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Caputo Type Fractional Differentiation for the Extended Generalized Mittag-Leffler Function

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Abstract: The goal of this paper is to develop some differential equation formulas for the extended generalized Mittag-Leffler function (EGMLF) using Caputo type Marichev-Saigo-Maeda (MSM) fractional derivative operators involving the third Appell function as kernel, with the results viewed in the context of extended Wright hypergeometric type function. Furthermore, we highlight their relation to previously known results.

Keywords: Extended Mittag-Leffler function, fractional integral and derivative operators, extended Wright-type hypergeometric functions.

1 Introduction and preliminaries

The Mittag-Leffler (M-L) function developed in 1903, different compositions and generalizations including major applications had been performed and investigated. The M-L function E_ζ and the extended function $E_{\zeta,\gamma}$ had been composed in different formulas which may apply in vast area of researches. Furthermore, applications of these functions are used in the biological, physical, earth sciences, and engineering fields to solve problems.
Fractional calculus is an intensively expanding field of mathematics that have an immense effect on daily life subject. It has fractional order derivatives and integration which includes complex numbers as well. Over some past years this field had drawn attention for many researchers due to its properties, extension complexity and vast uses of properties in almost every area of mathematics.
The M-L function defined in the form of series representation and its generalization were introduced and studied by Mittag-Leffler [1,2], Wiman [3,4], Agarwal [5], Humbert [6] and Humbert and Agrawal [7], in the following order:

$$E_\zeta(\propto) = \sum_{n=0}^{\infty} \frac{\propto^n}{\Gamma(\zeta n+1)} \quad (\zeta > 0, \propto \in \mathbb{C}), \tag{1}$$

$$E_{\zeta,\gamma}(\propto) = \sum_{n=0}^{\infty} \frac{\propto^n}{\Gamma(\zeta n+\gamma)} \quad (\zeta > 0, \gamma > 0, \propto \in \mathbb{C}), \tag{2}$$

where $\mathbb{C}$ is the set of complex numbers. Erdélyi et al. ([8], Section 18.1) books describes the basic characteristics of these functions. Dzherbashyan ([9], Chapter 2) provides a more comprehensive and detailed description of M-L functions.
The series representation of Eq. (2) is generalized by Prabhakar [10] as:

$$E^{\rho}_{\zeta,\gamma}(\propto) = \sum_{n=0}^{\infty} \frac{(\rho)_n}{\Gamma(\zeta n+\gamma)n!} \propto^n \quad (\zeta, \gamma, \rho \in \mathbb{C}; \Re(\zeta) > 0, \Re(\gamma) > 0), \tag{3}$$

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When $\rho=1$, it simplifies to the M-L function, which is provided in equation (2). It is entire function of order $[\Re(\zeta)]^{-1}$ and $(\rho)_n$ represents the pochhammer character, which is specified as:

$$(\rho)_n=\frac{\Gamma(\rho+n)}{\Gamma(\rho)}=\begin{cases}1, & n=0,\rho\in\mathbb{C}/\{0\},\\ \rho(\rho+1)\ldots(\rho+n-1), & (n\in\mathbb{C};\rho\in\mathbb{C}).\end{cases} \tag{4}$$

Salim [11] also introduced a new generalized M-L type function, which is specified as:

$$E_{\zeta,\gamma}^{\rho,\varepsilon}(\propto)=\sum_{n=0}^{\infty}\frac{(\rho)_n}{\Gamma(\zeta n+\gamma)}\frac{\propto^n}{(\varepsilon)_n} \tag{5}$$

$$(\zeta,\gamma,\rho,\varepsilon\in\mathbb{C};\Re(\zeta)>0,\Re(\gamma)>0,\Re(\rho)>0,\Re(\varepsilon)>0).$$

In this article, we will introduce the generalized M-L type function $E_{\zeta,\gamma}^{\rho,\varepsilon}(z)$ as follow:

$$E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(\propto;p)=\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho+n,\varepsilon-\rho;p)}{B(\rho,\varepsilon-\rho)}\frac{\propto^n}{\Gamma(\zeta n+\gamma)} \tag{6}$$

$$(\propto,\gamma,\rho,\varepsilon\in\mathbb{C};\Re(\zeta)>0,\Re(\gamma)>0,\Re(\varepsilon)>0,\Re(\rho)>1;p\geq 0),$$

which will be known as extended generalized Mittag-Leffler type function (EGMLF). Using the fact $\frac{(\rho)_n}{(\varepsilon)_n}=\frac{\mathrm{B}(\rho+n,\varepsilon-\rho)}{\mathrm{B}(\rho,\varepsilon-\rho)}$. Where $\mathfrak{B}^{((k_l)l\in\mathbb{N}_0)}(x,y;p)$ is the extended beta function described by [12] as:

$$\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(x,y;p)=\int_0^1\propto^{x-1}(1-\propto)^{y-1}\Im\left((k_l)_{l\in\mathbb{N}_0};\frac{-p}{\propto(1-\propto)}\right)d\propto \tag{7}$$

