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An Efficient Approach for Investigating the Solution of Fractional Singularly Integrodifferential Equations of First Order

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Abstract: In this study, fractional singularly integrodifferential equations of first order are investigated. This numerical technique is based on the reproducing kernel method. This technique gives an accurate results comparing the method in the literature review. Theoretical results such as the stably and the uniqueness of the solution for this class of problems are given. Some numerical examples are given.

Keywords: Singularly integrodifferential, RKM, solution.

1 Introduction

Integral equations have several special forms and some of them are the Volterra integral equations. Several application for them can be found in [1]. Numerous numerical strategies were utilized to take care of these issues. For example, Legendre wavelet method [2], Legendre wavelet method [3], collocation method [4], difference methods [5], and STWS approach [6]. Also, they were solved by Bernoulli polynomials [7]. The boundary layer was investigated by [8,9]. More methods can be found in [10]-[18].

The singularly perturbed problem has several applications in mathematics, physics, and engineers [19]. One of these applications is the nonlinear problems of plates and shells [20]. Other application appears in control problems [21,22,23, 24,25], More applications can be found in [27]-[38].

In this manuscript we explore the arrangement of the solution of

$$\xi D^{\mu} t+\omega(r,t)+\int_{0}^{r}\Psi(r,\zeta)\chi(\zeta,t)d\zeta=\rho(r), r\in(0,1), 0<\mu\leq 1, \tag{1}$$

$$t(0)=t_0 \tag{2}$$

with $\xi>0$, t_0 is constant, $\Psi(r,\zeta)$ and $\rho(r)$ are differentiable as the discussion required, and D^{μ} is the Caputo derivative. Next, we present the reproducing kernel (RKM) method for solving such problems.

The organisation of the manuscript is depicted below. In Section 2 we discuss the reproducing kernel method for first order initial value problems. Section 3 is devoted to reported analytical results. In Section 4 we deal with the method of solution. Section 5 presents the numerical examples. Finally, Section 6 presents our conclusions.

2 Reproducing kernel method for first order initial value problems

First, we define the fractional derivatives.

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Definition 1. *Let $\mu > 0$. Then, the Caputo derivative is*

$$D^{\mu}t(r) = \left\{ \begin{array}{ll} \frac{1}{\Gamma(m-\mu)} \int_0^r \frac{t^{(n)}(\tau)}{(r-\tau)^{\mu-m+1}} d\tau, & m-1 < \mu < m \in \mathbb{N}, \\ t^{(m)}(r), & m \in \mathbb{N} \end{array} \right\}.$$

For more details, see [26] and [37].

Definition 2. *Let $Q \neq \phi$. A function $\Theta : N \times N \to C$ is a RKHS G iff*

– $\Theta(.,r) \in G$ *for all* $r \in Q$,
– $(\pi(.), \Theta(.,r)) = \phi(r)$ *where* $r \in E$ *and* $\pi \in G$.

Consider

$$D^{\mu}t + \rho(t) = a, \; r \in [0,1], 0 < \mu \leq 1 \tag{3}$$

such that

$$t(0) = \varsigma \tag{4}$$

where a and ς are scalars. Let $\rho(t) = \eta(r)t$. Let $\omega = t - \varsigma$. Then

$$D^{\mu}\omega + \Upsilon(\omega) = a, \; r \in [0,1], 0 < \mu \leq 1 \tag{5}$$

subject to

$$\omega(0) = 0. \tag{6}$$

Let

$$H_2^1[0,1] = \{\omega(z) : \; \omega \text{ is absolutely continuous real value function (ACRVF)}, \; \omega' \in L^2[0,1]\}.$$

The inner product (IP) in $H_2^1[0,1]$ is

$$(\omega(t), \chi(t))_{H_2^1[0,1]} = \omega(0)\chi(0) + \int_0^1 \omega'(t)\chi'(t)dt,$$

with $\|\omega\|_{H_2^1[0,1]}$ is

$$\|\omega\|_{H_2^1[0,1]} = \sqrt{(\omega(t), \omega(t))_{H_2^1[0,1]}}$$

such that $\omega, \chi \in H_2^1[0,1]$.

Theorem 1. *$H_2^1[0,1]$ is a RKHS, i.e.; $\exists \; \Omega(z,t) \in H_2^1[0,1]$ such that*

$$(\omega(t), \Omega(z,t))_{H_2^1[0,1]} = \omega(z).$$

$\Omega(z,t)$ is

$$\Omega(z,t) = \left\{ \begin{array}{ll} 1+t, & t \leq z \\ 1+z, & t > z \end{array} \right\}.$$

Proof: Simple calculations imply that

$$\begin{aligned} (\omega(t), \Omega(z,t))_{H_2^1[0,1]} &= \omega(0)\Omega(z,0) + \int_0^1 \omega'(t) \frac{\partial \Omega}{\partial t}(z,t)dt \\ &= \omega(0)\Omega(z,0) + \omega(1)\frac{\partial \Omega}{\partial t}(z,1) - \omega(0)\frac{\partial \Omega}{\partial t}(z,0) - \int_0^1 \omega(t)\frac{\partial^2 \Omega}{\partial t^2}(z,t)dt. \end{aligned}$$

Since $\Omega(z,t)$ is RK of $H_2^1[0,1]$,

$$(\omega(t), \Omega(z,t))_{H_2^1[0,1]} = \omega(z)$$

which implies that

$$-\frac{\partial^2 \Omega}{\partial t^2}(z,t) = \delta(t-z), \tag{7}$$

$$\Omega(z,0)-\frac{\partial\Omega}{\partial t}(z,0)=0, \tag{8}$$

and

$$\frac{\partial\Omega}{\partial t}(z,1)=0. \tag{9}$$

Thus, $\Omega(z,t)$ is

$$\Omega(z,t)=\begin{cases} a_0(z)+a_1(z)t, & t\le z \\ b_0(z)+b_1(z)t, & t>z \end{cases}$$

