of the problem

We will introduce basic notation used in this paper. Most of them are well known and used in the literature. Let $\mathbb{Z}_+$ denote the set of all nonegative integers. Let the increasing sequence $\{n_i\}_{i=0}^{p+1}: \ n_i \in \mathbb{Z}_+, \ n_i \geq n_{i-1}+3, i=1,2,\dots,p$ and the sequence $\{d_i\}_{i=1}^{p}: \ d_i \in \mathbb{Z}_+, \ 1 \leq d_i \leq n_{i+1}-n_i-2, \ i=1,2,\dots,p$ be given. We denote $\mathbb{Z}[a,b]=\{z\in\mathbb{Z}_+ : a\leq z\leq b\}, \ a,b\in\mathbb{Z}_+, \ a<b$ and $I_k=\mathbb{Z}[n_k+d_k, n_{k+1}-2], \ k\in\mathbb{Z}[0,p-1], \ I_p=\mathbb{Z}[n_p+d_p, n_{p+1}-1]$ and $J_k=\mathbb{Z}[n_k+1, n_k+d_k], \ k\in\mathbb{Z}[1,p]$ where $d_0=0$.

Consider the *initial value problem (IVP)* for the nonlinear *difference equation with non-instantaneous impulses (NIDE)*

$$\begin{aligned} & x(n+1)=f(n,x(n),x(n+1)) \text{ for } n\in\bigcup_{k=0}^{p} I_k, \\ & x(n_k)=F(k,x(n_k-1)), \ k\in\mathbb{Z}[1,p] \\ & x(n)=g(n,x(n),x(n_k)) \text{ for } n\in\bigcup_{k=1}^{p} J_k, \\ & x(n_0)=x_0, \end{aligned} \tag{1}$$

where $x,x_0\in\mathbb{R}, \ f:\bigcup_{k=0}^{p} I_k\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, \ F:\mathbb{Z}[1,p]\times\mathbb{R}\to\mathbb{R}$, and $g:\bigcup_{k=1}^{p} J_k\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

3. Preliminaries results

Definition 1 *We will say that the function $\alpha:\mathbb{Z}[n_0,n_{p+1}]\to\mathbb{R}$ is a minimal(maximal) solution of the IVP for NIDE (1) in $\mathbb{Z}[n_0,n_{p+1}]$ if it is a solution of (1) and for any solution $u(n), n\in\mathbb{Z}[n_0,n_{p+1}]$ of (1) the inequalities $\alpha(n)\leq u(n)$ ($\alpha(n)\geq u(n)$) hold on $\mathbb{Z}[n_0,n_{p+1}]$*

Definition 2 *The function $\alpha:\mathbb{Z}[n_0,n_{p+1}]\to\mathbb{R}$ is called lower (upper) solutions of IVP for NIDE (1), if:*

$$\begin{aligned} & \alpha(n+1)\leq(\geq)f(n,\alpha(n),\alpha(n+1)), \textit{ for } n\in\bigcup_{k=0}^{p} I_k, \\ & \alpha(n_k)\leq(\geq)F(k,\alpha(n_k-1)), \ k\in\mathbb{Z}[1,p] \\ & \alpha(n)\leq(\geq)g(n,\alpha(n),\alpha(n_k)), \textit{ for } n\in\bigcup_{k=1}^{p} J_k \\ & \alpha(n_0)\leq(\geq)x_0 \end{aligned}$$

Consider the IVP for the linear NIDE of the type

$$\begin{aligned} & u(n+1)=Q_n u(n)+\sigma_n, \ \ n\in\bigcup_{k=0}^{p} I_k, \\ & u(n_k)=T_k u(n_k-1)+\mu_k, \ k\in\mathbb{Z}[1,p] \\ & u(n)=M_n u(n)+L_n u(n_k)+\gamma_n, \ \ n\in\bigcup_{k=1}^{p} J_k \\ & u(n_0)=x_0, \end{aligned} \tag{2}$$

where $u,x_0\in\mathbb{R}, \ Q_n: \ n\in\bigcup_{k=0}^{p} I_k, \ \sigma_n: \ n\in\bigcup_{k=0}^{p} I_k, \ L_n, \ M_n\neq 1, \gamma_n: \ n\in\bigcup_{k=1}^{p} J_k$, and $T_k,\mu_k: \ k\in\mathbb{Z}[1,p]$ are given real constants.

We will give an explicit formula to the solution of the IVP (2).