$$(\min\{\Re(x),\Re(y)\}>0;\Re(p)\geq 0),$$

and $\Im\left((k_l)_{l\in\mathbb{N}_0};\propto\right)$ is a function of an appropriately bounded sequence $(k_l)_{l\in\mathbb{N}_0}$ of arbitrary real or complex numbers defined as follows:

$$\Im\left((k_l)_{l\in\mathbb{N}_0};\propto\right)=\begin{cases}\sum_{l=0}^{\infty}(k_l)_{l\in\mathbb{N}_0}\frac{\propto^l}{l!}, & |\propto|<\Re;0<\Re<\infty;k_0=1,\\ M_0\propto^w e^{\propto}\left[1+O\left(\frac{1}{z}\right)\right], & \Re(\propto)\to\infty;M_0>0;w\in\mathbb{C}.\end{cases} \tag{8}$$

Some important special cases of EGMLF are enumerated below:

1.If we set $k_l=\frac{(\rho)_l}{(\sigma)_l}(l\in\mathbb{N}_0)$ in Eq. (6), we obtain another form of extended generalized M-L type function follow as:

$$E_{\zeta,\gamma}^{((\rho,\sigma);\rho,\varepsilon)}(\propto;p)=\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{(\rho,\sigma)}(\rho+n,\varepsilon-\rho;p)}{B(\rho,\varepsilon-\rho)}\frac{\propto^n}{\Gamma(\zeta n+\gamma)} \tag{9}$$

$$(\propto,\gamma,\rho,\varepsilon\in\mathbb{C};\Re(\sigma)>0,\Re(\rho)>0,\Re(\gamma)>0,\Re(\zeta)>0,\Re(\rho)>1,p\geq 0).$$

2.On setting $k_l=1\,(l\in\mathbb{N}_0)$, and $\varepsilon=1$, yield the recognized formula of Özarslan and Yilmaz [13] as:

$$E_{\zeta,\gamma}^{(\rho,1)}(\propto;p)=\sum_{n=0}^{\infty}\frac{\mathfrak{B}_p(\rho+n,1-\rho;p)}{B(\rho,1-\rho)}\frac{\propto^n}{\Gamma(\zeta n+\gamma)} \tag{10}$$

$$(\propto,\gamma,\rho\in\mathbb{C};\Re(\gamma)>0,\Re(\zeta)>0,\Re(\rho)>1,p\geq 0).$$

3.If we put $p=0$, Eq. (10) reduces to the Prabhakar's definition (Eq. (1.3)).

4.For $\gamma=\zeta=\varepsilon=1$; Eqs. (6), (9) and (10) can be written in terms of the extended confluent hypergeometric functions as follows:

$$E_{1,1}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,1\right)}(\propto;p)=\phi_p^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho;1;\propto), \tag{11}$$

$$E_{1,1}^{(\rho,\sigma;\rho,1)}(\propto;p)=\phi_p^{(\rho,\sigma)}(\rho;1;\propto), \tag{12}$$

$$E_{1,1}^{(\rho,1)}(\propto;p)=\phi_p(\rho;1;\propto). \tag{13}$$

For a comprehensive explanation of the many characteristics, developments, and implementations of this function, see; Dzherbashyan [9], Gorenflo et al. [14,15], Kilbas and Saigo [16,17], Mishra et al. [18], Nisar et al. [19], Purohit et al. [20], Saxena et al. [21], Suthar and Amsalu [22] and Suthar et al. [23,24,25,26].

Now, consider (see [27,28]) the generalized fractional integral operators (Marichev-Saigo-Maeda) with the Appell function $F_3(.)$ as a kernel (see, e.g., [29], p. 53, Equation (6)) defined for $\wp$, $\wp'$, μ, μ', $\varpi\in\mathbb{C}$ with $\Re(\varpi)>0$ and $x\in\mathbb{R}^+$ as described in the following:

$$\left(I_{0+}^{\wp,\wp',\mu,\mu',\varpi}f\right)(x)=\frac{x^{-\wp}}{\Gamma(\varpi)}\int_0^x(x-t)^{\varpi-1}t^{-\wp'}F_3\left(\wp,\wp',\mu,\mu';\varpi;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt \tag{14}$$

and

$$\left(I_{-}^{\wp,\wp',\mu,\mu',\varpi}f\right)(x)=\frac{x^{-\wp'}}{\Gamma(\varpi)}\int_x^\infty(t-x)^{\varpi-1}t^{-\wp}F_3\left(\wp,\wp',\mu,\mu';\varpi;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt. \tag{15}$$