Since $\frac{\partial^2\Omega}{\partial t^2}(z,t)=-\delta(t-z)$, we have

$$\Omega(z,z+0)-\Omega(z,z+0)=0\,, \tag{10}$$

$$\frac{\partial\Omega}{\partial t}(z,z+0)-\frac{\partial\Omega}{\partial t}(z,z+0)\ =-1. \tag{11}$$

Thus,

$$\begin{aligned} a_0(z)-a_1(z)&=0,\\ b_1(z)&=0,\\ a_0(z)+a_1(z)\,y&=b_0(z)+b_1(z)\,z,\\ b_1(z)-\ a_1(z)&=-1, \end{aligned} \tag{12}$$

which implies that

$$a_0(z)=1,\ a_1(z)=1, b_0(z)=1+z,\ b_1(z)=0.$$

Let

$H_2^2[0,1]=\{\rho(z):\ \rho\ \text{ is ACRVF},$
$\rho,\rho',\rho''\in L^2[0,1],\rho(0)=0\}.$

The IP in $H_2^2[0,1]$ is defined as

$$(\omega(t),\chi(t))_{H_2^2[0,1]}=\omega(0)\chi(0)+\omega(1)\chi(1)+\int_0^1\omega^{(2)}(t)\chi^{(2)}(t)dt,$$

with $\|\omega\|_{H_2^2[0,1]}$ is

$$\|\omega\|_{H_2^2[0,1]}=\sqrt{(\omega(t),\omega(t))_{H_2^2[0,1]}}$$

with $\omega,\chi\in H_2^2[0,1]$.

Theorem 2.$H_2^2[0,1]$ *is a RKHS, i.e.;* $\exists\ \Xi(z,t)\in H_2^2[0,1]$ *such that*

$$(\omega(t),\Xi(z,t))_{H_2^2[0,1]}=\omega(z).$$

$\Xi(z,t)$ *is*

$$\Xi(z,t)=\left\{\begin{array}{ll} \sum_{i=0}^3 a_i(z)t^i, & t\le z \\ \sum_{i=0}^3 b_i(z)t^i, & t>z \end{array}\right\}$$

where

$$a_0=0,\ a_1=\frac{1}{6}(8z-3z^2+z^3),\ a_2=0, a_3=\frac{1}{6}(z-1),$$
$$b_0=-\frac{z^3}{6},\ b_1=\frac{1}{6}(8z+z^3),\ b_2=-\frac{z}{2}, b_3=\ -\frac{z}{6}.$$

Proof: Simple calculations imply that

$$\begin{aligned}(\omega(t),\Xi(z,t))_{H_2^2[0,1]} &= \omega(0)\Xi(z,0)+\omega(1)\Xi(z,1)+\omega'(1)\Xi_{tt}(z,1)-\omega'(0)\Xi_{tt}(z,0)\\ &\quad -\omega(1)\frac{\partial^3\Xi}{\partial t^3}(z,1)+\omega(0)\frac{\partial^3\Xi}{\partial t^3}(z,0)+\int_0^1\omega(t)\frac{\partial^4\Xi}{\partial t^4}(z,t)dt.\end{aligned}$$

$\omega(t)$ and $\Xi(z,t)\in H_2^2[0,1]$ implies that

$$\omega(0)=0$$

and

$$\Xi(z,0)=0. \tag{13}$$

Hence,

$$\begin{aligned}(\omega(t),\Xi(z,t))_{H_2^2[0,1]} &= \omega(1)\Xi(z,1)+\omega'(1)\Xi_{tt}(z,1)-\omega'(0)\Xi_{tt}(z,0)\\ &\quad -\omega(1)\frac{\partial^3\Xi}{\partial t^3}(z,1)+\int_0^1\omega(t)\frac{\partial^4\Xi}{\partial t^4}(z,t)dt.\end{aligned}$$

Because $\Xi(z,t)$ is a RK of $H_2^2[0,1]$,

$$(\omega(t),\Xi(z,t))_{H_2^2[0,1]}=\omega(z)$$

then

$$\frac{\partial^4\Xi}{\partial t^4}(z,t)=\delta(t-z) \tag{14}$$

and

$$\Xi(z,1)-\frac{\partial^3\Xi}{\partial t^3}(z,1)=0, \tag{15}$$

$$\Xi_{tt}(z,1)=0, \tag{16}$$

$$\Xi_{tt}(z,0)=0. \tag{17}$$

Thus, $\Xi(z,t)$ is

$$\Xi(z,t)=\left\{\begin{array}{ll}\sum_{i=0}^3 a_i(z)t^i, & t\le z\\ \sum_{i=0}^3 b_i(z)t^i, & t>z\end{array}\right\}.$$

Since $\frac{\partial^3\Xi}{\partial t^3}(z,t)=\delta(z-t)$,

$$\frac{\partial^n\Xi}{\partial t^n}(z,z+0)=\frac{\partial^n\Xi}{\partial t^n}(z,z-0),\ n=0:2. \tag{18}$$

Integrate $\frac{\partial^6\Xi}{\partial t^z}(z,t)=\delta(z-t)$ from $z-\xi$ to $z+\xi$ w.r.t t and $\xi\to 0$,

$$\frac{\partial^3\Xi}{\partial t^3}(z,z+0)-\frac{\partial^3\Xi}{\partial t^3}(z,z-0)=1. \tag{19}$$