Lemma 1 *The IVP for NIDE (2) has an unique solution given by*

$$\begin{aligned} u(n)=N(n)\sum_{j=n_0-1}^{n-1}\Big(\prod_{i=j+1}^{n} R(i)\Big)\sigma_j\prod_{i=j+1}^{n-1} Q_i+N(n)\sum_{j=n_0}^{n}\Big(\prod_{i=j+1}^{n} R(i)\Big)\zeta(j)\prod_{i=j}^{n-1} Q_i+\tau(n) \\ \textit{for } n\in\mathbb{Z}[n_0,n_{p+1}] \end{aligned} \tag{3}$$

where $\sigma_{n_0-1} = x_0$, $\sigma_n = 0$, $Q_n = 1$ *for* $n \in \mathbb{Z}[n_0, n_{p+1}]/\bigcup_{k=0}^{p} I_k$,

$$N(n) = \begin{cases} \frac{L_n}{1-M_n} & \text{for } n \in \bigcup_{k=1}^{p} J_k \\ 1 & \text{otherwise} \end{cases} \tag{4}$$

$$\tau(n) = \begin{cases} \frac{\gamma_n}{1-M_n} & \text{for } n \in \bigcup_{k=1}^{p} J_k \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$R(n) = \begin{cases} N(n_k+d_k) = \frac{L_{n_k+d_k}}{1-M_{n_k+d_k}} & \text{for } n = n_k+d_k+1,\ k \in \mathbb{Z}[1,p] \\ T_k \ \text{for } n = n_k, & k \in \mathbb{Z}[1,p] \\ 1 & \text{otherwise} \end{cases} \tag{6}$$

$$\zeta(n) = \begin{cases} \tau(n_k+d_k) = \frac{\gamma_{n_k+d_k}}{1-M_{n_k+d_k}} & \text{for } n = n_k+d_k+1,\ k \in \mathbb{Z}[1,p] \\ \mu_k \ \text{for } n = n_k, & k \in \mathbb{Z}[1,p] \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Proof. We will use an induction with respect to the interval.

Let $n \in \mathbb{Z}[n_0+1, n_1-1]$. Then from the first equation of (2) we obtain $u(n) = \sum_{j=n_0-1}^{n-1} \sigma_j \prod_{i=j+1}^{n-1} Q_i$ for $n \in \mathbb{Z}[n_0+1, n_1-1]$.

Let $n = n_1$. Using $\sigma_{n_1-1} = 0,\ Q_{n_1-1} = 1$ we get

$$u(n_1) = T_1 u(n_1-1) + \mu_1 = T_1 \sum_{j=n_0-1}^{n_1-1} \sigma_j \prod_{i=j+1}^{n_1-1} Q_i + \mu_1 = \sum_{j=n_0-1}^{n_1-1} \Big(\prod_{i=j+1}^{n_1} R(i)\Big)\sigma_j \prod_{i=j+1}^{n_1-1} Q_i + \mu_1.$$

Let $n \in J_1 = \mathbb{Z}[n_1+1, n_1+d_1]$. Then using $\sigma_j = 0,\ Q_j = 1,\ j \in \mathbb{Z}[n_1, n_1+d_1]$ we get

$$u(n) = \frac{L_n}{1-M_n} \sum_{j=n_0-1}^{n-1} \Big(\prod_{i=j+1}^{n_1} R(i)\Big)\sigma_j \prod_{i=j+1}^{n-1} Q_i + \frac{L_n}{1-M_n}\mu_1 + \frac{\gamma_n}{1-M_n}.$$

$$= N(n) \sum_{j=n_0-1}^{n-1} \sigma_j \Big(\prod_{i=j+1}^{n} R(i)\Big) \prod_{i=j+1}^{n-1} Q_i + N(n)\mu_1 + \tau(n),\ n \in J_1.$$

Let $n \in \mathbb{Z}[n_1+d_1+1, n_2-1]$. Then using the first equation of (2) and the proved above we get

$$u(n) = N(n_1+d_1) \sum_{j=n_0-1}^{n_1+d_1-1} \sigma_j \Big(\prod_{i=j+1}^{n_1+d_1} R(i)\Big) \prod_{i=j+1}^{n_1+d_1-1} Q_i \prod_{i=n_1+d_1}^{n-1} Q_i$$
$$+ \mu_1 N(n_1+d_1) \prod_{i=n_1+d_1}^{n-1} Q_i + \tau(n_1+d_1) \prod_{i=n_1+d_1}^{n-1} Q_i + \sum_{j=n_1+d_1}^{n-1} \sigma_j \prod_{i=j+1}^{n-1} Q_i$$
$$= \sum_{j=n_0-1}^{n-1} \Big(\prod_{i=j+1}^{n} R(i)\Big)\sigma_j \prod_{i=j+1}^{n-1} Q_i + \sum_{j=n_0}^{n} \Big(\prod_{i=j+1}^{n} R(i)\Big)\zeta(j) \prod_{i=j}^{n-1} Q_i.$$