Marichev [27] introduced the integral operators of the types (14) and (15), which were later explored and expanded by Saigo and Maeda [28]. Furthermore, the equivalent fractional differential operators have the following forms:

$$\left(D_{0+}^{\wp,\wp',\mu,\mu',\varpi}f\right)(x)=\left(\frac{d}{dx}\right)^m\left(I_{0+}^{-\wp',-\wp,-\mu'+m,-\mu,-\varpi+m+1}f\right)(x) \tag{16}$$

and

$$\left(D_{-}^{\wp,\wp',\mu,\mu',\varpi}f\right)(x)=\left(-\frac{d}{dx}\right)^m\left(I_{-}^{-\wp',-\wp,-\mu',-\mu+m,-\varpi+m}f\right)(x). \tag{17}$$

Kataria and Vellaisamy [30] pioneered the Caputo-type fractional derivative, which employs third Appell function in the kernel as described in the following:

$$({}^cD_{0+}^{\wp,\wp',\mu,\mu',\varpi}f)(x)=(I_{-}^{-\wp',-\wp,-\mu'+m,-\mu,-\varpi+m}f^m)(x) \tag{18}$$

and

$$({}^cD_{-}^{\wp,\wp',\mu,\mu',\varpi}f)(x)=(-1)^m(I_{-}^{-\wp',-\wp,-\mu',-\mu+m,-\varpi+m}f^m)(x), \tag{19}$$

where $m=[\Re(\varsigma)]+1$ and f^m denotes the n-th derivative of f.
As a special cases of Caputo type fractional differential operators, Rao et al. [31], introduced Caputo-type fractional derivatives, which include a Gauss hypergeometric function in the kernel, as shown in:

$$({}^cD_{0+}^{\wp,\mu,\varpi}f)(x)=(I_{0+}^{-\wp+m,-\mu-m,\wp+\varpi-m}f^m)(x) \tag{20}$$

and

$$({}^cD_{-}^{\wp,\mu,\varpi}f)(x)=(-1)^m(I_{-}^{-\wp+m,-\mu-m,\wp+\varpi}f^m)(x). \tag{21}$$

Caputo type fractional differential operators (18) and (19) are connected to (20) and (21) as follows:

$$\left({}^cD_{0+}^{0,\wp',\mu,\mu',\varpi}f\right)(x)=\left({}^cD_{0+}^{\varpi,\wp'-\varpi,\mu'-\varpi}f\right)(x) \text{ and } \left({}^cD_{-}^{0,\wp',\mu,\mu',\varpi}f\right)(x)=\left({}^cD_{-}^{\varpi,\wp'-\varpi,\mu'-\varpi}f\right)(x).$$

To prove the proposed result, the following lemmas [30] will be required to establish the Caputo-type MSM fractional differential operator of the EGMLF (6).

Lemma 1. *Let* $\wp,\wp',\mu,\mu',\varpi,\varsigma\in\mathbb{C}$ *and* $m=[\Re(\varsigma)]+1$ *with* $\Re(\varsigma)-m>max\{0,\Re(-\wp+\mu),\Re(-\wp-\wp'-\mu'+\varpi)\}$ *and* $p\geq 0$. *Then*

$$\left({}^cD_{0+}^{\wp,\wp',\mu,\mu',\varpi}t^{\varsigma-1}\right)(x)=\frac{\Gamma(\varsigma+\wp-\mu-m)\Gamma(\varsigma+\wp+\wp'+\mu'-\varpi-m)\Gamma(\varsigma)}{\Gamma(\varsigma-\mu-m)\Gamma(\varsigma+\wp+\wp'-\varpi)\Gamma(\varsigma+\wp+\mu'-\varpi-m)}x^{\varsigma-\varpi+\wp+\wp'-1}.$$

Lemma 2. *Let $\wp, \wp', \mu, \mu', \varpi, \varsigma \in \mathbb{C}$ and $m = [\Re(\varsigma)] + 1$ with $\Re(\varsigma) + m > max\{\Re(-\mu'), \Re(\wp' + \mu - \varpi), \Re(\wp + \wp' - \varpi) + [\Re(\varpi)] + 1\}$. Then*

$$\left({}^{c}D_{-}^{\wp,\wp',\mu,\mu',\varpi} t^{-\varsigma}\right)(x) = \frac{\Gamma(\varsigma + \mu' + m)\Gamma(\varsigma - \wp - \wp' + \varpi)\Gamma(\varsigma - \wp' - \mu + \varpi + m)}{\Gamma(\varsigma - \wp' + \mu' + m)\Gamma(\varsigma - \wp - \wp' - \mu + \varpi + m)\Gamma(\varsigma)} x^{\wp + \wp' - \varpi - \varsigma}.$$

For our objectives, we should also recall that the Wright hypergeometric function is defined in Agarwal et al. ([32], Eq. 1.13) for $z, \rho \in \mathbb{C}, \Re(\varepsilon) > \Re(\rho) > 0; p \geq 0$ as follows:

$$ {}_{m+1}\Psi_{n+1}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(z;p) = {}_{m+1}\Psi_{n+1}^{\left((k_l)_{l\in\mathbb{N}_0}\right)} \left[\begin{matrix} (a_i,\alpha_i)_{1,m}, & (\rho,1); \\ (b_i,\beta_i)_{1,n}, & (\varepsilon,1); \end{matrix} \; (z;p) \right]$$

$$= \frac{1}{\Gamma(\varepsilon-\rho)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{m} \Gamma(a_i + k\alpha_i)}{\prod_{i=1}^{n} \Gamma(b_i + k\beta_i)} \frac{\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho+k, \varepsilon-\rho; p)}{k!} z^k. \tag{22}$$