By 13 and 15-19,

$$a_0(z)=0,\ \sum_{i=0}^3 b_i(z)-6b_3(z)=0,$$

$$6b_3(z)+2b_2(z)=0, a_2(z)=0,$$

$$\sum_{i=0}^3 a_i(z)z^i=\sum_{i=0}^3 b_i(z)z^i,$$

$$\sum_{i=1}^3 ia_i(z)z^{i-1}=\sum_{i=i}^3 ib_i(z)z^{i-1},$$

$$\sum_{i=2}^3 i(i-1)a_i(z)z^{i-2}=\sum_{i=1}^3 i(i-1)b_i(z)z^{i-2},$$

$$3!b_3(z) - 3!a_3(z) = 1.$$

Thus,

$$a_0 = 0,\ a_1 = \frac{1}{6}(8z - 3z^2 + z^3),\ a_2 = 0, a_3 = \frac{1}{6}(z-1),$$
$$b_0 = -\frac{z^3}{6},\ b_1 = \frac{1}{6}(8z + z^3),\ b_2 = -\frac{z}{2}, b_3 = -\frac{z}{6}.$$

Now, we present how to solve Problem 5-6

$$\sigma_i(z) = \Omega(z_i, z)$$

$i = 1, 2, \ldots$ where $\{z_i\}_{i=1}^{\infty}$ is dense on $[0,1]$. Thus, $\Pi : H_2^2[0,1] \to H_2^1[0,1]$ is bounded. Assume

$$F_i(z) = \Pi^* \sigma_i(z)$$

where $\Pi(\sigma_i(z)) = D^{\mu}\sigma_i(z) + \eta(z)\sigma_i(z)$ and Π^* is the adjoint operator. By GS, we get $\left\{\bar{F}_i(z)\right\}_{i=1}^{\infty}$ with

$$\bar{F}_i(s) = \sum_{j=1}^{i} \mu_{ij} F_j(z) \tag{20}$$

μ_{ij} are parameters of GS. In the following them, we demonstrate the presence of the arrangement of Problem (5)-(6).

Theorem 3. *If $\{z_i\}_{i=1}^{\infty}$ is dense,*

$$\omega(z) = a \sum_{i=1}^{\infty} \sum_{j=1}^{i} \mu_{ij} \bar{F}_i(s). \tag{21}$$

Proof: To begin with, we need to demonstrate that $\{F_i(z)\}_{i=1}^{\infty}$ is complete system with $F_i(z) = \Pi(\Omega(z, z_i))$. Thus, $F_i(z) \in H_2^2[0,1]$ for $i = 1, 2, \ldots$. Thus,

$$\begin{aligned} F_i(z) &= L^* \sigma_i(z) = (\Pi^* \sigma_i(z), \Omega(z,t))_{H_2^2[0,1]} \\ &= (\sigma_i(z), \Pi(\Omega(z,t)))_{H_2^2[0,1]} \\ &= \Pi(\Omega(z, z_i)). \end{aligned}$$

For $\omega(z) \in H_2^2[0,1]$, let

$$(\omega(z), F_i(z))_{H_2^2[0,1]} = 0,\ i = 1, 2, \ldots.$$

Then

$$\begin{aligned} (\omega(z), F_i(z))_{H_2^2[0,1]} &= (\omega(z), \Pi^* \sigma_i(z))_{H_2^2[0,1]} \\ &= (\Pi\rho(z), \sigma_i(z))_{H_2^2[0,1]} \\ &= \Pi\omega(z_i) = 0. \end{aligned}$$

Since $\{z_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, $\Pi\omega(z) = 0$. Since Π^{-1} exists, $\omega(z) = 0$. Thus, $\{F_i(z)\}_{i=1}^{\infty}$ is the complete system of $H_2^2[0,1]$.

Second, we prove Equation 21. Thus,

$$\begin{aligned}
\omega(z) &= \sum_{i=1}^{\infty}(\omega(z),\bar{F}_i(z))_{H_2^2[0,1]}\bar{F}_i(z)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\mu_{ij}(\omega(z),\Pi^*(\Omega(z,z_j)))_{H_2^2[0,1]}\bar{F}_i(z)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\mu_{ij}(\Pi\rho(z),\Omega(z,z_j))_{H_2^2[0,1]}\bar{F}_i(z)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\mu_{ij}(a,\Omega(z,z_j))_{H_2^2[0,1]}\bar{F}_i(z)\\
&= a\sum_{i=1}^{\infty}\sum_{j=1}^{i}\mu_{ij}\bar{F}_i(s).
\end{aligned}$$

Thus,

$$\omega_N(z) = a\sum_{i=1}^{N}\sum_{j=1}^{i}\mu_{ij}\bar{F}_i(z). \tag{22}$$

Theorem 4. $\left\{\frac{d^l\omega_N(z)}{dz^l}\right\}_{N=1}^{\infty}$ *converges uniformly to* $\frac{d^l\omega(z)}{dz^l}$ *for* $l=0,1$.

Proof: For $l=0$. For any $s\in[0,1]$,

$$\begin{aligned}
\|\omega(z)-\omega_N(z)\|^2_{H_2^2[0,1]} &= (\omega(z)-\omega_N(z),\omega(z)-\omega_N(z))_{H_2^2[0,1]}\\
&= \sum_{i=N+1}^{\infty}((\omega(z),\bar{F}_i(z))_{H_2^2[0,1]}\bar{F}_i(z),(\omega(z),\bar{F}_i(z))_{H_2^2[0,1]}\bar{F}_i(z))_{H_2^2[0,1]}\\
&= \sum_{i=N+1}^{\infty}(\omega(z),\bar{F}_i(z))^2_{H_2^2[0,1]}.
\end{aligned}$$

Thus,

$$\underset{z\in[0,1]}{Sup}\|\omega(z)-\omega_N(z)\|^2_{H_2^2[0,1]} = \underset{z\in[0,1]}{Sup}\sum_{i=N+1}^{\infty}(\omega(z),\bar{F}_i(z))^2_{H_2^2[0,1]}.$$

By Theorem (6), $\sum_{i=1}^{\infty}(\omega(z),\bar{F}_i(z))_{H_2^2[0,1]}\bar{F}_i(z)$ converges uniformly to $\omega(z)$. Hence,

$$\underset{N\to\infty}{Lim}\ \underset{z\in[0,1]}{Sup}\|\omega(z)-\omega_N(z)\|_{H_2^2[0,1]}=0$$

which implies that $\{\omega_N(z)\}_{N=1}^{\infty}$ converges uniformly to $\omega(z)$.