Let $n = n_2$. Then we get

$$u(n_2) = T_1 u(n_2-1) + \mu_2$$
$$= \sum_{j=n_0-1}^{n_2-1} \Big(\prod_{i=j+1}^{n_2} R(i)\Big)\sigma_j \prod_{i=j+1}^{n_2-1} Q_i + \sum_{j=n_0}^{n_2} \Big(\prod_{i=j+1}^{n_2} R(i)\Big)\zeta(j) \prod_{i=j}^{n_2-1} Q_i.$$

Let $n \in J_2 = \mathbb{Z}[n_2+1, n_2+d_2]$. Then

$$u(n) = N(n)u(n_2) + \tau(n)$$
$$= N(n) \sum_{j=n_0-1}^{n_2-1} \Big(\prod_{i=j+1}^{n_2} R(i)\Big)\sigma_j \prod_{i=j+1}^{n_2-1} Q_i + N(n) \sum_{j=n_0}^{n_2} \Big(\prod_{i=j+1}^{n_2} R(i)\Big)\zeta(j) \prod_{i=j}^{n_2-1} Q_i + \tau(n)$$
$$= N(n) \sum_{j=n_0-1}^{n-1} \Big(\prod_{i=j+1}^{n} R(i)\Big)\sigma_j \prod_{i=j+1}^{n-1} Q_i + N(n) \sum_{j=n_0}^{n} \Big(\prod_{i=j+1}^{n} R(i)\Big)\zeta(j) \prod_{i=j}^{n-1} Q_i + \tau(n).$$

Continue this process step by step w.r.t. the interval we prove the IVP for NIDE (2) has an unique solution given by (3) for all $n \in \mathbb{Z}[n_0, n_{p+1}]$.

Lemma 2 *Let $m : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ satisfy the linear difference inequalities*

$$\begin{aligned}
&m(n+1) \le Q_n m(n), \ \ n \in \bigcup_{k=0}^{p} I_k, \\
&m(n_k) \le T_k m(n_k - 1), \ \ k \in \mathbb{Z}[1, p] \\
&m(n) \le M_n m(n) + L_n m(n_k), \ \ n \in \bigcup_{k=1}^{p} J_k \\
&m(n_0) \le 0,
\end{aligned} \tag{8}$$

where $Q_n > 0$, $\left(n \in \bigcup_{k=0}^{p} I_k\right)$, $T_k > 0$, $\left(k \in \mathbb{Z}[1, p]\right)$ *and* $L_n > 0$, $M_n < 1$, $\left(n \in \bigcup_{k=0}^{p} J_k\right)$.

Then $m(n) \le 0$ for every $n \in \mathbb{Z}[n_0, n_{p+1}]$.

The proof is based on an induction w.r.t. the interval and we omit it.

4. Main results

For any pair of function $\alpha, \beta : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ such that $\alpha(n) \le \beta(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$ we define the sets

$$\begin{aligned}
&S(\alpha, \beta) = \{u : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R} : \alpha(n) \le u(n) \le \beta(n), \ \ n \in \mathbb{Z}[n_0, n_{p+1}]\} \\
&\Omega_1(\alpha, \beta) = \{u \in \mathbb{R} : \min_{n \in \bigcup_{k=0}^{p} I_k} \alpha(n) \le u \le \max_{n \in \bigcup_{k=0}^{p} I_k} \beta(n)\} \\
&\Omega_2(\alpha, \beta) = \{u \in \mathbb{R} : \min_{n \in \bigcup_{k=0}^{p-1} I_k} \alpha(n+1) \le u \le \max_{n \in \bigcup_{k=0}^{p-1} I_k} \beta(n+1)\} \\
&\Lambda(\alpha, \beta) = \{u \in \mathbb{R} : \min_{n \in \bigcup_{k=1}^{p} J_k} \alpha(n) \le u \le \max_{n \in \bigcup_{k=1}^{p} J_k} \beta(n)\} \\
&\Gamma(\alpha, \beta) = \{y \in \mathbb{R} : \min_{k \in \mathbb{Z}[1,p]} \alpha(n_k) \le y \le \max_{k \in \mathbb{Z}[1,p]} \beta(n_k)\} \\
&\Upsilon(\alpha, \beta) = \{z \in \mathbb{R} : \min_{k \in \mathbb{Z}[1,p]} \alpha(n_k - 1) \le z \le \max_{k \in \mathbb{Z}[1,p]} \beta(n_k - 1)\}
\end{aligned}$$