2 Caputo-type MSM fractional differentiation of EGMLF

Theorem 1. *Let $\wp, \mu, \wp', \mu', \rho, \varepsilon, \varpi, \gamma, \varsigma \in \mathbb{C}$, $m = [\Re(\varsigma)] + 1$ with $\Re(\varpi) > 0$, $\Re(\zeta) > 0$, $\Re(\gamma) > 0$, $\Re(\varepsilon) > 0$, $\Re(\rho) > 0$ and $\Re(\varsigma) - m > max\{0, \Re(-\wp + \mu), \Re(-\wp - \wp' - \mu' + \varpi)\}$; $p \geq 0$. Also let $x \in \mathbb{R}^+$, then*

$$\left({}^{c}D_{0+}^{\wp,\wp',\mu,\mu',\varpi} t^{\varsigma-1} E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(t;p)\right)(x) = x^{\varsigma+\wp+\wp'-\varpi-1} \frac{\Gamma(\varepsilon)}{\Gamma(\rho}$$

$$\times {}_5\Psi_5^{\left((k_l)_{l\in\mathbb{N}_0}\right)} \left[\begin{matrix} (\varsigma,1),(\varsigma-\varpi+\wp+\wp'+\mu'-m,1),(\varsigma-\mu+\wp-m,1),(\rho,1),(1,1); \\ (\varsigma-\mu-m,1),(\varsigma-\varpi+\wp+\mu'-m,1),(\varsigma-\varpi+\wp+\wp',1),(\gamma,\zeta),(\varepsilon,1); \end{matrix} \; (x;p) \right]. \tag{23}$$

Proof. Let $\Im_1$ be LHS of (23), then using (6), we have

$$\Im_1 = \left({}^{c}D_{0+}^{\wp,\wp',\mu,\mu',\varpi} t^{\varsigma-1} E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(t;p)\right)(x)$$

$$= \left({}^{c}D_{0+}^{\wp,\wp',\mu,\mu',\varpi} t^{\varsigma-1} \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho+n,\varepsilon-\rho;p)}{B(\rho,\varepsilon-\rho)} \frac{t^n}{\Gamma(\zeta n+\gamma)}\right)(x),$$

This theorem's condition requires the changing of summation and integration order, we get

$$\Im_1 = \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho+n,\varepsilon-\rho;p)}{B(\rho,\varepsilon-\rho)} \frac{1}{\Gamma(\zeta n+\gamma)} \left({}^{c}D_{0+}^{\wp,\wp',\mu,\mu',\varpi} t^{\varsigma+n-1}\right)(x),$$

Using Lemma (1), we obtain

$$\Im_1 = \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho+n,\varepsilon-\rho;p)}{B(\rho,\varepsilon-\rho)} \frac{1}{\Gamma(\zeta n+\gamma)} x^{\varsigma+n-\varpi+\wp+\wp'-1}$$
$$\times \frac{\Gamma(\varsigma+n)\Gamma(\varsigma-\varpi+\wp+\wp'+\mu'-m+n)\Gamma(\varsigma-\mu+\wp-m+n)}{\Gamma(\varsigma-\mu-m+n)\Gamma(\varsigma-\varpi+\wp+\wp'+n)\Gamma(\varsigma-\varpi+\wp+\mu'-m+n)}$$
$$= \frac{x^{\varsigma-\varpi+\wp+\wp'-1}}{\Gamma(\rho)} \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho+n,\varepsilon-\rho;p)}{\Gamma(\varepsilon-\rho)} \frac{\Gamma(\varepsilon)}{\Gamma(\zeta n+\gamma)}$$
$$\times \frac{\Gamma(\varsigma+n)\Gamma(\varsigma-\varpi+\wp+\wp'+\mu'-m+n)\Gamma(\varsigma-\mu+\wp-m+n)}{\Gamma(\varsigma-\mu-m+n)\Gamma(\varsigma-\varpi+\wp+\wp'+n)\Gamma(\varsigma-\varpi+\wp+\mu'-m+n)} x^n,$$

Thus, we get the desired result by using (22).