Since $\frac{d^l\Omega(z,t)}{dz^l}$ is bounded,

$$\left\|\frac{d^l\Omega(z,t)}{dz^l}\right\|_{H_2^2[0,1]}\le\gamma_l,\ l=1.$$

For $z\in[0,1]$,

$$\begin{aligned}
\left|\omega^{(l)}(z)-\omega_N^{(l)}(z)\right| &= \left|(\omega(z)-\omega_N(z),\frac{d^l\Omega(z,t)}{dz^l})_{H_2^2[0,1]}\right|\\
&\le \|\omega(z)-\omega_N(z)\|_{H_2^2[0,1]}\left\|\frac{d^l\Omega(z,t)}{dz^l}\right\|_{H_2^2[0,1]}\\
&\le \gamma_l\|\omega(z)-\omega_N(z)\|_{H_2^2[0,1]}\\
&\le \gamma_l\underset{z\in[0,1]}{Sup}\|\omega(z)-\omega_N(z)\|_{H_2^2[0,1]}.
\end{aligned}$$

Hence,

$$\underset{z\in[0,1]}{Sup}\left\|\omega^{(l)}(z)-\omega_N^{(l)}(z)\right\|_{H_2^2[0,1]}\leq\gamma_l\underset{z\in[0,1]}{Sup}\left\|\omega(z)-\omega_N(z)\right\|_{H_2^2[0,1]}$$

which implies that

$$\underset{N\to\infty}{Lim}\ \underset{z\in[0,1]}{Sup}\left\|\omega^{(l)}(z)-\omega_N^{(l)}(z)\right\|_{H_2^2[0,1]}=0.$$

Therefore, $\left\{\frac{d^l\omega_N(z)}{dz^l}\right\}_{N=1}^{\infty}$ converges uniformly to $\frac{d^l\omega(z)}{dz^l}$ for $l=1$.

Let $\Pi(t(r))=D^{\mu}t(r)-a$ and $N(t(r))=\rho(t)$. Let

$$\Delta(t,v)=\Pi(t(r))+vN(t(r))=0 \tag{23}$$

with $v\in[0,1]$. If $v=0$,

$$D^{\mu}t(r)-a=0$$

and $t(r)=a\frac{r^{\mu}}{\Gamma(1+\mu)}$. If $v=1$, Equation 3 is produced. Let

$$t=t_0+vt_1+v^2t_2+v^3t_3+\dots. \tag{24}$$

From Equation 24 and 23, we get

$$\begin{aligned}
&v^0: D^{\mu}t_0(r)=a\ ,t_0(0)=\varsigma,\\
&v^1: D^{\mu}t_1(r)=-N(\sum_{i=0}^{\infty}v^it_i(r))\ |_{v=0}\ ,\ t_1(0)=0,\\
&v^2: D^{\mu}t_2(r)=-\frac{dN(\sum_{i=0}^{\infty}v^it_i(r))}{dv}\ |_{v=0}\ ,t_2(0)0,\\
&v^3: D^{\mu}t_3(r)=-\frac{d^2N(\sum_{i=0}^{\infty}v^it_i(r))}{dv^2}\ |_{v=0},\ t_3(0)=0,\\
&\quad\vdots\\
&v^k: D^{\mu}t_k(r)=-\frac{d^{k-1}N(\sum_{i=0}^{\infty}v^it_i(r))}{dv^{k-1}}\ |_{v=0},t_k(0)=0.
\end{aligned}$$

By RKM,

$$t_k(r)=\sum_{i=1}^{\infty}\sum_{j=1}^{i}v_{ij}\Upsilon_k(r_j)\bar{F}_i(z)\ ,k=0,1,\dots \tag{25}$$

where

$$\begin{aligned}
&\Upsilon_0(r)=a\\
&\Upsilon_1(r)=-N(\sum_{i=0}^{\infty}v^it_i(r))\ |_{v=0}\\
&\qquad\vdots\\
&\Upsilon_k(r)=-\frac{d^{k-1}N(\sum_{i=0}^{\infty}\Upsilon^it_i(r))}{dv^{k-1}}\ |_{v=0},k>1.
\end{aligned}$$

By 25,

$$t(r)=\sum_{k=0}^{\infty}t_k(r)\ =\sum_{k=0}^{\infty}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{i}\mu_{ij}\Upsilon_k(r_j)\bar{F}_i(r)\right). \tag{26}$$

Thus,

$$t_{n,m}(r)=\sum_{k=0}^{m}\left(\sum_{i=1}^{n}\sum_{j=1}^{i}\mu_{ij}\Upsilon_k(r_j)\bar{F}_i(r)\right). \tag{27}$$

3 Analytical results

In this part, we prove some of our theoretical results. Define

$$\Delta t : \xi D^{\mu} t + \omega(r,t) + \int_0^r \Psi(r,\xi)\chi(\xi,t)d\xi = \rho(r), r \in (0,1), 0 < \mu \leq 1, \tag{28}$$

$$t(0) = t_0 \tag{29}$$

$$-\vartheta_2 \geq \omega(r,t) \geq -\vartheta_1, \tag{30}$$

$$0 \geq \chi(r,t) \geq -\vartheta 3, \tag{31}$$

$$\Psi(r,\xi) \geq \vartheta_4 \geq 0, \tag{32}$$

for all $r \in [0,1]$, where $\vartheta_1, \vartheta_2, \vartheta 3$, and ϑ_4 are positive with $t \in C^1(0,1) \cup C[0,1]$.