Theorem 1 *Let the following conditions be fulfilled:*

1. *The functions $\alpha, \beta : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ are lower and upper solutions of the IVP for NIDE (1) and $\alpha(n) \le \beta(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$.*
2. *The function $f : \bigcup_{k=0}^{p} I_k \times \Omega_1(\alpha, \beta) \times \Omega_2(\alpha, \beta)$ is continuous in its second and third arguments and there exist functions $K : \bigcup_{k=0}^{p} I_k \to (-\infty, 1)$ and $P : \bigcup_{k=0}^{p} I_k \to (0, \infty)$ such that for any $n \in \bigcup_{k=0}^{p} I_k$ and $x_1, x_2 \in \Omega_1(\alpha, \beta)$, with $x_1 \le x_2$, and $x_3, x_4 \in \Omega_2(\alpha, \beta)$, with $x_3 \le x_4$ the inequality*

$$f(n, x_1, x_3) - f(n, x_2, x_4) \le P(n)(x_1 - x_2) + K(n)(x_3 - x_4)$$

   holds.
3. *The function $F : \mathbb{Z}[1, p] \times \mathbb{R} \to \mathbb{R}$ is continuous in its second argument and there exists a function $T : \mathbb{Z}[1, p] \to (0, \infty)$ such that for any $k \in \mathbb{Z}[1, p]$ and $z_1, z_2 \in \Upsilon(\alpha, \beta)$ with $z_1 \le z_2$*

$$F(k, z_1) - F(k, z_2) \le T(k)(z_1 - z_2).$$

4. *The function $g : \bigcup_{k=1}^{p} J_k \times \Lambda(\alpha, \beta) \times \Gamma(\alpha, \beta) \to \mathbb{R}$ is continuous in its second and third arguments and there exist functions $M : \bigcup_{k=1}^{p} J_k \to (-\infty, 1)$ and $L : \bigcup_{k=1}^{p} J_k \to (0, \infty)$ such that for any $n \in \bigcup_{k=1}^{p} J_k$ and $y_1, y_2 \in \Lambda(\alpha, \beta)$, with $y_1 \le y_2$, and $y_3, y_4 \in \Gamma(\alpha, \beta)$, with $y_3 \le y_4$ the inequality*

$$g(n, y_1, y_3) - g(n, y_2, y_4) \le M(n)(y_1 - y_2) + L(n)(y_3 - y_4)$$

   holds.

Then there exist two sequences of discrete functions $\{\alpha^{(j)}(n)\}_0^\infty$ *and* $\{\beta^{(j)}(n)\}_0^\infty$, $n \in \mathbb{Z}[n_0, n_{p+1}]$ *with* $\alpha^{(0)} = \alpha$ *and* $\beta^{(0)} = \beta$ *such that:*

a) The inequalities

$$\alpha(n) \le \alpha^{(j)}(n) \le \alpha^{(j+1)}(n) \le \beta^{(j+1)}(n) \le \beta^{(j)}(n) \le \beta(n),\ for\ \ n \in \mathbb{Z}[n_0, n_{p+1}],\ \ j \in \mathbb{Z}$$

hold;

b) The functions $\alpha^{(j)}(n)$ *and* $\beta^{(j)}(n)$, $j \in \mathbb{Z}$ *are lower and upper solutions of the IVP for NIDE* (1)*, respectively;*

c) Both sequences are convergent on $\mathbb{Z}[n_0, n_{p+1}]$*;*

d) The limits $\lim_{j\to\infty} \alpha^{(j)}(n) = A(n)$, $\lim_{j\to\infty} \beta^{(j)}(n) = B(n)$ *are the minimal and maximal solutions of IVP for NIDE* (1) *in* $S(\alpha, \beta)$*, respectively;*

e) If IVP for NIDE (1) *has an unique solution* $u(n) \in S(\alpha, \beta)$*, then* $A(n) \equiv u(n) \equiv B(n)$ *for* $n \in \mathbb{Z}[n_0, n_{p+1}]$.