Corollary 1. *Let $\wp,\mu,\rho,\varepsilon,\varpi,\gamma,\varsigma\in\mathbb{C}$, $m=[\Re(\wp)]+1$ with $\Re(\varpi)>0$, $\Re(\zeta)>0$, $\Re(\gamma)>0$, $\Re(\varepsilon)>0$, $\Re(\rho)>0$ and $\Re(\varsigma)-m>max\{0,\Re(-\wp-\mu-\varpi)\}$; $p\geq 0$. Also let $x\in\mathbb{R}^+$, then*

$$\left({}^{c}D_{0+}^{\wp,\mu,\varpi}t^{\varsigma-1}E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(t;p)\right)=\frac{\Gamma(\varepsilon)}{\Gamma(\rho)}x^{\varsigma+\mu-1}$$
$$\times{}_4\Psi_4^{\left((k_l)_{l\in\mathbb{N}_0}\right)}\left[\begin{matrix}(\varsigma,1),(\varsigma+\wp+\mu+\varpi-m,1),(\rho,1),(1,1);\\(\varsigma+\varpi-m,1),(\varsigma+\mu,1),(\gamma,\zeta),(\varepsilon,1);\end{matrix}\;(x;p)\right]. \tag{24}$$

Corollary 2. *Let $\wp,\rho,\varepsilon,\varpi,\gamma,\varsigma\in\mathbb{C}$, $m=[\Re(\wp)]+1$ with $\Re(\varpi)>0$, $\Re(\zeta)>0$, $\Re(\gamma)>0$, $\Re(\varepsilon)>0$, $\Re(\rho)>0$ and $\Re(\varsigma)-m>max\{0,\Re(-\wp-\varpi)\}$; $p\geq 0$. Also let $x\in\mathbb{R}^+$, then*

$$\left({}^{c}D_{0+}^{\wp,\varpi}t^{\varsigma-1}E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(t;p)\right)(x)=\frac{\Gamma(\varepsilon)}{\Gamma(\rho)}x^{\varsigma-1}$$
$$\times{}_3\Psi_3^{\left((k_l)_{l\in\mathbb{N}_0}\right)}\left[\begin{matrix}(\varsigma+\wp+\varpi-m,1),(\rho,1),(1,1);\\(\varsigma+\varpi-m,1),(\gamma,\zeta),(\varepsilon,1);\end{matrix}\;(x;p)\right]. \tag{25}$$

Corollary 3. *Let $\wp,\rho,\varepsilon,\gamma,\varsigma\in\mathbb{C}$, $m=[\Re(\wp)]+1$ with $\Re(\zeta)>0$, $\Re(\gamma)>0$, $\Re(\varepsilon)>0$, $\Re(\rho)>0$ and $\Re(\varsigma)-m>max\{\Re(-\wp)\}$; $p\geq 0$. Also let $x\in\mathbb{R}^+$, then*

$$\left({}^{c}D_{0+}^{\wp}t^{\varsigma-1}E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(t;p)\right)(x)=\frac{\Gamma(\varepsilon)}{\Gamma(\rho)}x^{\varsigma-1}$$
$$\times{}_3\Psi_3^{\left((k_l)_{l\in\mathbb{N}_0}\right)}\left[\begin{matrix}(\varsigma+\wp-m,1),(\rho,1),(1,1);\\(\varsigma-m,1),(\gamma,\zeta),(\varepsilon,1);\end{matrix}\;(x;p)\right]. \tag{26}$$

Theorem 2. *Let $\wp,\mu,\wp',\mu',\rho,\varepsilon,\varpi,\gamma,\varsigma\in\mathbb{C}$, $m=[\Re(\varsigma)]+1$ with $\Re(\varpi)>0$ $\Re(\zeta)>0$, $\Re(\gamma)>0$, $\Re(\varepsilon)>0$, $\Re(\rho)>0$ and $\Re(\rho)+m<max\{\Re(-\mu'),\Re(-\mu+\wp'-\varpi),\Re(\wp+\wp'-\varpi)+m,\}$; $p\geq 0$. Also let $x\in\mathbb{R}^+$, then*

$$\left({}^{c}D_{-}^{\wp,\wp',\mu,\mu',\varpi}t^{-\varsigma}E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(1/t;p)\right)(x)=x^{\wp+\wp'-\varpi-\rho}\frac{\Gamma(\varepsilon)}{\Gamma(\rho)}$$

$$\times{}_5\Psi_5^{\left((k_l)_{l\in\mathbb{N}_0}\right)}\left[\begin{matrix}(\varsigma+\mu'+m,1),(\varsigma+\varpi-\wp-\wp',1),(\varsigma-\mu-\wp'+\varpi+m,1),(\rho,1),(1,1);\\(\varsigma,1),(\varsigma-\wp'+\mu'+m,1),(\varsigma+\varpi-\mu-\wp-\wp'+m,1),(\gamma,\zeta),(\varepsilon,1);\end{matrix}\;(\tfrac{1}{x};p)\right]. \tag{27}$$

Proof. Let $\Im_2$ be the LHS of (27), and then taking (6), we have

$$\begin{aligned}\Im_2&=\left({}^{c}D_{-}^{\wp,\wp',\mu,\mu',\varpi}t^{-\varsigma}E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(1/t;p)\right)(x)\\&=\left({}^{c}D_{-}^{\wp,\wp',\mu,\mu',\varpi}t^{-\varsigma}\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho+n,\varepsilon-\rho;p)}{B(\rho,\varepsilon-\rho)}\frac{t^{-n}}{\Gamma(\zeta n+\gamma)}\right)(x),\end{aligned}$$