Theorem 5. *Let $\Lambda \urcorner \geq 0$ and $\urcorner(0) \geq 0$. then, $\urcorner(r) \geq 0$ in $[0,1]$.*

Proof. Let $\urcorner(r) < 0$ with $r \in [0,1]$. Thus, $\urcorner(r)$ attend its minimum at r_0 with $r_0 \in (0,1]$. Thus,

$$\begin{aligned}
\Lambda \urcorner(r_0) &= \xi D^{\mu} \urcorner(r_0) + \omega(r_0, \urcorner) + \int_0^{x_0} K(x_0,t)v(t,\phi)dt \\
&\leq \xi \frac{r_0^{-\mu}}{\Gamma(1-\mu)}(\urcorner(r_0) - \urcorner(0)) + \omega(r_0, \urcorner) + \int_0^{r_0} \Psi(r_0,t)\chi(t,\urcorner)dt \\
&\leq 0.
\end{aligned}$$

Thus, $\urcorner(r) \geq 0$ in $[0,1]$.

Theorem 6. *Let $\omega = \omega(r)$ and $\chi = \chi(r)$. If $t(r)$ is differentiable function as the discussion required,*

$$\|t\| = \frac{1}{\varepsilon} \max\{|t(r)| : r \in [0,1]\} \leq \frac{1}{\xi} \max\left\{|t_0|, \max_{r\in[0,1]} |\Lambda t|\right\}.$$

Proof. Let

$$\vartheta_0 = \max\left\{|t_0|, \max_{r\in[0,1]} |\Delta t|\right\} = \max\left\{|t_0|, \max_{r\in[0,1]} |\rho(r)|\right\}$$

with

$$z^{\pm}(r) = \frac{\vartheta_0}{\xi}(1 + \frac{r^{\mu}}{\Gamma(1+\mu)}) \pm t(r), r \in [0,1].$$

Hence,

$$\begin{aligned}
\Delta z^{\pm}(r) &= \xi D^{\mu}\left(\frac{\vartheta_0}{\xi}(1 + \frac{r^{\mu}}{\Gamma(1+\mu)}) \pm t(r)\right) + \omega(r) + \int_0^r \Psi(r,t)\chi(t)dt \\
&= \xi \frac{\vartheta_0}{\xi} \pm \Delta t(r) = \vartheta_0 \pm \Delta t(r) \geq 0
\end{aligned}$$

with$r \in [0,1]$. Furthermore,

$$z^{\pm}(0) = \frac{\vartheta_0}{\xi} \pm t(0) > \vartheta_0 \pm t_0 \geq 0$$

since $0 < \xi << 1$. By Theorem 3,

$$\|t\| \leq \max_{r\in[0,1]}\left\{\frac{\vartheta_0}{\xi}(1 - \frac{r^{\mu}}{\Gamma(1+\mu)})\right\} \leq \frac{\vartheta_0}{\xi} = \frac{1}{\xi}\max\left\{|t_0|, \max_{r\in[0,1]} |\Delta t|\right\}.$$

Theorem 7. *Let $\omega = \omega(r)$ with $\chi = \chi(r)$. If t_1 and t_2 satisfy Eqs. 1-2, $t_1(r) = t_2(r)$ with $r \in [0,1]$.*

Proof. Assume $\chi(r) = t_1(r) - t_2(r)$. Thus,

$$\Delta\phi = 0,\ \phi(0) = 0,$$

$$\Delta(-\phi) = 0,\ -\phi(0) = 0.$$

By Theorem 9, $\chi(r) \geq 0$ with $\chi(r) \leq 0$ for $r \in [0,1]$. Thus, $t_1(r) = t_2(r)$ for all $r \in [0,1]$.

4 Method of solution

Let

$$\xi D^{\mu} t + \omega(r,t) + \int_0^r \Psi(r,z)\chi(z,t)dz = \rho(r), r \in (0,1), 0 < \mu \leq 1,$$

such that

$$t(0) = t_0$$

with $\xi > 0$. Thus,

Step 1: Let $\xi = 0$, then

$$\omega(r,t_1) + \int\limits_0^r \Psi(r,z)v(z,t_1)dz = \rho(r),\ r \in [0,1]. \tag{33}$$

Step 2: Choose $r = \xi^{\frac{1}{\mu}} z^{\frac{1}{\mu}}$ to get

$$\begin{aligned}
D^{\mu}t(r) &= \frac{1}{\Gamma(1-\mu)} \int_0^r (r-z)^{-\alpha} t'(z)dz \\
&= \frac{1}{\Gamma(1-\mu)} \int_0^{\xi^{\frac{1}{\mu}} z^{\frac{1}{\mu}}} (\xi^{\frac{1}{\mu}} z^{\frac{1}{\mu}} - z)^{-\mu} t'(z)dz \\
&= \frac{1}{\xi\Gamma(1-\mu)} \int_0^{\xi^{\frac{1}{\mu}} s^{\frac{1}{\mu}}} (z^{\frac{1}{\mu}} - \frac{z}{\xi^{\frac{1}{\mu}}})^{-\mu} t'(z)dz.
\end{aligned}$$

Assume $\rho = \frac{z}{\xi^{\frac{1}{\mu}}}$. Then, $dz = \xi^{\frac{1}{\mu}} d\rho$ and

$$\frac{dt}{dz} = \frac{dt}{d\rho}\frac{d\rho}{dz} = \frac{1}{\xi^{\frac{1}{\mu}}}\frac{dt}{d\rho}.$$