*Proof.*For any arbitrary fixed function $\eta \in S(\alpha, \beta)$, we consider the IVP for the linear NIDE

$$\begin{aligned}
&u(n+1) = P(n)u(n) + K(n)u(n+1) + \psi(n, \eta(n), \eta(n+1)),\ \ n \in \bigcup_{k=0}^{p} I_k \\
&u(n_k) = T(k)u(n_k - 1) + \upsilon(k, \eta(n_k - 1)),\ \ k \in \mathbb{Z}[1, p] \\
&u(n) = M(n)u(n) + L(n)u(n_k) + \xi(n, \eta(n), \eta(n_k)),\ \ n \in \bigcup_{k=1}^{p} J_k \\
&u(n_0) = x_0,
\end{aligned} \tag{9}$$

where $u, x_0 \in \mathbb{R}$, and

$$\begin{aligned}
&\psi(n,x,y) = f(n,x,y) - P(n)x - K(n)y,\ \ n \in \bigcup_{k=0}^{p} I_k, \\
&\upsilon(k,x) = F(k,x) - T(k)x,\ k \in \mathbb{Z}[1, p] \\
&\xi(n,x,y) = g(n,x,y) - M(n)x - L(n)y,\ \ n \in \bigcup_{k=1}^{p} J_k.
\end{aligned} \tag{10}$$

According to Lemma 1 the IVP for linear NIDE (9) has an unique solution given by (3) with $\sigma_n = \frac{\psi(n,\eta(n),\eta(n+1))}{1-K(n)}$, $Q_n = \frac{P(n)}{1-K(n)}$, $\gamma_n = \xi(n, \eta(n), \eta(n_j))$, $\mu_k = \upsilon(k, \eta(n_k - 1))$, $T_k = T(k), M_n = M(n), L_n = L(n)$.

For any function $\eta \in S(\alpha, \beta)$ we define the operator $W : S(\alpha, \beta) \to S(\alpha, \beta)$ by $W\eta = u$, where u is the unique solution of IVP for the linear NIDE (9) for the function η. The operator W has the following properties:

(P1) $\alpha \le W\alpha$, $\beta \ge W\beta$

(P2) W is a monotone nondecreasing operator in $S(\alpha, \beta)$.

To prove (P1) set $W\alpha = \alpha^{(1)}$, where $\alpha^{(1)}$ is the unique solution of (9) with $\eta = \alpha$ and let $m(n) = \alpha(n) - \alpha^{(1)}(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$

For any $n \in \bigcup_{k=0}^{p} I_k$ we obtain the inequality

$$\begin{aligned}
m(n+1) &= \alpha(n+1) - P(n)\alpha^{(1)}(n) - K(n)\alpha^{(1)}(n+1) \\
&\quad - \psi(n, \alpha(n), \alpha(n+1)) \\
&\le P(n)(\alpha(n) - \alpha^{(1)}(n)) + K(n)(\alpha(n+1) - \alpha^{(1)}(n+1)) \\
&= P(n)m(n) + K(n)m(n+1).
\end{aligned}$$

Hence the inequality $m(n+1) \le \frac{P(n)}{1-K(n)}\ m(n)$ holds for $n \in \bigcup_{k=0}^{p} I_k$.

For any $n = n_k$ we obtain

$$\begin{aligned}
m(n_k) &\le F(k, \alpha(n_k - 1)) - T(k)\alpha^{(1)}(n_k - 1) - F(k, \alpha(n_k - 1)) \\
&\quad + T(k)\alpha(n_k - 1) = T(k)m(n_k - 1).
\end{aligned}$$

For any $n \in \bigcup_{k=1}^{p} J_k$ we get

$$\begin{aligned} m(n) &\leq g(n,\alpha(n),\alpha(n_k)) - M(n)\alpha^{(1)}(n) - L(n)\alpha^{(1)}(n_k) \\ &\quad - g(n,\alpha(n),\alpha(n_k)) + M(n)\alpha(n) + L(n)\alpha(n_k) \\ &= M(n)m(n) + L(n)m(n_k) \end{aligned}$$

Therefore, the function $m(n)$ satisfies the inequalities (8) with $Q_n = \frac{P(n)}{1-K(n)}$, $T_k = T(k), M_n = M(n), L_n = L(n)$. According to Lemma 2 the function $m(n)$ is non-positive in $\mathbb{Z}[n_0, n_{p+1}]$, i.e. $\alpha \leq W\alpha$. Analogously it can be proved that the inequality $\beta \geq W\beta$ holds.

To prove (P2) we consider two arbitrary function $\eta_1,\ \eta_2 \in S(\alpha,\beta)$ such that $\eta_1(n) \leq \eta_2(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$. Let $u^{(1)} = W\eta_1$ and $u^{(2)} = W\eta_2$. Denote $m(n) = u^{(1)}(n) - u^{(2)}(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in \bigcup_{k=0}^{p} I_k$ we obtain the inequality

$$\begin{aligned} m(n+1) &= P(n)u^{(1)}(n) + K(n)u^{(1)}(n+1) + f(n,\eta_1(n),\eta_1(n+1)) \\ &\quad - P(n)\eta_1(n) - K(n)\eta_1(n+1) - P(n)u^{(2)}(n) - K(n)u^{(2)}(n+1) \\ &\quad - f(n,\eta_2(n),\eta_2(n+1)) + P(n)\eta_2(n) + K(n)\eta_2(n+1) \\ &\leq P(n)m(n) + K(n)m(n+1) \end{aligned}$$

Hence the inequality $m(n+1) \leq \frac{P(n)}{1-K(n)}\, m(n)$ holds for $n \in I_k,\ k \in \mathbb{Z}[0,p]$.

