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A Competitive Version of Equivalence Scheme Based on Atangana-Baleanu Fractional Derivative

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Abstract: In the present paper, a novel and fundamental approach is presented as an extension of existence and uniqueness theorems of ordinary fractional difference (FDE) which was applied to the reducing Cauchy type problems (CTP). Results present a new interpretation that helps analyze the fractional calculus which implies an elegant superiority of the suggested derivative.

Keywords: Atangana-Baleanu fractional derivative, Equivalence scheme, Existence, Fractional calculus, Uniqueness.

1 Introduction

Fractional calculus (FC) is an interesting and important part of science addressing applying integrals and derivatives of real and complex orders. Three centuries ago, FC was considered in many important researches [1,2]. Nowadays, with the rapid development of computational softwares, FC has extended and scientists have found that related FC topics can analyze the various real-world phenomena such as signal processing, hydrology, rheology, acoustics, control, continuum, damping law, robotics, turbulence, viscoelasticity, thermal engineering, etc [3, 4]. For instance, the oscillation of an earthquake or the frequency dependence of the damping materials can be described with fractional derivatives very well [5,6]. For an interesting background and applications of FC, see [7–16].

Several interpretations for fractional derivatives are expressed. Some of these exegesises include Grunwald-Letnikov, Riemann-Liouville, Katugampola, Weyl, Hilfer, Marchaud, Caputo, Nishimoto, Riesz, Coimbra and Jumarie's fractional operators [17]. Most of the attempts in FC contributed on the existence and uniqueness of solutions for FDE. Recently, using the property of Mittaga Leffler function, Atangana and Baleanu proposed a new fractional derivative [18]. Their definitions have all the properties of the Fabrizio and Caputo, Riemanna Liouville and Caputo with a nonlocal and non-singular kernel respectively.

In this study, using properties of the suggested fractional derivative by Atangana and Baleanu, the existence and uniqueness of the solution for ordinary FDE based on the CTP are discussed. The present paper is organized, as follows: Section 2 introduces the essential results, basic definitions and properties of FC. Section 3 is dedicated to equivalence of the Volterra integral equation and the CTP. Section 4 discusses the existence and uniqueness analysis of the CTP. Conclusion is presented in Section 5.

2 Basic notations and preliminaries

Definition 1 *The Sobolev space H of order one in (t_0,t_q) is introduced as follows [18]:*

$$H^1(t_0,t_q) = \{h, h' \in L_2(t_0,t_q)\}. \tag{1}$$

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Definition 2 *Let $\Omega = [t_0,t_q](-\infty < t_0 < t_q < \infty)$ be a finite interval on $\mathbb{R}$. The Riemann-Liouville integrals of fractional order $\alpha \in \mathbb{C}, \Re(\alpha) > 0$ (named as $\mathscr{I}^{\alpha}_{t_0+}f$ and $\mathscr{I}^{\alpha}_{t_q-}f$) are determined by [1]*

$$(\mathscr{I}^{\alpha}_{t_0+}h)(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{h(x)dx}{(t-x)^{1-\alpha}},\ Re(\alpha)>0,\ t>t_0, \tag{2}$$

and

$$(\mathscr{I}^{\alpha}_{t_q-}h)(t) = \frac{1}{\Gamma(\alpha)}\int_{t}^{t_q}\frac{h(x)dx}{(x-t)^{1-\alpha}},\ Re(\alpha)>0,\ t<t_q. \tag{3}$$

Definition 3 *Let $\Omega = [t_0,t_q](-\infty < t_0 < t_q < \infty)$ be a finite space on $\mathbb{R}$. Let a function $v \in H^1(t_0,t_q)$. For arbitrary function v with a based point t_0, the Atangana-Baleanu fractional derivative in the Caputo sense of order α is defined in the following form [18]*

$$^{ABC}_{t_0}D^{\alpha}_{t}v(t) = \frac{N(\alpha)}{1-\alpha}\int_{t_0}^{t}v'(s)E_{\alpha}[\frac{\alpha}{\alpha-1}(t-s)^{\alpha}]ds. \tag{4}$$

Here $N(\alpha)$ is a normalization function such that $N(0)=N(1)=1$. Also, E_{α} is the Mittag-Leffler series presented in the following form

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+\beta)},\ \alpha,\beta>0. \tag{5}$$

For more details regarding the properties of mentioned derivative the reader is advised to see the studies presented in [19–23].