Thus,

$$\begin{aligned}
D^{\mu}t(r) &= \frac{1}{\xi\Gamma(1-\mu)} \int_0^{z^{\frac{1}{\mu}}} (z^{\frac{1}{\mu}} - \rho)^{-\mu} \frac{1}{\xi^{\frac{1}{\mu}}}\frac{dt}{d\rho}\xi^{\frac{1}{\mu}} d\rho \\
&= \frac{1}{\xi\Gamma(1-\mu)} \int_0^{z^{\frac{1}{\mu}}} (z^{\frac{1}{\mu}} - \rho)^{-\mu} \frac{dt}{d\rho} d\rho \\
&= \frac{1}{\xi} D^{\mu} t(z^{\frac{1}{\mu}}).
\end{aligned} \tag{34}$$

Hence, Eq.1 becomes

$$D^{\mu}t + \omega(\xi^{\frac{1}{\mu}} z^{\mu}, t) + \int\limits_0^{\xi^{\frac{1}{\mu}} z^{\mu}} \Psi(\xi^{\frac{1}{\mu}} z^{\mu} z, \sigma)\chi(\sigma,t)d\sigma = \rho(\xi^{\frac{1}{\mu}} z^{\mu} z). \tag{35}$$

Putting $\xi = 0$ in Eq. 35 to get

$$D^{\mu}t + \omega(0,t) = \rho(0). \tag{36}$$

The solution has the form $t_1(0) + t_2(r)$. Substitute

$$t(r) = t_1(0) + t_2(r)$$

in Eq. 36, we obtain

$$D^{\mu}t_2(z^{\frac{1}{\mu}}) + \omega(0, t_1(0) + t_2(z^{\frac{1}{\mu}})) = \rho(0). \tag{37}$$

Thus,

$$t(r) = t_1(r) + t_2(\frac{r^{\frac{1}{\mu}}}{\xi}), \tag{38}$$

with

$$t_0 = t(0) = t_1(0) + t_2(0)$$

or

$$t_2(0) = t_0 - t_1(0). \tag{39}$$

Using RKM, we get the solution.

5 Numerical examples

Tow examples will be presented in this part.
Example 1: Consider

$$\xi D^{\frac{1}{2}}t(r) + t(r) + \int_0^r t(\sigma)d\sigma = \rho(r),\ 0 \le r \le 1,\ 0 < \xi << 1, \tag{40}$$

with

$$t(0) = 2 \tag{41}$$

such that

$$\rho(r) = \frac{2}{\sqrt{\pi}}r^{1/2} - r^{1/2}E_{1,3/2}(\frac{-r}{\varepsilon}) + \frac{r^2}{2} + 2r + (2-\xi)e^{-r/\xi} + (1+\xi).$$

When $\xi \to 0$,

$$t_1(r) + \int_0^r t_1(\sigma)d\sigma = \frac{r^2}{2} + 2r + 1 \tag{42}$$

since $\lim\limits_{\xi\to 0} E_{1,3/2}(\frac{-r}{\xi}) = 0$. Thus,

$$t_1'(r) + t_1(r) = r + 2.$$

Hence,

$$t_1(r) = 1 + r + ae^{-r}. \tag{43}$$

Substitute Eq. 43 into Eq. 42 to get

$$1 + r + ae^{-r} + \frac{r^2}{2} + r - ae^{-r} + a = \frac{r^2}{2} + 2r + 1.$$

Thus, $a = 0$ and

$$t_1(r) = r + 1.$$

Let $r = \xi^2 z^2$, we get

$$D^{1/2}t_2(z^2) + 1 + t_2(z^2) = 1$$

or

$$D^{1/2}t_2(z^2)+t_2(z^2)=0$$

subject to

$$t_2(0)=t_0-t_1(0)=1.$$

Using the RKM, we get

$$t_2(z^\mu) = 1-\frac{z}{1}+\frac{z^2}{2!}-\frac{z^3}{3!}+\ldots$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}=e^{-z}.$$

Thus,

$$t(r) = t_1(r)+t_2(\frac{\sqrt{r}}{\xi})$$
$$= r+1+e^{-\frac{\sqrt{r}}{\xi}}.$$

Figures 1-3 show our solution to $\xi = 0.1,\ 0.01, ,\ 0.001$.

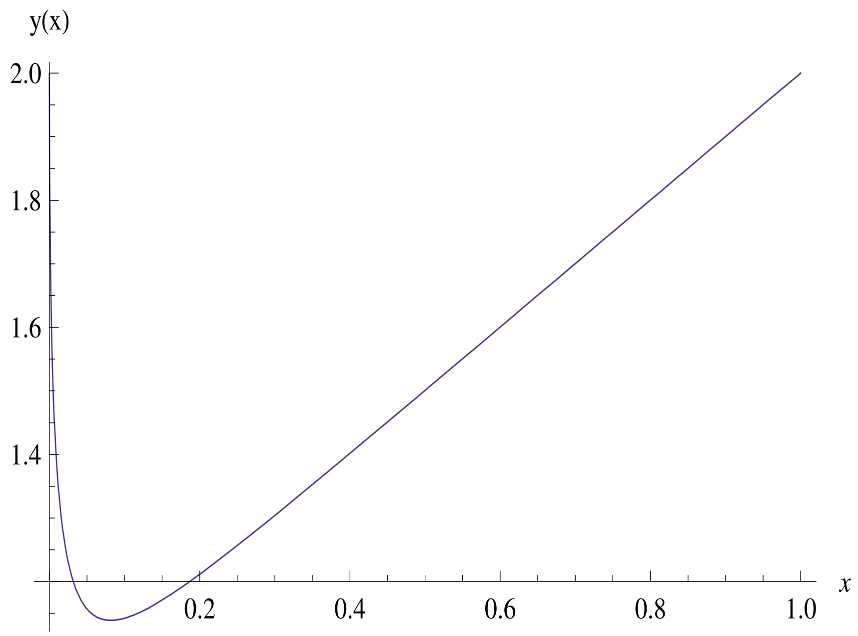

Figure 1. Proposed solution to $\xi = 0.1$.