For any $n = n_k, k \in \mathbb{Z}[1,p]$ we get

$$\begin{aligned} m(n_k) &= T(k,u^{(1)}(n_k-1) - u^{(2)}(n_k-1)) - T(k)(\eta_1(n_k-1) - \eta_2(n_k-1)) \\ &\quad + F(k,\eta_1(n_k-1)) - F(k,\eta_2(n_k-1)) \leq T(k)m(n_k-1) \end{aligned}$$

For any $n \in \bigcup_{k=1}^{p} J_k$ we obtain

$$\begin{aligned} m(n) &= M_(n)u^{(1)}(n) + L(n)u^{(1)}(n_k) + g(n,\eta_1(n),\eta_1(n_k)) \\ &\quad - M(n)\eta_1(n) - L(n)\eta_1(n_k) - M(n)u^{(2)}(n) - L(n)u^{(2)}(n_k) \\ &\quad - g(n,\eta_2(n),\eta_2(n_k)) + M(n)\eta_2(n) + L(n)\eta_2(n_k) \\ &\leq M(n)m(n) + L(n)m(n_k) \end{aligned}$$

According to Lemma 2 with $Q_n = \frac{P(n)}{1-K(n)}$, $T_k = T(k), M_n = M(n), L_n = L(n)$ the function $m(n) \leq 0$, i.e. $W\eta_1 \leq W\eta_2$, for $\eta_1(n) \leq \eta_2(n), n \in \mathbb{Z}[n_0, n_{p+1}]$.

Let $\eta \in S(\alpha,\beta)$ be a lower solution of (1). We consider the function $W\eta = m$. According to the proved $\eta(n) \leq m(n), n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in \bigcup_{k=0}^{p} I_k$ we get the inequality

$$\begin{aligned} m(n+1) &= P(n)m(n) + K(n)m(n+1) + f(n,\eta(n),\eta(n+1)) \\ &\quad - P(n)\eta(n) - K(n)\eta(n+1) \\ &\leq f(n,m(n),m(n+1)) \end{aligned} \tag{11}$$

For any $n = n_k, k \in \mathbb{Z}[1,p]$ we obtain

$$\begin{aligned} m(n_k) &= F(k,m(n_k-1)) - F(k,m(n_k-1)) + T(k)m(n_k-1) \\ &\quad + F(k,\eta(n_k-1)) - T(k)\eta(n_k-1) \\ &\leq F(k,m(n_k-1)) \end{aligned} \tag{12}$$

For any $n \in \bigcup_{k=1}^{p} J_k$ we obtain

$$\begin{aligned} m(n) &= g(n,m(n),m(n_k)) - g(n,m(n),m(n_k)) + M(n)m(n) \\ &\quad + L(n)m(n_k) + f(n,\eta(n),\eta(n_k)) - M(n)\eta(n) - L(n)\eta(n_k) \\ &\leq g(n,m(n),m(n_k)) \end{aligned} \tag{13}$$

Inequalities (11),(12) and (13) prove the function m is a lower solution of NIDE (1). Similarly, if $\eta \in S(\alpha,\beta)$ is an upper solution of NIDE (1) then the function $m = W\eta$ is an upper solution of (1).

We define the sequences of functions $\{\alpha^{(j)}(n)\}_0^\infty$ and $\{\beta^{(j)}(n)\}_0^\infty$ by the equalities $\alpha^{(0)} = \alpha,\ \beta^{(0)} = \beta,\ \alpha^{(j)} = W\alpha^{(j-1)},\ \beta^{(j)} = W\beta^{(j-1)}$. The functions $\alpha^{(s)}(n)$ and $\beta^{(s)}(n)$ satisfy the initial value problem (9) with $\eta(n) = \alpha^{(s-1)}(n)$ and $\eta(n) = \beta^{(s-1)}(n),\ n \in \mathbb{Z}[n_0,n_{p+1}]$, respecticely.