This theorem's condition requires the changing of summation and integration order, we get

$$\Im_2=\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho+n,\varepsilon-\rho;p)}{B(\rho,\varepsilon-\rho)}\frac{1}{\Gamma(\zeta n+\gamma)}\left({}^{c}D_{-}^{\wp,\wp',\mu,\mu',\varpi}t^{-\varsigma-n}\right)(x),$$

Using Lemma (2), we obtain

$$\mathfrak{I}_2=\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho+n,\varepsilon-\rho;p)}{B(\rho,\varepsilon-\rho)}\frac{1}{\Gamma(\zeta n+\gamma)}x^{-\varsigma-n-\varpi+\wp+\wp'}$$
$$\times\frac{\Gamma(\varsigma+\mu'+m+n)\Gamma(\varsigma+\varpi-\wp-\wp'+n)\Gamma(\varsigma-\wp'-\mu+\varpi+m+n)}{\Gamma(\varsigma+n)\Gamma(\varsigma+\mu'-\wp'+m+n)\Gamma(\varsigma+\varpi-\wp-\wp'-\mu+m+n)}$$

$$=\frac{x^{\wp+\wp'-\varpi-\varsigma}}{\Gamma(\rho)}\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho+n,\varepsilon-\rho;p)}{\Gamma(\varepsilon-\rho)}\frac{\Gamma(\varepsilon)}{\Gamma(\zeta n+\gamma)}$$

$$\times\frac{\Gamma(\varsigma+\mu'+m+n)\Gamma(\varsigma+\varpi-\wp-\wp'+n)\Gamma(\varsigma-\wp'-\mu+\varpi+m+n)}{\Gamma(\varsigma+n)\Gamma(\varsigma+\mu'-\wp'+m+n)\Gamma(\varsigma+\varpi-\wp-\wp'-\mu+m+n)}x^{-n},$$

Thus, we get the desired result by using (22).

Corollary 4. *Let $\wp,\mu,\rho,\varepsilon,\varpi,\gamma,\varsigma\in\mathbb{C}$, $m=[\Re(\wp)]+1$ with $\Re(\varpi)>0$, $\Re(\zeta)>0$, $\Re(\gamma)>0$, $\Re(\varepsilon)>0$, $\Re(\rho)>0$ and $\Re(\rho)+m>max\{\Re(\mu)+m,\Re(-\varpi-\wp)$; $p\geq 0$. Also let $x\in\mathbb{R}^+$, then*

$$\left({}^{c}D_{-}^{\wp,\mu,\varpi}t^{-\varsigma}E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(1/t;p)\right)(x)=x^{\mu-\rho}\frac{\Gamma(\varepsilon)}{\Gamma(\rho)}$$
$$\times{}_4\Psi_4^{\left((k_l)_{l\in\mathbb{N}_0}\right)}\left[\begin{matrix}(\varsigma+\varpi+\wp+m,1),(\varsigma-\mu,1),(\rho,1),(1,1);\\(\varsigma,1),(\varsigma+\varpi-\mu+m,1),(\gamma,\zeta),(\varepsilon,1);\end{matrix}\left(\tfrac{1}{x};p\right)\right].\tag{28}$$

Corollary 5. *Let $\wp,\rho,\varepsilon,\varpi,\gamma,\varsigma\in\mathbb{C}$, $m=[\Re(\wp)]+1$ with $\Re(\varpi)>0$ $\Re(\zeta)>0$, $\Re(\gamma)>0$, $\Re(\varepsilon)>0$, $\Re(\rho)>0$ and $\Re(\rho)+m>max\{m,\Re(-\varpi-\wp),$; $p\geq 0$. Also let $x\in\mathbb{R}^+$, then*

$$\left({}^{c}D_{-}^{\wp,\varpi}t^{\varsigma-1}E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(1/t;p)\right)(x)=x^{-\rho}\frac{\Gamma(\varepsilon)}{\Gamma(\rho)}$$
$$\times{}_3\Psi_3^{\left((k_l)_{l\in\mathbb{N}_0}\right)}\left[\begin{matrix}(\varsigma+\varpi+\wp+m,1),(\rho,1),(1,1);\\(\varsigma+\varpi+m,1),(\gamma,\zeta),(\varepsilon,1);\end{matrix}\left(\tfrac{1}{x};p\right)\right].\tag{29}$$