Assume that the nonlinear FDE of order α ($\Re(\alpha)>0$) on $[t_0,t_q]\subseteq\mathbb{R}$ has the following form

$$^{ABC}_{t_0}D^{\alpha}_{t}v(t) = h(v(t)), \Re(\alpha)>0,\ t>t_0, \tag{6}$$

with initial conditions

$$^{ABC}_{t_0}D^{(\alpha-r)}_{t}v(t) = e_r,\ \ e_r\in\mathbb{C},\ r=1,2,\cdots,q, \tag{7}$$

where

$$\begin{cases} q=\Re(\alpha)+1 & \alpha\notin\mathbb{N}, \\ \alpha=q & \alpha\in\mathbb{N}. \end{cases}$$

Thus, the problem (6)-(7) is called a CTP.

Corollary 1 *Let $[t_0,t_q](-\infty < t_0 < t_q < \infty)$ be a finite interval and $\mathscr{AC}[t_0,t_q]$ be the space of arbitrary functions f which are absolutely continuous on $[t_0,t_q]$. Consequently*

$$h(t)\in\mathscr{AC}[t_0,t_q]\Leftrightarrow h(t)=c+\int_{t_0}^{t}\eta(x)dx,\ \eta(t)\in L(t_0,t_q),$$

$$\forall q\in\mathbb{N},\ \mathscr{AC}^q[t_0,t_q]=\Big\{h:[t_0,t_q]\to\mathbb{C},\ [(\mathscr{D}^{q-1}h)(t)]\in\mathscr{AC}[t_0,t_q]\Big\}.$$

Definition 4

$$d(v_1,v_2)=\|v_1-v_2\|_1=\int_{t_0}^{t_1}|v_1(t)-v_2(t)|dt. \tag{8}$$

3 Equivalence of the integral equation of Volterra type and the CTP

In what follows, we prove that the following nonlinear Volterra integral equation and the CTP (6)-(7) are equivalent:

$$^{ABC}_{t_0}D^{\alpha}_{t}v(t) = \sum_{j=1}^{q}\frac{l_j}{\Gamma(\alpha-j+1)}(t-t_0)^{\alpha-j}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f(v(u))du}{(t-u)^{1-\alpha}},\ t>t_0. \tag{9}$$

The problem is proven by assuming that a function $h[t,v]\in L(t_0,t_q)$ for any $v\in G\subset\mathbb{C}$. Now, we consider the following lemmas and corollaries.

Lemma 1 *([1]) The support space $\mathscr{AC}^q[t_0,t_q]$ consists of only those functions $h(x)$ which can be written in the following form:*

$$h(t) = (\mathscr{I}^q_{t_0+})\eta(t) + \sum_{k=0}^{q-1} c_k (t-t_0)^k, \tag{10}$$

where $\eta(t) \in L(t_0,t_q)$ and c_k are constants, and

$$(\mathscr{I}^q_{t_0+}\eta)(t) = \frac{1}{(q-1)!}\int_{t_0}^{t} (t-x)^{q-1}\eta(x)dx. \tag{11}$$

Lemma 2 *If $\Re(\alpha) > 0$ and $h(x) \in L_p(t_0,t_q)$ $(1 \leqq p \leqq \infty)$, then*

$$\forall t \in [t_0,t_q] : \ ({}^{ABC}_{t_0}D^{\alpha}_{t} \mathscr{I}^{\alpha}_{t_0+} h)(t) = ({}^{ABC}_{t_q}D^{\alpha}_{t} \mathscr{I}^{\alpha}_{t_q-} h)(t) = h(t), (\Re(\alpha) > 0). \tag{12}$$

Proof. The proof is similar to that of Lemma 1 in [24].

Lemma 3 *Suppose that $\Re > 0$, $q = [\Re]+1$ and $h_{q-\alpha}(x) = (\mathscr{I}^{q-\alpha}_{t_0+}h)(x)$ are the fractional integral (2) of order $-\alpha+q$*
a) If $1 \leqq p \leqq \infty$ and $h(t) \in \mathscr{I}^{\alpha}_{t_0+}(L_p)$, then

$$(\mathscr{I}^{\alpha}_{t_0+} {}^{ABC}_{t_0}D^{\alpha}_{t} h)(t) = h(t). \tag{13}$$

b) If $h(x) \in L_1(t_0,t_q)$ and $h_{q-\alpha}(x) \in \mathscr{AC}^q[t_0,t_q]$, then

$$\forall t \in [t_0,t_q] : \ (\mathscr{I}^{\alpha}_{t_0+} {}^{ABC}_{t_0}D^{\alpha}_{t} h)(t) = h(t) - \sum_{j=1}^{q} \frac{h^{(q-j)}_{q-\alpha}(t_0)}{\Gamma(\alpha-j+1)}(t-t_0)^{\alpha}. \tag{14}$$

Proof. The proof is similar to that of proof of Lemma 2 in [24].