Example 2: Let

$$\xi D^{\frac{1}{4}}t(r)-\frac{1}{2}t^2+\int_0^r t(\sigma)d\sigma = 0,\ 0\le r\le 1,\ 0<\xi<<1, \tag{44}$$

subject to

$$t(0)=1. \tag{45}$$

When $\xi\to 0$,

$$-\frac{1}{2}t_1^2(r)+\int_0^r t_1(\sigma)d\sigma=0. \tag{46}$$

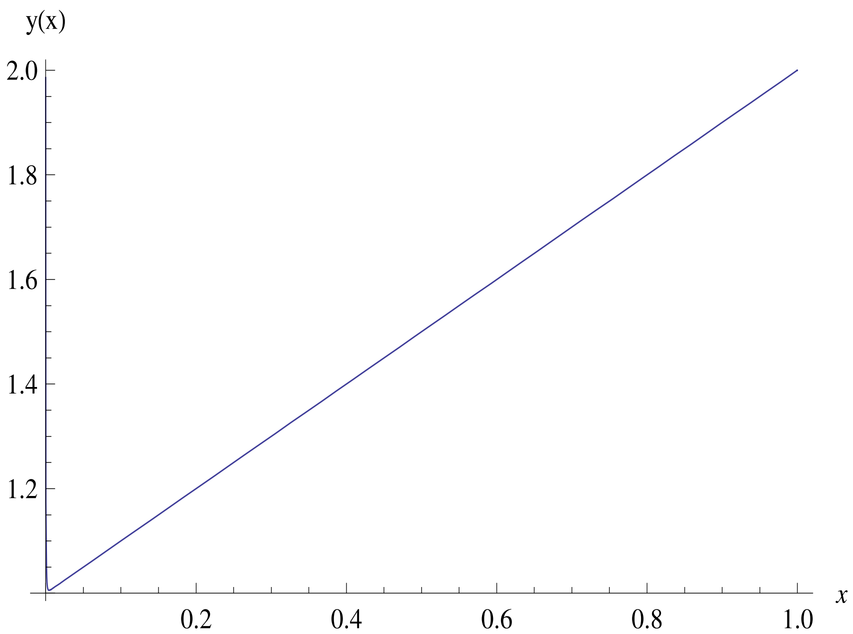

Figure 2. The proposed solution to $\xi = 0.01$.

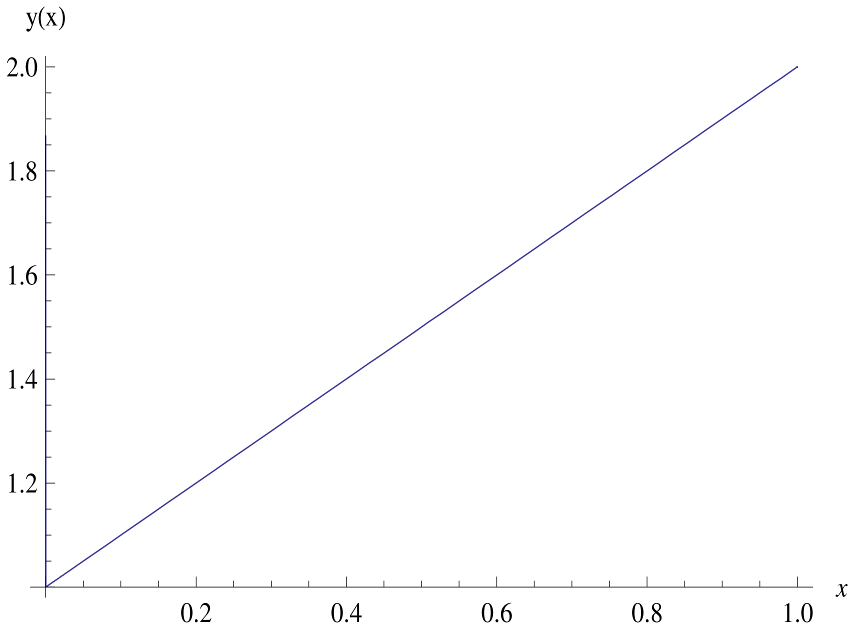

Figure 3. The proposed solution to $\xi = 0.001$.

Thus,

$$-t_1'(r)t_1(r)+t_1(r)=0.$$

Hence,

$$t_1(r)=a+r. \tag{47}$$

Substitute Eq. 47 into Eq. 46 to get

$$-\frac{1}{2}(a+r)^2+\frac{1}{2}(a+r)^2-\frac{1}{2}a^2=0$$

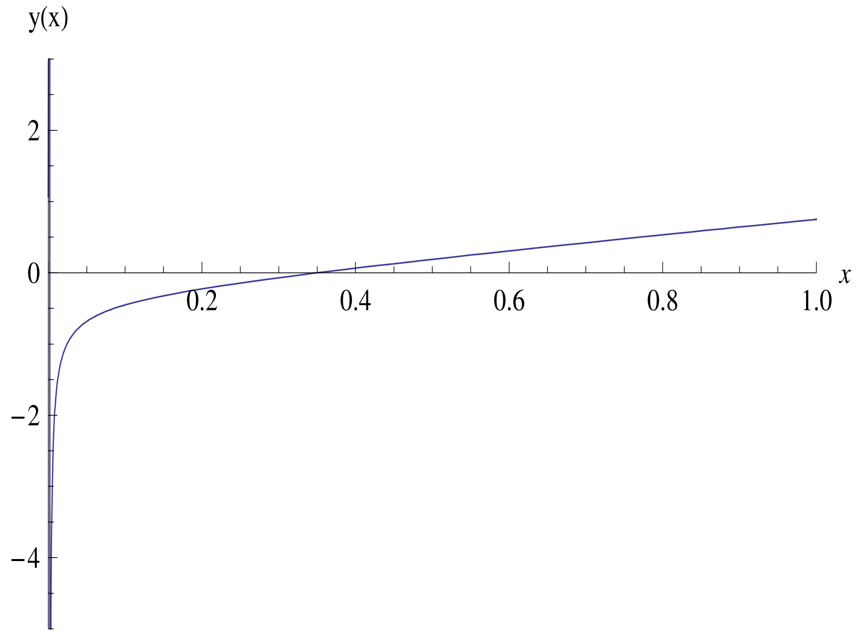

Figure 4. The proposed solution to $\xi = 0.1$.