Corollary 6. *Let $\wp,\rho,\varepsilon,\gamma,\varsigma\in\mathbb{C}$, $m=[\Re(\wp)]+1$ with $\Re(\zeta)>0$, $\Re(\gamma)>0$, $\Re(\varepsilon)>0$, $\Re(\rho)>0$ and $\Re(\rho)+m>max\{\Re(-\wp),$; $p\geq 0$. Also let $x\in\mathbb{R}^+$, then*

$$\left({}^{c}D_{-}^{\wp}t^{\varsigma-1}E_{\zeta,\gamma}^{\left((k_l)_{l\in\mathbb{N}_0};\rho,\varepsilon\right)}(1/t;p)\right)(x)=x^{-\rho}\frac{\Gamma(\varepsilon)}{\Gamma(\rho)}$$
$$\times{}_3\Psi_3^{\left((k_l)_{l\in\mathbb{N}_0}\right)}\left[\begin{matrix}(\varsigma+\wp+m,1),(\rho,1),(1,1);\\(\varsigma+m,1),(\gamma,\zeta),(\varepsilon,1);\end{matrix}\left(\tfrac{1}{x};p\right)\right].\tag{30}$$

3 Special cases

In this section, we derive following Corollaries with some specialized to yield the corresponding formulas by using known extended generalized M-L type function, generalized M-L type function and extended confluent hypergeometric functions.

(i) If we employ the same method as in proofs of Theorems 1-2, we obtain the following two Corollaries with the help of (9) which is known another form of extended generalized M-L type function. For the conditions of $k_l=\frac{(\rho)_l}{(\sigma)_l}$ $(l\in\mathbb{N}_0)$, the above Theorems reduce to:

Corollary 7. *Let $\wp,\mu,\wp',\mu',\rho,\varepsilon,\varpi,\gamma,\varsigma\in\mathbb{C}$, $m=[\Re(\varsigma)]+1$ with $\Re(\rho)>0,\Re(\sigma)>0,\Re(\varpi)>0$, $\Re(\zeta)>0$, $\Re(\gamma)>0$, $\Re(\varepsilon)>0$, $\Re(\rho)>0$ and $\Re(\varsigma)-m>max\{0,\Re(-\wp+\mu),\ \Re(-\wp-\wp'-\mu'+\varpi)\}$; $p\geq 0$. Moreover take $x\in\mathbb{R}^+$, then*

$$\left({}^{c}D_{0+}^{\wp,\wp',\mu,\mu',\varpi}t^{\varsigma-1}E_{\zeta,\gamma}^{((\rho,\sigma);\rho,\varepsilon)}(t;p)\right)(x)=x^{\varsigma+\wp+\wp'-\varpi-1}\frac{\Gamma(\varepsilon)\Gamma(\sigma)}{\Gamma(\rho)\Gamma(\rho)}{}_6\Psi_6^{\left((k_l)_{l\in\mathbb{N}_0}\right)}$$

$$\times\left[\begin{matrix}(\varsigma,1),(\varsigma-\varpi+\wp+\wp'+\mu'-m,1),(\varsigma-\mu+\wp-m,1),(\rho,1),(\rho,1),(1,1);\\(\varsigma-\mu-m,1),(\varsigma-\varpi+\wp+\mu'-m,1),(\varsigma-\varpi+\wp+\wp',1),(\gamma,\zeta),(\sigma,1),(\varepsilon,1);\end{matrix}(x;p)\right].\qquad(31)$$

Corollary 8. *Let $\wp,\mu,\wp',\mu',\rho,\varepsilon,\varpi,\gamma,\varsigma\in\mathbb{C}$, $m=[\Re(\varsigma)]+1$ with $\Re(\rho)>0,\Re(\sigma)>0,\Re(\varpi)>0$ $\Re(\zeta)>0$, $\Re(\gamma)>0$, $\Re(\varepsilon)>0$, $\Re(\rho)>0$ and $\Re(\rho)+m<max\{\Re(-\mu'),\ \Re(-\mu+\wp'-\varpi),R(\wp+\wp'-\varpi)+m,\}$; $p\geq 0$. Furthermore assume $x\in\mathbb{R}^+$, so*

$$\left({}^{c}D_{-}^{\wp,\wp',\mu,\mu',\varpi}t^{-\varsigma}E_{\zeta,\gamma}^{((\rho,\sigma);\rho,\varepsilon)}(1/t;p)\right)(x)=x^{\wp+\wp'-\varpi-\rho}\frac{\Gamma(\varepsilon)\Gamma(\sigma)}{\Gamma(\rho)\Gamma(\rho)}{}_6\Psi_6^{\left((k_l)_{l\in\mathbb{N}_0}\right)}$$

$$\times\left[\begin{matrix}(\varsigma+\mu'+m,1),(\varsigma+\varpi-\wp-\wp',1),(\varsigma-\mu-\wp'+\varpi+m,1),(\rho,1),(\rho,1),(1,1);\\(\varsigma,1),(\varsigma-\wp'+\mu'+m,1),(\varsigma+\varpi-\mu-\wp-\wp'+m,1),(\sigma,1),(\gamma,\zeta),(\varepsilon,1);\end{matrix}(\tfrac{1}{x};p)\right].\qquad(32)$$

(ii) If we put $k_l=1\,(l\in\mathbb{N}_0)$ and $\varepsilon=1$, the similar way as in proofs of Theorems 1-2, we obtain the following two corollaries with the help of (10) below as:

Corollary 9. *If the requirement of Theorem 1 is satisfied, the differential formula follows:*

$$\left({}^{c}D_{0+}^{\wp,\wp',\mu,\mu',\varpi}t^{\varsigma-1}E_{\zeta,\gamma}^{(\rho,1)}(t;p)\right)(x)=\frac{x^{\varsigma+\wp+\wp'-\varpi-1}}{\Gamma(\rho)}$$

$$\times{}_5\Psi_5\left[\begin{matrix}(\varsigma,1),(\varsigma-\varpi+\wp+\wp'+\mu'-m,1),(\varsigma-\mu+\wp-m,1),(\rho,1),(1,1);\\(\varsigma-\mu-m,1),(\varsigma-\varpi+\wp+\mu'-m,1),(\varsigma-\varpi+\wp+\wp',1),(\gamma,\zeta),(\varepsilon,1);\end{matrix}(x;p)\right].\qquad(33)$$

Corollary 10. *If the constraint of Theorem 2 is achieved, the differential formula follows:*

$$\left({}^{c}D_{-}^{\wp,\wp',\mu,\mu',\varpi}t^{-\varsigma}E_{\zeta,\gamma}^{(\rho,1)}(1/t;p)\right)(x)=\frac{x^{\wp+\wp'-\varpi-\rho}}{\Gamma(\rho)}$$

$$\times{}_5\Psi_5\left[\begin{matrix}(\varsigma+\mu'+m,1),(\varsigma+\varpi-\wp-\wp',1),(\varsigma-\mu-\wp'+\varpi+m,1),(\rho,1),(1,1);\\(\varsigma,1),(\varsigma-\wp'+\mu'+m,1),(\varsigma+\varpi-\mu-\wp-\wp'+m,1),(\gamma,\zeta),(\varepsilon,1);\end{matrix}(\tfrac{1}{x};p)\right].\qquad(34)$$

(iii) Similarly, we derive the fractional differentiation formulae using the Prabhakar-type [10] M-L function by placing $p=0$ in Corollaries 9-10. We take out all the details.

(iv) If we set $\gamma=\zeta=\varepsilon=1$ in proofs of Theorems 1-2, we obtain the following two Corollaries with the help of (11) which is known extended Confluent hypergeometric functions as follows:

Corollary 11. *Let the condition of Theorem 1 be achieved, the resulting differential equation holds:*

$$\left({}^{c}D_{0+}^{\wp,\wp',\mu,\mu',\varpi}t^{\varsigma-1}\phi_p^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho;1;t)\right)(x)=\frac{x^{\varsigma+\wp+\wp'-\varpi-1}}{\Gamma(\rho)}$$

$$\times{}_4\Psi_4^{\left((k_l)_{l\in\mathbb{N}_0}\right)}\left[\begin{matrix}(\varsigma,1),(\varsigma-\varpi+\wp+\wp'+\mu'-m,1),(\varsigma-\mu+\wp-m,1),(\rho,1);\\(\varsigma-\mu-m,1),(\varsigma-\varpi+\wp+\mu'-m,1),(\varsigma-\varpi+\wp+\wp',1),(1,1);\end{matrix}(x;p)\right].\qquad(35)$$

Corollary 12. *Let the condition of Theorem 2 be fulfilled, the subsequent differential formula holds:*

$$\left({}^{c}D_{-}^{\wp,\wp',\mu,\mu',\varpi}t^{-\varsigma}\phi_{p}^{\left((k_l)_{l\in\mathbb{N}_0}\right)}(\rho;1;1/t)\right)(x)=\frac{x^{\wp+\wp'-\varpi-\rho}}{\Gamma(\rho)}$$

$$\times{}_4\psi_4^{\left((k_l)_{l\in\mathbb{N}_0}\right)}\left[\begin{matrix}(\varsigma+\mu'+m,1),(\varsigma+\varpi-\wp-\wp',1),(\varsigma-\mu-\wp'+\varpi+m,1),(\rho,1);\\(\varsigma,1),(\varsigma-\wp'+\mu'+m,1),(\varsigma+\varpi-\mu-\wp-\wp'+m,1),(\varepsilon,1);\end{matrix}\ \left(\tfrac{1}{x};p\right)\right]. \tag{36}$$

4 Conclusion

In this article, we established images of Caputo type MSM fractional differential operators, applied on newly defined extended generalized M-L type function. Several variants of the derived outcomes in context of fractional differential (Saigo operators, Erdélyi-Kober operators and Riemann operators) and another type of generalized M-L functions, extended M-L functions and extended confluent hypergeometric functions are investigated by taking suitable values of parameters involved. If we set $\varepsilon = 1$ in (6), we obtain the generalized M-L type function due to Parmar [22]. In this sequel, The insights described in this paper build on a number of recent advances in the theory of special functions, which has a wide spectrum of uses in physics and engineering.

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