Corollary 2 *The fractional operator ${}^{ABC}_{t_0}D^{\alpha}_{t}$ (wherein $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$) is bounded in $L(t_0,t_q)$:*

$$\|{}^{ABC}_{t_0}D^{\alpha}_{t} v(t)\|_1 \leqq \frac{(t_q-t_0)^{\Re(\alpha)}}{\Re(\alpha)|\Gamma(\alpha)|}\|v\|_1. \tag{15}$$

Again, consider a CTP (6)-(7) where

$${}^{ABC}_{t_0}D^{\alpha}_{t} v(t) = h(v(t)), \ \alpha > 0, \tag{16}$$

$${}^{ABC}_{t_0}D^{(\alpha-p)}_{t} v(t) = e_r, \ e_r \in \mathbb{R}, \ r = 1,\cdots,q = -[-\alpha]. \tag{17}$$

According to the these equations, the results are shown below:

Theorem 1 *Let $\alpha > 0, q = -[-\alpha]$. If $G \subset \mathbb{R}$ is an open set, also $h : (t_0,t_q] \times G \to \mathbb{R}$ is an arbitrary function that $\forall v \in G, v(t) \in L(t_0,t_q) \Rightarrow h(v(t)) \in L(t_0,t_q)$. Then, $v(t)$ satisfies the relations (16) and (17) $\Leftrightarrow$ $v(t)$ satisfies the integral equation (9).*

Proof. The proof is similar to that of Theorem 1 in [24].

Corollary 3 *If $v(t) \in L(t_0,t_q)$, then $v(t)$ satisfies the relations in (17) with $e_r \in \mathbb{R}$ $(r = 1,\cdots,q)$ $\Leftrightarrow$ $v(t)$ satisfies the following equation*

$$v(t) \cong \sum_{j=1}^{q} \frac{(t-t_0)^{q-j}}{(q-j)!} + \frac{1}{(q-1)!}\int_{t_0}^{t} (t-x)^{q-1} h(v(t))dx. \tag{18}$$

4 On existence and uniqueness analysis of the CTP

Hereunder, the existence of a unique solution to CTP (6)-(7) is demonstrated.

Theorem 2 *Let $\alpha > 0$, $q = -[-\alpha]$. If $G \subset \mathbb{R}$ is an open set and $h : [t_0,t_q] \times G \to \mathbb{R}$ is an arbitrary function that $\forall v \in G$ $h[t,v] \in L(a,b)$ and the following condition*

$$\forall t \in (t_0,t_q], \forall v_1,v_2 \in G \subset \mathbb{C}, \ |f(v_1) - f(v_2)| \leqq A|v_1 - v_2|, \ A > 0, \tag{19}$$

where A does not depend on $t \in [t_0,t_q]$, is satisfied. Consequently, there has to be a unique solution $v(t)$ to CTP (16)-(17) in the space $L_{\alpha}(t_0,t_q)$.

Proof. See [24].

Theorem 3 *Assume that $\alpha \in \mathbb{C},\ q-1 < \Re(\alpha) < q$. If G is an open set in $\mathbb{C}$ and $h : (t_0, t_q] \times G \longrightarrow \mathbb{C}$ is an arbitrary function that $h(v(t)) \in L(t_0, t_q)$ for any $v \in G$ and the condition (7) holds. Then, there has to be a unique solution $v(t)$ to CTP (6)-(7) in the space $L_\alpha(t_0, t_q)$. Specially, if $0 < \Re(\alpha) < 1$, then there has to be a unique solution $v(t)$ to the following type equation*

$$ {}_{t_0}^{ABC}D_t^{\alpha} v(t) = h(v(t)),\ 0 < \Re(\alpha) < 1, \tag{20}$$

in the space $L_\alpha(t_0, t_q)$.

Proof. The proof is similar to that of Theorem 3 in [24], if we benefit from the inequality

$$A \frac{(t_1 - t_0)^{\Re(\alpha)}}{\Re(\alpha)|\Gamma(\alpha)|} < \limsup_{t} \left\|E_{\alpha,\beta}(z)\right\|_\infty (t_q - t_0), \tag{21}$$

instead of the one in (4) and only choose the upper bound $(v(t) - v_0)\frac{B(\alpha)}{1-\alpha} M(t_{\max})$ where $M := \sup \left\|E_{\alpha,\beta}(z)\right\|_\infty$.

5 Conclusion

In the current work, a new approach for existence and uniqueness of ordinary FDE based on the reducing CTP ([25]; pp. 135-219) and Atangana-Baleanu derivative in fractional Caputo sense thoroughly investigated. The suggested scheme was based mainly upon the reduction to Volterra integral equations. The results showed that, the suggested derivative has many useful advantages for analyzing the CTP. Furthermore, [24] can be considered a special case of our strategy, so the proofs are shorter and easier.

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