According to Lemma 2 the following representations are valid:

$$\begin{aligned}\alpha^{(s)}(n) = N(n)&\sum_{j=n_0-1}^{n-1}\Big(\prod_{i=j+1}^{n}R(i)\Big)\frac{\psi(j,\alpha^{(s-1)}(j),\alpha^{(s=1)}(j+1))}{1-K(j)}\\ &\times\prod_{i=j+1}^{n-1}\frac{P(i)}{1-K(i)} + \sum_{j=n_0}^{n}\Big(\prod_{i=j+1}^{n}R(i)\Big)\zeta(j)\prod_{i=j}^{n-1}\frac{P(i)}{1-K(i)} + \tau(n),\\ &\text{for } n \in \mathbb{Z}[n_0,n_{p+1}],\end{aligned} \tag{14}$$

where $\tau(n)$ is given by (5) for $\gamma_n = \xi(n,\alpha^{(s-1)}(n),\alpha^{(s-1)}(n_j)),\ n \in \bigcup_{k=1}^{p} J_k,\ j \in \mathbb{Z}[1,p]$ and $\zeta(n)$ is given by (7) for $\gamma_{n_k+d_k} = \xi(n_k+d_k,\alpha^{(s-1)}(n_k+d_k),\alpha^{(s-1)}(n_k)),\ \mu_k = \upsilon(k,\alpha^{(s-1)}(n_k-1)),\ k \in \mathbb{Z}[1,p]$.

$$\begin{aligned}\beta^{(s)}(n) = N(n)&\sum_{j=n_0-1}^{n-1}\Big(\prod_{i=j+1}^{n}R(i)\Big)\frac{\psi(j,\beta^{(s-1)}(j),\beta^{(s-1)}(j+1))}{1-K(j)}\\ &\times\prod_{i=j+1}^{n-1}\frac{P(i)}{1-K(i)} + \sum_{j=n_0}^{n}\Big(\prod_{i=j+1}^{n}R(i)\Big)\zeta(j)\prod_{i=j}^{n-1}\frac{P(i)}{1-K(i)} + \tau(n),\\ &\text{for } n \in \mathbb{Z}[n_0,n_{p+1}],\end{aligned} \tag{15}$$

where $\tau(n)$ is given by (5) for $\gamma_n = \xi(n,\beta^{(s-1)}(n),\beta^{(s-1)}(n_j)),\ j \in \mathbb{Z}[1,p]$ and $\zeta(n)$ is given by (7) for $\gamma_{n_k+d_k} = \xi(n_k+d_k,\beta^{(s-1)}(n_k+d_k),\beta^{(s-1)}(n_k)),\ \mu_k = \upsilon(k,\beta^{(s-1)}(n_k-1)),\ k \in \mathbb{Z}[1,p]$.

According to the above proved, functions $\alpha^{(s)}(n)$ and $\beta^{(s)}(n)$ are lower and upper solutions of NIDE (1), respectively and they satisfy for $n \in \mathbb{Z}[n_0,n_{p+1}]$ the following inequalities

$$\alpha^{(0)}(n) \le \alpha^{(1)}(n) \le \ldots \le \alpha^{(s)}(n) \le \beta^{(s)}(n) \le \ldots \le \beta^{(1)}(n) \le \beta^{(0)}(n) \tag{16}$$

Both sequences of discrete functions being monotonic and bounded are convergent on $\mathbb{Z}[n_0,n_{p+1}]$.

Let $A(n) = \lim_{s\to\infty}\alpha^{(s)}(n), B(n) = \lim_{s\to\infty}\beta^{(s)}(n)$.

Take a limit in (14) for $s \to \infty$ we obtain (3) with $u(n) = A(n),\ \sigma_j = \frac{\psi(j,A(j),A(j+1))}{1-K(j)},\ Q_i = \frac{P(i)}{1-K(i)}$,

$$\begin{aligned}A(n) = N(n)&\sum_{j=n_0-1}^{n-1}\Big(\prod_{i=j+1}^{n}R(i)\Big)\frac{\psi(j,A(j),A(j+1))}{1-K(j)}\prod_{i=j+1}^{n-1}\frac{P(i)}{1-K(i)}\\ &+\sum_{j=n_0}^{n}\Big(\prod_{i=j+1}^{n}R(i)\Big)\zeta(j)\prod_{i=j}^{n-1}\frac{P(i)}{1-K(i)} + \tau(n),\ \text{for } n \in \mathbb{Z}[n_0,n_{p+1}],\end{aligned} \tag{17}$$

where $\tau(n)$ is given by (5) for $\gamma_n = \xi(n,A(n),A(n_j)),\ j \in \mathbb{Z}[1,p]$ and $\zeta(n)$ is given by (7) for $\gamma_{n_k+d_k} = \xi(n_k+d_k,A(n_k+d_k),A(n_k)),\ \mu_k = \upsilon(k,A(n_k-1)),\ k \in \mathbb{Z}[1,p]$.