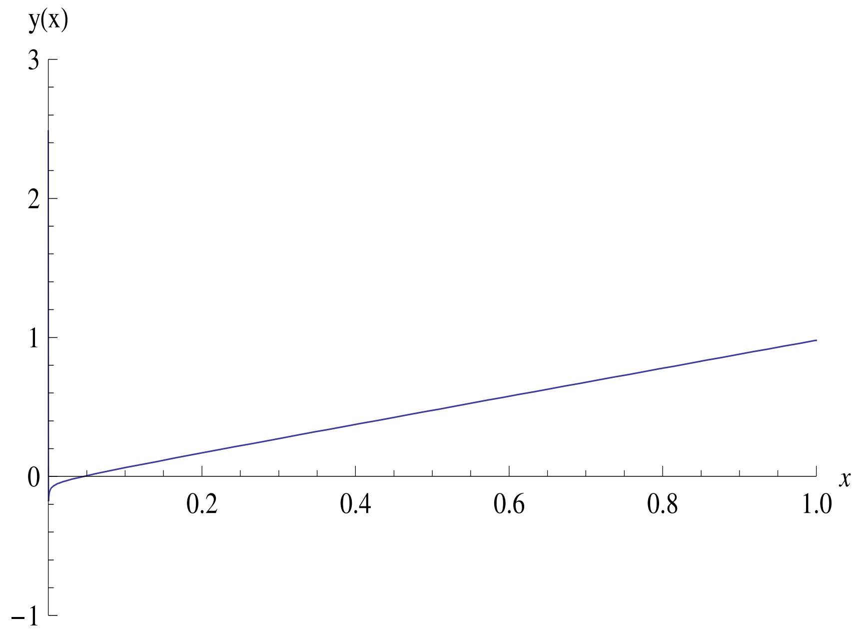

Figure 5. The proposed solution to $\xi = 0.01$.

which implies that $a = 0$ and

$$t_1(r) = r.$$

Using the change of variable $r = \xi^4 z^4$, we get

$$D^{1/4} t_2(z^4) - \frac{1}{2} t_2^2(z^4) = 0$$

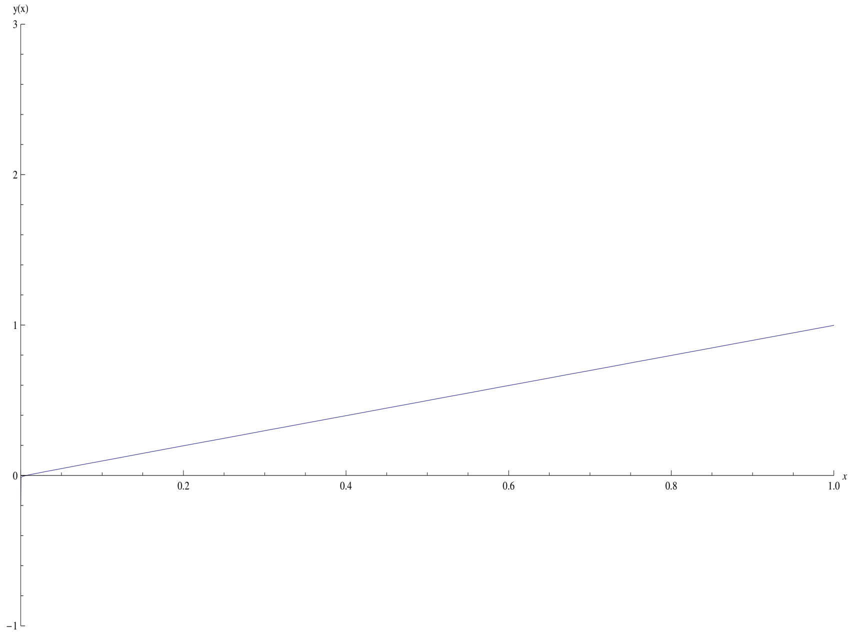

Figure 6. The proposed solution to $\xi = 0.001$.

subject to

$$t_2(0) = t_0 - t_1(0) = 1.$$

Using the RKM, we get

$$\begin{aligned} t_2(z^4) &= 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots \\ &= \sum_{k=0}^{\infty} \frac{z^k}{2^k} = \frac{1}{1-\frac{r}{2}} = \frac{2}{2-r}. \end{aligned}$$

Thus,

$$\begin{aligned} t(r) &= t_1(r) + t_2(\frac{\sqrt[4]{r}}{\xi}) \\ &= r + \frac{2\xi}{2\xi - \sqrt[4]{r}}. \end{aligned}$$

Figures 4-6 show the proposed solutions to $\xi = 0.1,\ 0.01$, and 0.001.

6 Conclusions

In this article, we study singularly perturbed problem of first order with fractional derivative. We present a numerical scheme based on RKM. We prove several theorems related to this topic. Two examples are presented. The outcomes demonstrate that the proposed technique is promising and give precise outcomes. Six figures are presented to support this conclusion.

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