From (17) it follows the function $A(n)$ is a solution of NIDE (1).

Similarly, we prove the function $B(n)$ is a solution of NIDE (1).

Let $u \in S(\alpha,\beta)$ be a solution of IVP for NIDE (1). From inequalities (16) it follows there exists a natural number p such that $p \in \mathbb{N}$:

$$\alpha^{(p)}(n) \le u(n) \le \beta^{(p)}(n) \ \text{ for } \ n \in \mathbb{Z}[n_0,n_{p+1}].$$

We introduce the notation $m(n) = \alpha^{(p+1)}(n) - u(n),\ \ n \in \mathbb{Z}[n_0,n_{p+1}]$.

For any $n \in \bigcup_{k=0}^{p} I_k$ we get the inequality

$$\begin{aligned}m(n+1) &= P(n)\alpha^{(p+1)}(n) + K(n)\alpha^{(p+1)}(n+1) + f(n,\alpha^{(p)}(n),\alpha^{(p)}(n+1))\\ &\quad - P(n)\alpha^{(p)}(n) - K(n)\alpha^{(p)}(n+1) - f(n,u(n),u(n+1))\\ &\le P(n)m(n) + K(n)m(n+1).\end{aligned} \tag{18}$$

Hence the inequality $m(n+1) \le \frac{P(n)}{1-K(n)} m(n)$ holds for $n \in \bigcup_{k=0}^{p} I_k$.
For any $n = n_k, \ k \in \mathbb{Z}[1,p]$ we obtain

$$\begin{aligned} m(n_k) &= T(k)\alpha^{(p+1)}(n_k-1) + T(k)u(n_k-1) - T(k)u(n_k-1) \\ &\quad + F(k,(\alpha^{(p)}(n_k-1)) - T(k)\alpha^{(p)}(n_k-1) - F(k,u(n_k-1)) \\ &\le T(k)m(n_k-1). \end{aligned} \tag{19}$$

For any $n \in \bigcup_{k=1}^{p} J_k$ we obtain

$$\begin{aligned} m(n) &= M(n)\alpha^{(p+1)}(n) + L(n)\alpha^{(p+1)}(n_k) + g(n,\alpha^{(p)}(n),\alpha^{(p)}(n_k)) \\ &\quad - M(n)\alpha^{(p)}(n) - K(n)\alpha^{(p)}(n_k) - g(n,u(n),u(n_k)) \\ &\le M(n)m(n) + L(n)m(n_k). \end{aligned}$$

According to Lemma 2 with $Q_n = \frac{P(n)}{1-K(n)}$, $T_k = T(k), M_n = M(n), L_n = L(n)$ the function $m(n)$ is nonpositive, i.e. $\alpha^{(p+1)}(n) \le u(n), \ n \in \mathbb{Z}[n_0,n_{p+1}]$. Similarly $\beta^{(p+1)}(n) \ge u(n), \ n \in \mathbb{Z}[n_0,n_{p+1}]$, and hence $\alpha^{(j+1)} \le u(n) \le \beta^{(j+1)}, \ n \in \mathbb{Z}[n_0,n_{p+1}]$. Since $\alpha^{(0)}(n) \le u(n) \le \beta^{(0)}(n)$ this proves by induction that $\alpha^{(j)}(n) \le u(n) \le \beta^{(j)}(n), \ n \in \mathbb{Z}[n_0,n_{p+1}]$, for every $j \in \mathbb{Z}$.

Taking the limit as $j \to \infty$ we conclude $A(n) \le u(n) \le B(n), \ n \in \mathbb{Z}[n_0,n_{p+1}]$. Hence $A(n)$ and $B(n)$ are minimal and maximal solutions of IVP for NIDE (1), respectively.

Let the IVP for NIDE (1) has an unique solution $u(n) \in S(\alpha,\beta)$.

Then from above it follows $A(n) \equiv u(n) \equiv B(n), \ n \in \mathbb{Z}[n_0,n_{p+1}]$.

5. Conclusion

In the paper we study the initial value problem for a nonlinear scalar difference equation with so called non-instantaneous impulses, i.e. impulses which duration of acting is not negligible. An iterative technique combined with the method of lower ad upper solution is suggested to solve the studied equation. Two sequences of discrete valued functions are constructed. Each term of the sequences is s solution of a linear discrete equation with non- instantaneous impulses with an explicit formula for the solution, obtained in the paper.

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Conflict of Interests

There is no conflict of interests by authors regarding the publication of this manuscript.

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